Initial Maclaurin Coefficients Bounds for New Subclasses of Bi-univalent Functions

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Abstract
In this work we introduce the subclasses $\mathcal{L}_2(\theta,\alpha)$ and $\mathcal{L}_2(\theta,\gamma)$ of bi-univalent functions. Furthermore, we obtain coefficient bounds involving the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these classes. The results presented in this paper would generalize those in related works of several earlier authors.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions $f$ which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ with the conditions $f(0) = 0$ and $f'(0) = 1$ and having form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \quad (z \in \mathcal{U}).$$

Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

For each $\theta$, $-\pi < \theta \leq \pi$, Silverman and Silvia (Silverman & Silvia, 1999) introduced the class

$$\mathcal{L}(\theta) = \left\{ f \in \mathcal{A} : \text{Re} \left( f'(z) + \frac{1}{2} e^{i\theta} z f''(z) \right) > 0, \quad z \in \mathcal{U} \right\}$$

and they proved that $\mathcal{L}(\theta) \subset \mathcal{L}(\pi)$, where $\mathcal{L}(\pi)$ is the well known class $\mathcal{R}$ that consists of univalent functions in whose derivatives have positive real part in $\mathcal{U}$ (Alexander, 1915). The class $\mathcal{L}(0)$

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was studied by Singh and Singh (Singh & Singh, 1989), Lewandowski et al. (Lewandowski et al., 1976), Chichra (Chichra, 1977), and Silverman (Silverman, 1994).

It is well known that every function \( f \in S \) has an inverse \( f^{-1} \), defined by
\[
f^{-1}(f(z)) = z \quad (z \in U)
\]
and
\[
f(f^{-1}(w)) = w \quad (|w| < r_0(f); \ r_0(f) \geq \frac{1}{4})
\]
where
\[
f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \).

Let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by (1.1). For a brief history and interesting examples in the class \( \Sigma \), see (Srivastava et al., 2010).

Brannan and Taha (Brannan & Taha, 1988) (see also (Taha, 1981)) introduced certain subclasses of the bi-univalent function class \( \Sigma \) similar to the familiar subclasses \( S^\alpha \) and \( K(\alpha) \) of starlike and convex functions of order \( \alpha \) \((0 \leq \alpha < 1)\), respectively (see (Brannan & Taha, 1988)).

Thus, following Brannan and Taha (Brannan & Taha, 1988) (see also (Taha, 1981)), a function \( f \in A \) is in the class \( S^\alpha \Sigma \) \([\alpha]\) of strongly bi-starlike functions of order \( \alpha \) \((0 < \alpha \leq 1)\) if each of the following conditions is satisfied:
\[
f \in \Sigma \text{ and } \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ z \in U)
\]
and
\[
\left| \arg \left( \frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, \ w \in U),
\]
where \( g \) is the extension of \( f^{-1} \) to \( U \). The classes \( S^\alpha_2(\alpha) \) and \( K_2(\alpha) \) of bi-starlike functions of order \( \alpha \) and bi-convex functions of order \( \alpha \), corresponding (respectively) to the function classes \( S^\alpha \) and \( K(\alpha) \), were also introduced analogously. For each of the function classes \( S^\alpha_2(\alpha) \) and \( K_2(\alpha) \), they found non-sharp estimates on the first two Taylor–Maclaurin coefficients \(|a_2|\) and \(|a_3|\) (for details, see (Brannan & Taha, 1988; Taha, 1981)).

Recently, Srivastava et al. (Srivastava et al., 2010), Frasin (Frasin, 2014), Frasin and Aouf (Frasin & Aouf, 2011), Goyal and Goswami (Goyal & P.Goswami, 2012), Li and Wang (Li & Wang, 2012), Siregar and Raman (Siregar & Raman, 2012) and Caglar et al.(Caglar et al., 2012) introduced new subclasses of bi-univalent functions and found estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in these classes.

The object of the present paper is to introduce two new subclasses of the function class \( \Sigma \) and find estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in these new subclasses of the function class \( \Sigma \).

In order to establish our main results, we shall require the following lemma:
Lemma 1. (Pommerenke, 1975) If \( p \in \mathcal{P} \), then \( |c_k| \leq 2 \) for each \( k \), where \( \mathcal{P} \) is the family of all functions \( p \) analytic in \( \mathcal{U} \) for which
\[
\text{Re}(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathcal{U}).
\]

2. Coefficient bounds for the function class \( L_{\Sigma}(\theta, \alpha) \)

We now introduce the subclass \( L_{\Sigma}(\theta, \alpha) \) of the functions in the class \( \mathcal{A} \) as follows.

**Definition 2.1.** A function \( f(z) \) given by (1.1) is said to be in the class \( L_{\Sigma}(\theta, \alpha) \) where \( 0 < \alpha \leq 1 \) and \( \theta \in (-\pi, \pi] \), if the following conditions are satisfied:
\[
f \in \Sigma \text{ and } \left| \arg \left( f'(z) + \frac{1 + e^{i\theta}}{2}zf''(z) \right) \right| < \frac{\alpha \pi}{2} \quad (z \in \mathcal{U}) \quad (2.1)
\]
and
\[
\left| \arg \left( g'(w) + \frac{1 + e^{i\theta}}{2}wg''(w) \right) \right| < \frac{\alpha \pi}{2} \quad (w \in \mathcal{U}), \quad (2.2)
\]
where the function \( g \) is given by
\[
g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \quad (2.3)
\]

We first state and prove the following result.

**Theorem 1.** Let \( f(z) \) given by (1.1) be in the function class \( L_{\Sigma}(\theta, \alpha) \) where \( 0 < \alpha \leq 1 \) and \( \theta \in (-\pi, \pi] \). Then
\[
|a_2| \leq \frac{2\alpha}{(3\alpha + 9 + (1 - \alpha)\cos 2\theta + 6\cos \theta)^2 + ((1 - \alpha)\sin 2\theta + 6\sin \theta)^2}^{1/4} \quad (2.4)
\]
and
\[
|a_3| \leq \frac{2\alpha^2}{5 + 3\cos \theta} + \frac{2\alpha}{3 \sqrt{5 + 4\cos \theta}}. \quad (2.5)
\]

**Proof.** It follows from (2.1) and (2.2) that
\[
f''(z) + \left( \frac{1 + e^{i\theta}}{2} \right)zf'''(z) = [p(z)]^\alpha \quad (2.6)
\]
and
\[
g'(w) + \left( \frac{1 + e^{i\theta}}{2} \right)wg''(w) = [q(w)]^\alpha \quad (2.7)
\]
where \( p(z) \) and \( q(w) \) are in \( \mathcal{P} \) and have the forms
\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (2.8)
\]
and
\[ q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \]  
(2.9)
Now, equating the coefficients in (2.6) and (2.7), we get
\[
(3 + e^{i \theta}) a_2 = \alpha p_1, 
\]  
(2.10)
\[
3(2 + e^{i \theta}) a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2, 
\]  
(2.11)
\[-(3 + e^{i \theta}) a_2 = \alpha q_1, 
\]  
(2.12)
and
\[
3(2 + e^{i \theta}) (2a_2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2. 
\]  
(2.13)
From (2.10) and (2.12), we get
\[
p_1 = -q_1 
\]  
(2.14)
and
\[
2(3 + e^{i \theta}) a_2^2 = \alpha^2 (p_1^2 + q_1^2). 
\]  
(2.15)
Now from (2.11), (2.13) and (2.15), we obtain
\[
6(2 + e^{i \theta}) a_2^2 = \alpha (p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} (p_1^2 + q_1^2) 
= \alpha (p_2 + q_2) + \frac{\alpha(\alpha - 1)(3 + e^{i \theta})^2}{\alpha} a_2^2. 
\]  
Thus
\[
a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{6 \alpha (2 + e^{i \theta}) - (\alpha - 1)(3 + e^{i \theta})^2} 
\]  
that is
\[
|a_2^2| = \frac{\alpha^2 |p_2 + q_2|}{|6 \alpha (2 + e^{i \theta}) - (\alpha - 1)(3 + e^{i \theta})^2|}. 
\]
Applying Lemma 1 for the coefficients \( p_2 \) and \( q_2 \), we have
\[
|a_2| \leq \frac{2 \alpha}{[(3 \alpha + 9 + (1 - \alpha) \cos 2 \theta + 6 \cos \theta)^2 + ((1 - \alpha) \sin 2 \theta + 6 \sin \theta)^2]^{1/4}}. 
\]
This gives the bound on \( |a_2| \) as asserted in (2.4).
Next, in order to find the bound on \( |a_3| \), by subtracting (2.13) from (2.11), we get
\[
6(2 + e^{i \theta}) a_3 - 6(2 + e^{i \theta}) a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 - (\alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2). 
\]  
(2.16)
Upon substituting the value of \( a_2^2 \) from (2.15) and observing that \( p_1^2 = q_1^2 \), it follows that
\[
a_3 = \frac{\alpha^2 p_1^2}{(3 + e^{i \theta})^2} + \frac{\alpha (p_2 - q_2)}{6(2 + e^{i \theta})}. 
\]
Applying Lemma 1 once again for the coefficients $p_1$, $p_2$, $q_1$ and $q_2$, we readily get

$$|a_3| \leq \frac{2\alpha^2}{5 + 3\cos \theta} + \frac{2\alpha}{3 \sqrt{5 + 4\cos \theta}},$$

which completes the proof of Theorem 1.

Choosing $\theta = \pi$ in Theorem 1, we obtain the following particular case due to Srivastava et al. (Srivastava et al., 2010):

**Corollary 2.1.** (Srivastava et al., 2010) Let $f(z)$ given by (1.1) be in the function class $L_\Sigma(\pi, \alpha)$; $0 < \alpha \leq 1$. Then

$$|a_2| \leq \alpha \sqrt{\frac{2}{\alpha + 1}} \quad (2.17)$$

and

$$|a_3| \leq \frac{\alpha(3\alpha + 2)}{3} \quad (2.18)$$

Putting $\theta = 0$ in Theorem 1, we obtain the following particular case due to Frasin (Frasin, 2014):

**Corollary 2.2.** (Frasin, 2014) Let $f(z)$ given by (1.1) be in the function class $L_\Sigma(0, \alpha)$, $0 < \alpha \leq 1$. Then

$$|a_2| \leq \alpha \sqrt{\frac{2}{\alpha + 8}} \quad (2.19)$$

and

$$|a_3| \leq \frac{9\alpha^2 + 8\alpha}{36} \quad (2.20)$$

3. **Coefficient bounds for the function class $L_\Sigma(\theta, \gamma)$**

**Definition 3.1.** A function $f(z)$ given by (1.1) is said to be in the class $L_\Sigma(\theta, \gamma)$ where $0 \leq \gamma < 1$, $\theta \in (-\pi, \pi]$, if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \Re\left(\frac{f'(z)}{2} + \frac{1 + e^{i\theta}}{2}zf''(z)\right) > \gamma \quad (z \in \mathcal{U}) \quad (3.1)$$

and

$$\Re\left(\frac{g'(w)}{2} + \frac{1 + e^{i\theta}}{2}wg''(w)\right) > \gamma \quad (w \in \mathcal{U}), \quad (3.2)$$

where the function $g$ is given by (2.3).
Theorem 2. Let \( f(z) \) given by (1.1) be in the class \( \mathcal{L}_\Sigma(\theta, \gamma) \), where \( 0 \leq \gamma < 1, \theta \in (-\pi, \pi] \). Then

\[
|a_2| \leq \sqrt{\frac{4(1 - \gamma)}{6 \sqrt{5 + 4 \cos \theta}}}
\]

and

\[
|a_3| \leq \frac{2(1 - \gamma)^2}{5 + 3 \cos \theta} + \frac{2(1 - \gamma)}{3 \sqrt{5 + 4 \cos \theta}}.
\]

Proof. It follows from (3.1) and (3.2) that there exist \( p \) and \( q \in P \) such that

\[
f'(z) + \left( \frac{1 + e^{i\theta}}{2} \right) z f''(z) = \gamma + (1 - \gamma)p(z)
\]

and

\[
g'(w) + \left( \frac{1 + e^{i\theta}}{2} \right) wg''(w) = \gamma + (1 - \gamma)q(w)
\]

where \( p(z) \) and \( q(w) \) have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

\[
(3 + e^{i\theta})a_2 = (1 - \gamma)p_1,
\]

\[
3(2 + e^{i\theta})a_3 = (1 - \gamma)p_2,
\]

\[
-(3 + e^{i\theta})a_2 = (1 - \gamma)q_1
\]

and

\[
3(2 + e^{i\theta})(2a_2^2 - a_3) = (1 - \gamma)q_2
\]

From (3.7) and (3.9), we get

\[
p_1 = -q_1
\]

and

\[
2(3 + e^{i\theta})^2a_2^2 = (1 - \gamma)^2(p_1^2 + q_1^2).
\]

Also, from (3.8) and (3.10), we find that

\[
6(2 + e^{i\theta})a_2^2 = (1 - \gamma)(p_2 + q_2).
\]

Thus, we have

\[
|a_2^2| \leq \frac{(1 - \gamma)}{6 |2 + e^{i\theta}|(|p_2| + |q_2|)}
\]

\[
\leq \frac{4(1 - \gamma)}{6 \sqrt{5 + 4 \cos \theta}}
\]

which is the bound on \( |a_2| \) as given in (3.3).
Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get
\[ 6(2 + e^{i\theta})a_3 - 6(2 + e^{i\theta})a_2^2 = (1 - \gamma)(p_2 - q_2) \]
or, equivalently,
\[ a_3 = a_2^2 + \frac{(1 - \gamma)(p_2 - q_2)}{6(2 + e^{i\theta})}. \]

Upon substituting the value of $a_2^2$ from (3.12), we obtain
\[ a_3 = \frac{(1 - \gamma)^2(p_1^2 + q_1^2)}{2(3 + e^{i\theta})^2} + \frac{(1 - \gamma)(p_2 - q_2)}{6(2 + e^{i\theta})}. \]

Applying Lemma 1 for the coefficients $p_1, p_2, q_1$ and $q_2$, we readily get
\[ |a_3| \leq \frac{2(1 - \gamma)^2}{5 + 3 \cos \theta} + \frac{2(1 - \gamma)}{3 \sqrt{5 + 4 \cos \theta}} \]
which is the bound on $|a_3|$ as asserted in (3.4).

Choosing $\theta = \pi$ in Theorem 2, we obtain the following particular case due to Srivastava et al. (Srivastava et al., 2010):

**Corollary 3.1.** (Srivastava et al., 2010) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(0, \gamma)$, $0 \leq \gamma < 1$. Then
\[ |a_2| \leq \sqrt{\frac{2(1 - \gamma)}{3}} \]
and
\[ |a_3| \leq \frac{(1 - \gamma)(5 - 3\gamma)}{3}. \] (3.14)

Putting $\theta = 0$ in Theorem 2, we obtain the following particular case due to Frasin (Frasin, 2014):

**Corollary 3.2.** (Frasin, 2014) Let $f(z)$ given by (1.1) be in the function class $\mathcal{L}_\Sigma(0, \gamma)$, $0 \leq \gamma < 1$. Then
\[ |a_2| \leq \frac{1}{3} \sqrt{2(1 - \gamma)} \]
and
\[ |a_3| \leq \frac{(1 - \gamma)(9(1 - \gamma) + 8)}{36}. \] (3.16)

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References


