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Further on Fuzzy Pseudo Near Compactness and *ps-ro* Fuzzy Continuous Functions

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Abstract

Main objective of this paper is to study further properties of fuzzy pseudo near compactness via *ps-ro* closed fuzzy sets, fuzzy nets and fuzzy filterbases. It is shown by an example that *ps-ro* fuzzy continuous and fuzzy continuous functions do not imply each other. Several characterizations of *ps-ro* fuzzy continuous function are obtained in terms of a newly introduced concept of *ps-ro* interior operator, *ps-ro q-nbd* and its graph.

Keywords: Fuzzy pseudo near compactness, fuzzy net, fuzzy filterbase, *ps-ro* interior, *ps-ro q-nbd*.
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1. Introduction

In (Ray & Chettri, 2010), while finding interplay between a fuzzy topological space (fts, for short) (X, τ) and its corresponding strong α-level topology(general) on X, the concept of pseudo regular open(closed) fuzzy sets and *ps-ro* fuzzy topology on X was introduced, members of which are called *ps-ro* open fuzzy sets and their complements are *ps-ro* closed fuzzy sets on (X, τ). In (Ray & Chettri, 2011), in terms of above fuzzy sets, a fuzzy continuous type function called *ps-ro* fuzzy continuous function and a compact type notion called fuzzy pseudo near compactness were introduced and different properties were studied.

In this paper, fuzzy pseudo near compactness has been studied via *ps-ro* closed fuzzy sets, fuzzy nets and fuzzy filterbases. Further, it is shown by an example that *ps-ro* fuzzy continuous and fuzzy continuous functions are independent of each other. An interior-type operator called *ps-ro* interior is introduced and several properties of such functions are studied interms of this operator, *ps-ro q-nbd* and its graph.

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We state a few known definitions and results here that we require subsequently. A fuzzy point x_α is said to q-coincident with a fuzzy set A, denoted by $x_\alpha qA$ if $\alpha + A(x) > 1$. If A and B are not q-coincident, we write $A \not{q} B$. A fuzzy set A is said to be a q-neighbourhood (in short, q-*nbd*.) of a fuzzy point x_α if there is a fuzzy open set B such that $x_\alpha qB \leqslant A$ (Pao-Ming & Ying-Ming, 1980). Let f be a function from a set X into a set Y. Then the following holds:
(i) $f^{-1}(1 - B) = 1 - f^{-1}(B)$, for any fuzzy set B on Y.
(ii) $A_1 \leqslant A_2 \Rightarrow f(A_1) \leqslant f(A_2)$, for any fuzzy sets A_1 and A_2 on X. Also, $B_1 \leqslant B_2 \Rightarrow f^{-1}(B_1) \leqslant f^{-1}(B_2)$, for any fuzzy sets B_1 and B_2 on Y.
(iii) $ff^{-1}(B) \leqslant B$, for any fuzzy set B on Y and the equality holds if f is onto. Also, $f^{-1}f(A) \geqslant A$, for any fuzzy set A on X, equality holds if f is one-to-one (Chang, 1968). For a function $f : X \to Y$, the graph $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$, for each $x \in X$, where X and Y are any sets. Let X, Y be fts and $g : X \to X \times Y$ be the graph of the function $f : X \to Y$. Then if A, B are fuzzy sets on X and Y respectively, $g^{-1}(A \times B) = A \wedge f^{-1}(B)$(Azad, 1981). Let Z, X, Y be fts and $f_1 : Z \to X$ and $f_2 : Z \to Y$ be two functions. Let $f : Z \to X \times Y$ be defined by $f(z) = (f_1(z), f_2(z))$ for $z \in Z$, where $X \times Y$ is provided with the product fuzzy topology. Then if B, U_1 U_2 are fuzzy sets on Z, X, Y respectively such that $f(B) \leqslant U_1 \times U_2$, then $f_1(B) \leqslant U_1$ and $f_2(B) \leqslant U_2$ (Bhattacharyya & Mukherjee, 2000). A function f from a fts (X, τ) to fts (Y, σ) is said to be fuzzy continuous, if $f^{-1}(\mu)$ is fuzzy open on X, for all fuzzy open set μ on Y (Chang, 1968). For a fuzzy set μ in X, the set $\mu^\alpha = \{x \in X : \mu(x) > \alpha\}$ is called the strong α-level set of X. In a fts (X, τ), the family $i_\alpha(\tau) = \{\mu^\alpha : \mu \in \tau\}$ for all $\alpha \in I_1 = [0, 1)$ forms a topology on X called strong α-level topology on X (Lowen, 1976), (Kohli & Prasannan, 2001). A fuzzy open(closed) set μ on a fts (X, τ) is said to be pseudo regular open(closed) fuzzy set if the strong α-level set μ^α is regular open(closed) in $(X, i_\alpha(\tau))$, $\forall \alpha \in I_1$. The family of all pseudo regular open fuzzy sets form a fuzzy topology on X called *ps-ro* fuzzy topology on X which is coarser than τ. Members of *ps-ro* fuzzy topology are called *ps-ro* open fuzzy sets and their complements are known as *ps-ro* closed fuzzy sets on (X, τ) (Ray & Chettri, 2010). A function f from a fts (X, τ_1) to another fts (Y, τ_2) is pseudo fuzzy *ro* continuous (in short, *ps-ro* fuzzy continuous) if $f^{-1}(U)$ is *ps-ro* open fuzzy set on X for each pseudo regular open fuzzy set U on Y. For a fuzzy set A, $\wedge\{B : A \leqslant B$, B is *ps-ro* closed fuzzy set on $X\}$ is called fuzzy *ps*-closure of A. In a fts (X, τ), a fuzzy set A is said to be a *ps-ro nbd*. of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha \in B \leqslant A$. In addition, if A is *ps-ro* open fuzzy set, the *ps-ro nbd*. is called *ps-ro* open *nbd*. A fuzzy set A is called *ps-ro* quasi neighborhood or simply *ps-ro q-nbd*. of a fuzzy point x_α, if there is a *ps-ro* open fuzzy set B such that $x_\alpha qB \leqslant A$. In addition, if A is *ps-ro* open, the *ps-ro q-nbd*. is called *ps-ro* open *q-nbd*. Let $\{S_n : n \in D\}$ be a fuzzy net on a fts X. i.e., for each member n of a directed set $(D, \leqslant)$, S_n be a fuzzy set on X. A fuzzy point x_α on X is said to be a fuzzy *ps*-cluster point of the fuzzy net if for every $n \in D$ and every *ps-ro* open *q-nbd*. V of x_α, there exists $m \in D$, with $n \leqslant m$ such that $S_m qV$. A collection $\mathcal{B}$ of fuzzy sets on a fts (X, τ) is said to form a fuzzy filter base in X if for every finite subcollection $\{B_1, B_2, ..., B_n\}$ of $\mathcal{B}$, $\bigwedge_{i=1}^n B_i \neq 0$ (Ray & Chettri, 2011).

2. Fuzzy Pseudo Near Compactness

It is easy to observe, as pseudo regular open fuzzy sets form a base for *ps-ro* fuzzy topology, replacing *ps-ro* open cover by pseudo regular open cover, we may obtain pseudo near compactness.

Definition 2.1. Let x_α be a fuzzy point on a fts X. A fuzzy net $\{S_n : n \in (D, \geqslant)\}$ on X is said to ps-converge to x_α, written as $S_n \xrightarrow{ps} x_\alpha$ if for each *ps-ro* open *q-nbd*. W of x_α, there exists $m \in D$ such that $S_n qW$ for all $n \geqslant m$,$(n \in D)$.

Definition 2.2. Let x_α be a fuzzy point on a fts X. A fuzzy filterbase $\mathcal{B}$ is said to
(i) ps-adhere at x_α written as $x_\alpha \leqslant ps\text{-}ad.\mathcal{B}$ if for each *ps-ro* open *q-nbd*. U of x_α and each $B \in \mathcal{B}$, BqU.
(ii) ps-converge to x_α, written as $\mathcal{B} \xrightarrow{ps} x_\alpha$ if for each *ps-ro* open *q-nbd*. U of x_α, there coresponds some $B \in \mathcal{B}$ such that $B \leqslant U$.

Theorem 2.1. A fts (X, τ) is fuzzy pseudo nearly compact iff every $\{B_\alpha : \alpha \in \Lambda\}$ of *ps-ro* closed fuzzy sets on X with $\wedge_{\alpha\in\Lambda} B_\alpha = 0$, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha\in\Lambda_0} B_\alpha = 0$.
Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a *ps-ro* open cover of X. Now, $\wedge_{\alpha\in\Lambda}(1 - U_\alpha) = (1 - \vee_{\alpha\in\Lambda} U_\alpha) = 0$. As $\{1 - U_\alpha : \alpha \in \Lambda\}$ is a collection of *ps-ro* closed fuzzy sets on X, by given condition, there exist a finite subset Λ_0 of Λ such that $\wedge_{\alpha\in\Lambda_0}(1 - U_\alpha) = 0 \Rightarrow 1 - \vee_{\alpha\in\Lambda_0} U_\alpha = 0$. i.e., $1 = \vee_{\alpha\in\Lambda_0} U_\alpha$. So, X is fuzzy pseudo nearly compact.
Conversely, let $\{B_\alpha : \alpha \in \Lambda\}$ be a family of *ps-ro* closed fuzzy sets on X with $\wedge_{\alpha\in\Lambda} B_\alpha = 0$. Then $1 = 1 - \wedge_{\alpha\in\Lambda} B_\alpha \Rightarrow 1 = \vee_{\alpha\in\Lambda}(1 - B_\alpha)$. By given condition there exist a finite subset Λ_0 of Λ such that $1 = \vee_{\alpha\in\Lambda_0}(1 - B_\alpha) \Rightarrow 1 = (1 - \wedge_{\alpha\in\Lambda_0} B_\alpha)$. Hence, $\wedge_{\alpha\in\Lambda_0} B_\alpha \leqslant (\wedge_{\alpha\in\Lambda_0} B_\alpha) \wedge (1 - \wedge_{\alpha\in\Lambda_0} B_\alpha) = 0$. Consequently, $\wedge_{\alpha\in\Lambda_0} B_\alpha = 0$.

Theorem 2.2. For a fuzzy set A on a fts, the following are equivalent:
(a) Every fuzzy net in A has fuzzy ps-cluster point in A.
(b) Every fuzzy net in A has a ps-convergent fuzzy subnet.
(c) Every fuzzy filterbase in A ps-adheres at some fuzzy point in A.
Proof. $(a) \Rightarrow (b)$**:** Let $\{S_n : n \in (D, \geqslant)\}$ be a fuzzy net in A having fuzzy ps-cluster point at $x_\alpha \leqslant A$. Let $Q_{x_\alpha} = \{A : A$ is *ps-ro* open *q-nbd*. of $x_\alpha\}$. For any $B \in Q_{x_\alpha}$, some $n \in D$ can be chosen such that $S_n qB$. Let E denote the set of all ordered pairs (n, B) with the property that $n \in D$, $B \in Q_{x_\alpha}$ and $S_n qB$. Then $(E, \succ)$ is a directed set where $(m, C) \succ (n, B)$ iff $m \geqslant n$ in D and $C \leqslant B$. Then $T : (E \succ) \to (X, \tau)$ given by $T(n, B) = S_n$, is a fuzzy subnet of $\{S_n : n \in (D, \geqslant)\}$. Let V be any *ps-ro* open *q-nbd*. of x_α. Then there exists $n \in D$ such that $(n, V) \in E$ and hence $S_n qV$. Now, for any $(m, U) \succ (n, V)$, $T(m, U) = S_m qU \leqslant V \Rightarrow T(m, U)qV$. Hence, $T \xrightarrow{ps} x_\alpha$.
$(b) \Rightarrow (a)$ If a fuzzy net $\{S_n : n \in (D, \geqslant)\}$ in A does not have any fuzzy ps-cluster point, then there is a *ps-ro* open *q-nbd*. U of X_α and $n \in D$ such that $S_n \not{q} U, \forall\, m \geqslant n$. Then clearly no fuzzy subnet of the fuzzy net can ps-converge to x_α.
$(c) \Rightarrow (a)$ Let $\{S_n : n \in (D, \geqslant)\}$ be a fuzzy net in A. Consider the fuzzy filter base $\mathcal{F} = \{T_n : n \in D\}$ in A, generated by the fuzzy net, where $T_n = \{S_m : m \in (D, \geqslant)$ and $m \geqslant n\}$. By (c), there exist a fuzzy point $a_\alpha \leqslant A \wedge (ps\text{-}ad\mathcal{F})$. Then for each *ps-ro* open *q-nbd*. U of a_α and each $F \in \mathcal{F}$, UqF, i.e., $UqT_n, \forall\, n \in D$. Hence, the given fuzzy net has fuzzy ps-cluster point a_α.
$(a) \Rightarrow (c)$ Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a fuzzy filterbase in A. For each $\alpha \in \Lambda$, choose a fuzzy point $x_{F_\alpha} \leqslant F_\alpha$, and construct the fuzzy net $S = \{x_{F_\alpha} : F_\alpha \in \mathcal{F}\}$ in A with $(\mathcal{F}, \succ\succ)$ as domain, where for two members $F_\alpha, F_\beta \in \mathcal{F}$, $F_\alpha \succ\succ F_\beta$ iff $F_\alpha \leqslant F_\beta$. By (a), the fuzzy net has a fuzzy ps-cluster

point say $x_t \leqslant A$, where $0 < t \leqslant 1$. Then for any *ps-ro* open *q-nbd*. U of x_t and any $F_\alpha \in \mathcal{F}$, there exists $F_\beta \in \mathcal{F}$ such that $F_\beta >> F_\alpha$ and $x_{F_\beta} qU$. Then $F_\beta qU$ and hence $F_\alpha qU$. Thus $\mathcal{F}$ adheres at x_t.

Theorem 2.3. If a fts is fuzzy pseudo nearly compact, then every fuzzy filterbase on X with at most one ps-adherent point is ps-convergent.

Proof. Let $\mathcal{F}$ be a fuzzy filterbase with at most one ps-adherent point in a fuzzy pseudo nearly compact fts X. Then by Theorem (2.2), $\mathcal{F}$ has at least one ps-adherent point. Let x_α be the unique ps-adherent point of $\mathcal{F}$. If $\mathcal{F}$ does not ps-converge to x_α, then there is some *ps-ro* open *q-nbd*. U of x_α such that for each $F \in \mathcal{F}$ with $F \leqslant U$, $F \wedge (1 - U) \neq 0$. Then $\mathcal{G} = \{F \wedge (1 - U) : F \in \mathcal{F}\}$ is a fuzzy filterbase on X and hence has a ps-adherent point y_t(say) in X. Now, $U \not{q} G$, for all $G \in \mathcal{G}$, so that $x_\alpha \neq y_t$. Again, for each *ps-ro* open *q-nbd*. V of y_t and each $F \in \mathcal{F}$, $Vq(F \wedge (1 - U)) \Rightarrow VqF \Rightarrow y_t$ is a ps-adherent point of $\mathcal{F}$, where $x_\alpha \neq y_t$. This shows that y_t is another ps-adherent point of $\mathcal{F}$, which is not the case.

3. *ps-ro* Fuzzy Continuous Functions

We begin this section by introducing an interior-type operator, called ps-interior operator and observe a few useful properties of that operator.

Definition 3.1. The union of all *ps-ro* open fuzzy sets, each contained in a fuzzy set A on a fts X is called fuzzy ps-interior of A and is denoted by $ps\text{-}int(A)$. So, $ps\text{-}int(A) = \vee\{B : B \leqslant A,\ B$ is *ps-ro* open fuzzy set on $X\}$

Some properties of *ps-int* operator are furnished below. The proofs are straightforward and hence omitted.

Theorem 3.1. For any fuzzy set A on a fts (X, τ), the following hold:
(a) $ps\text{-}int(A)$ is the largest *ps-ro* open fuzzy set contained in A.
(b) $ps\text{-}int(0) = 0$, $ps\text{-}int(1) = 1$.
(c) $ps\text{-}int(A) \leqslant A$.
(d) A is *ps-ro* open fuzzy set iff $A = ps\text{-}int(A)$.
(e) $ps\text{-}int(ps\text{-}int(A)) = ps\text{-}int(A)$.
(f) $ps\text{-}int(A) \leqslant ps\text{-}int(B), if A \leqslant B$.
(g) $ps\text{-}int(A \wedge B) = ps\text{-}int(A) \wedge ps\text{-}int(B)$.
(h) $ps\text{-}int(A \vee B) \geqslant ps\text{-}int(A) \vee ps\text{-}int(B)$.
(i) $ps\text{-}int(ps\text{-}int(A)) = ps\text{-}int(A)$.
(j) $1 - ps\text{-}int(A) = ps\text{-}cl(1 - A)$.
(k) $1 - ps\text{-}cl(A) = ps\text{-}int(1 - A)$.

Now, we recapitulate the definition of *ps-ro* fuzzy continuous functions.

Definition 3.2. A function f from fts (X, τ_1) to fts (Y, τ_2) is pseudo fuzzy *ro* continuous (in short, *ps-ro* fuzzy continuous) if $f^{-1}(U)$ is *ps-ro* open fuzzy set on X for each pseudo regular open fuzzy set U on Y.

The following Example shows that *ps-ro* fuzzy continuity and fuzzy continuity do not imply each other.

Example 3.1. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Let A, B and C be fuzzy sets on X defined by $A(a) = 0.2, A(b) = 0.4, A(c) = 0.4$, $B(t) = 0.4, \forall t \in X$ and $C(t) = 0.2, \forall t \in X$. Let D and E be fuzzy sets on Y defined by $D(t) = 0.2, \forall t \in Y$ and $E(x) = 0.6, E(y) = 0.7, E(z) = 0.7$. Clearly, $\tau_1 = \{0, 1, A, B, C\}$ and $\tau_2 = \{0, 1, D, E\}$ are fuzzy topologies on X and Y respectively. In the corresponding topological space $(X, i_\alpha(\tau_1)), \forall \alpha \in I_1 = [0, 1)$, the open sets are $\phi, X, A^\alpha, B^\alpha$ and C^α, where $A^\alpha = \begin{cases} X, & \text{for } \alpha < 0.2 \\ \{b, c\}, & \text{for } 0.2 \leqslant \alpha < 0.4 \\ \phi, & \text{for } \alpha \geqslant 0.4 \end{cases}$, $B^\alpha = \begin{cases} X, & \text{for } \alpha < 0.4 \\ \phi, & \text{for } \alpha \geqslant 0.4 \end{cases}$ and $C^\alpha = \begin{cases} X, & \text{for } \alpha < 0.2 \\ \phi, & \text{for } \alpha \geqslant 0.2 \end{cases}$

For $0.2 \leqslant \alpha < 0.4$, the closed sets are on $(X, i_\alpha(\tau_1))$ are ϕ, X and $\{a\}$. Therefore, $int(cl(A^\alpha)) = X$. So, A^α is not regular open on $(X, i_\alpha(\tau_1))$ and hence, A is not pseudo regular open fuzzy sets on (X, τ_1) for $0.2 \leqslant \alpha < 0.4$. Similarly, it can be seen that $0, 1, B$ and C are pseudo regular open fuzzy set on (X, τ_1). Therefore, *ps-ro* fuzzy topology on X is $\{0, 1, B, C\}$. Again, E is not pseudo regular open fuzzy set for $0.6 \leqslant \alpha < 0.7$ on Y. Therefore, *ps-ro* fuzzy topology on Y is $\{0, 1, D\}$. Now, $ps\text{-}cl(B) = 1 - B$ and $ps\text{-}cl(C) = 1 - B$ where, $(1 - B)(t) = 0.6, \forall t \in X$. Define a function $f : X \to Y$ by $f(a) = x, f(b) = y$ and $f(c) = z$. Then, $f^{-1}(D)(t) = 0.2 = C(t), \forall t \in X$. Hence, $f^{-1}(U)$ is *ps-ro* open fuzzy set on X, for every *ps-ro* open fuzzy set U on Y. Therefore, f is *ps-ro* fuzzy continuous function. But, f is not fuzzy continuous as $f^{-1}(E)$ is not fuzzy open on X. Clearly, every *ps-ro* open fuzzy set is fuzzy open but not conversely, as for an example here A is fuzzy open but not *ps-ro* open fuzzy on X. This implies that a fuzzy continuous function need not be *ps-ro* fuzzy continuous. Hence, *ps-ro* fuzzy continuous and fuzzy continuous functions are independent of each other.

The following couple of results give characterizations of *ps-ro* fuzzy continuous functions.

Theorem 3.2. Let (X, τ) and (Y, σ) be two fts. For a function $f : X \to Y$, the following are equivalent:
(a) f is *ps-ro* fuzzy continuous.
(b) Inverse image of each *ps-ro* open fuzzy set on Y under f is *ps-ro* open on X.
(c) For each fuzzy point x_α on X and each *ps-ro* open *nbd*. V of $f(x_\alpha)$, there exists a *ps-ro* open fuzzy set U on X, such that $x_\alpha \leqslant U$ and $f(U) \leqslant V$.
(d) For each *ps-ro* closed fuzzy set F on Y, $f^{-1}(F)$ is *ps-ro* closed on X.
(e) For each fuzzy point x_α on X, the inverse image under f of every *ps-ro nbd*. of $f(x_\alpha)$ on Y is a *ps-ro nbd*. of x_α on X.
(f) For all fuzzy set A on X, $f(ps\text{-}cl(A)) \leqslant ps\text{-}cl(f(A))$.
(g) For all fuzzy set B on Y, $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(B))$.
(h) For all fuzzy set B on Y, $f^{-1}(ps\text{-}int(B)) \leqslant ps\text{-}int(f^{-1}(B))$.

Proof. $(a) \Rightarrow (b)$ Let f be *ps-ro* fuzzy continuous and μ be any *ps-ro* open fuzzy set on Y. Then $\mu = \vee \mu_i$, where μ_i is pseudo regular open fuzzy set on Y, for each i. Now, $f^{-1}(\mu) = f^{-1}(\vee_i \mu_i) = \vee_i f^{-1}(\mu_i)$. f being *ps-ro* fuzzy continuous, $f^{-1}(\mu_i)$ is *ps-ro* open fuzzy set and consequently,

$f^{-1}(\mu)$ is *ps-ro* open fuzzy set on X.
$(b) \Rightarrow (a)$ Let the inverse image of each *ps-ro* open fuzzy set on Y under f be *ps-ro* open fuzzy set on X. Let U be a pseudo regular open fuzzy set on Y. Every pseudo regular open fuzzy set being *ps-ro* open fuzzy set, the result follows.
$(b) \Rightarrow (c)$ Let V be any *ps-ro* open *nbd*. of $f(x_\alpha)$ on Y. Then there is a *ps-ro* open fuzzy set V_1 on Y such that $f(x_\alpha) \leqslant V_1 \leqslant V$. By hypothesis, $f^{-1}(V_1)$ is *ps-ro* open fuzzy set on X. Again, $x_\alpha \leqslant f^{-1}(V_1) \leqslant f^{-1}(V)$. So, $f^{-1}(V)$ is a *ps-ro nbd*. of x_α, such that $f(f^{-1}(V)) \leqslant V$, as desired.
$(c) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y and $x_\alpha \leqslant f^{-1}(V)$. Then $f(x_\alpha) \leqslant V$ and so by given condition, there exists *ps-ro* open fuzzy set U on X such that $x_\alpha \leqslant U$ and $f(U) \leqslant V$. Hence, $x_\alpha \leqslant U \leqslant f^{-1}(V)$. i.e., $f^{-1}(V)$ is a *ps-ro nbd*. of each of the fuzzy points contained in it. Thus $f^{-1}(V)$ is *ps-ro* open fuzzy set on X.
$(b) \Leftrightarrow (d)$ Obvious.
$(b) \Rightarrow (e)$ Suppose, W is a *ps-ro* open *nbd*. of $f(x_\alpha)$. Then there exists a *ps-ro* open fuzzy set U on Y such that $f(x_\alpha) \leqslant U \leqslant W$. Then $x_\alpha \leqslant f^{-1}(U) \leqslant f^{-1}(W)$. By hypothesis, $f^{-1}(U)$ is *ps-ro* open fuzzy set on X and hence the result is obtained.
$(e) \Rightarrow (b)$ Let V be any *ps-ro* open fuzzy set on Y. If $x_\alpha \leqslant f^{-1}(V)$ then $f(x_\alpha) \leqslant V$ and so $f^{-1}(V)$ is a *ps-ro nbd*. of x_α.
$(d) \Rightarrow (f)$ $ps\text{-}cl(f(A))$ being a *ps-ro* closed fuzzy set on Y, $f^{-1}(ps\text{-}cl(f(A)))$ is *ps-ro* closed fuzzy set on X. Again, $f(A) \leqslant ps\text{-}cl(f(A))$. So, $A \leqslant f^{-1}(ps\text{-}cl(f(A)))$. As $ps\text{-}cl(A)$ is the smallest *ps-ro* closed fuzzy set on X containing A, $ps\text{-}cl(A) \leqslant f^{-1}(ps\text{-}cl(f(A)))$. Hence, $f(ps\text{-}cl(A)) \leqslant ff^{-1}(ps\text{-}cl(f(A))) \leqslant ps\text{-}cl(f(A))$.
$(f) \Rightarrow (d)$ For any *ps-ro* closed fuzzy set B on Y, $f(ps\text{-}cl(f^{-1}(B))) \leqslant ps\text{-}cl(f(f^{-1}(B))) \leqslant ps\text{-}cl(B) = B$. Hence, $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(B) \leqslant ps\text{-}cl(f^{-1}(B))$. Thus, $f^{-1}(B)$ is *ps-ro* closed fuzzy set on X.
$(f) \Rightarrow (g)$ For any fuzzy set B on Y, $f(ps\text{-}cl(f^{-1}(B))) \leqslant ps\text{-}cl(f(f^{-1}(B))) \leqslant ps\text{-}cl(B)$. Hence, $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(B))$.
$(g) \Rightarrow (f)$ Let $B = f(A)$ for some fuzzy set A on X. Then $ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(B)) \Rightarrow ps\text{-}cl(A) \leqslant ps\text{-}cl(f^{-1}(B)) \leqslant f^{-1}(ps\text{-}cl(f(A)))$. So, $f(ps\text{-}cl(A)) \leqslant ps\text{-}cl(f(A))$.
$(b) \Rightarrow (h)$ For any fuzzy set B on Y, $f^{-1}(ps\text{-}int(B))$ is *ps-ro* open fuzzy set on X. Also, $f^{-1}(ps\text{-}int(B)) \leqslant f^{-1}(B)$. So, $f^{-1}(ps\text{-}int(B)) \leqslant ps\text{-}int(f^{-1}(B))$.
$(h) \Rightarrow (b)$ Let B be any *ps-ro* open fuzzy set on Y. So, $ps\text{-}int(B) = B$. Now, $f^{-1}(ps\text{-}int(B)) \leqslant ps\text{-}int(f^{-1}(B)) \Rightarrow f^{-1}(B) \leqslant ps\text{-}int(f^{-1}(B)) \leqslant f^{-1}(B)$. Hence, $f^{-1}(B)$ is *ps-ro* open fuzzy set on X.

Theorem 3.3. Let (X, τ) and (Y, σ) be two fts. A function $f : X \to Y$ is f is *ps-ro* fuzzy continuous iff for every fuzzy point x_α on X and every *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$ there exists a *ps-ro* open fuzzy set U on X with $x_\alpha qU$ and $f(U) \leqslant V$.
Proof. Let f be *ps-ro* fuzzy continuous and x_α a fuzzy point on X, V a *ps-ro* open fuzzy set V on Y with $f(x_\alpha)qV$. So, $V(f(x)) + \alpha > 1 \Rightarrow f^{-1}(V)(x) + \alpha > 1$. So, $x_\alpha q(f^{-1}(V))$. Now, $ff^{-1}(V) \leqslant V$ is always true. Choosing $U = f^{-1}(V)$ we have, $f(U) \leqslant V$ with $x_\alpha qU$.
Conversely, let the condition hold. Let V be any *ps-ro* open fuzzy set on Y. To prove $f^{-1}(V)$ is *ps-ro* open fuzzy set on X, we shall prove $1 - f^{-1}(V)$ is *ps-ro* closed fuzzy set on X. Let x_α be any fuzzy point on X such that $x_\alpha > 1_X - f^{-1}(V)$. So, $(1 - f^{-1}(V))(x) < \alpha \Rightarrow V(f(x)) + \alpha > 1$. So,

$f(x_\alpha)qV$. By given condition, there exists a *ps-ro* open fuzzy set on U such that $x_\alpha qU$ and $f(U) \leqslant V$. Now, $U(t) + (1 - f^{-1}(V))(t) \leqslant V(f(t)) + 1 - V(f(t)) = 1, \forall\, t$. Hence, $U \not{q}(1 - f^{-1}(V))$. Consequently, x_α is not a fuzzy *ps*-cluster point of $1 - f^{-1}(V)$. This proves $1 - f^{-1}(V)$ is a *ps-ro* closed fuzzy set on X

Theorem 3.4. Let X, Y, Z be *fts*. For any functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, a function $f : Z \to X \times Y$ is defined as $f(x) = (f_1(x), f_2(x))$ for $x \in Z$, where $X \times Y$ is endowded with the product fuzzy topology. If f is *ps-ro* fuzzy continuous then f_1 and f_2 are both *ps-ro* fuzzy continuous.

Proof. Let U_1 be a *ps-ro q-nbd*. of $f_1(x_\alpha)$ on X, for any fuzzy point x_α on Z. Then $U_1 \times 1_Y$ is a *ps-ro q-nbd*. of $f(x_\alpha) = (f_1(x_\alpha), f_2(x_\alpha))$ on $X \times Y$. By *ps-ro* continuity of f, there exists *ps-ro q-nbd*. V of x_α on Z such that $f(V) \leqslant U_1 \times 1_Y$. Then $f(V)(t) \leqslant (U_1 \times 1_Y)(t) = U_1(t) \wedge 1_Y(t) = U_1(t), \forall\, t \in Z$. So, $f_1(V) \leqslant U_1$. Hence, f_1 is *ps-ro* fuzzy continuous. Similarly, it can be shown that f_2 is also *ps-ro* fuzzy continuous.

Theorem 3.5. Let $f : X \to Y$ be a function from a *fts* X to another *fts* Y and $g : X \to X \times Y$ be the graph of the function f. Then f is *ps-ro* fuzzy continuous if g is so.

Proof. Let g be *ps-ro* fuzzy continuous and B be *ps-ro* open fuzzy set on Y. By Lemma 2.4 of (Azad, 1981), $f^{-1}(B) = 1_X \wedge f^{-1}(B) = g^{-1}(1_X \times B)$. Now, as $1_X \times B$ is *ps-ro* open fuzzy set on $X \times Y$, $f^{-1}(B)$ becomes *ps-ro* open fuzzy set on X. Hence, f is *ps-ro* fuzzy continuous.

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Coefficient Estimates for New Subclasses of m-Fold Symmetric Bi-univalent Functions

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Abstract

In this paper, we introduce and investigate two subclasses $\mathcal{A}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$ of Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disc $\mathbb{U}$. For functions in each of the subclasses introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$.

Keywords: Univalent functions, Bi-univalent functions, Coefficient estimates, m-fold symmetric bi-univalent functions.
2010 MSC: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : \; z \in \mathbb{C} \text{ and } |z| < 1\}$$

and normalized by the conditions $f(0) = 0$, $f'(0) = 1$ and having the following form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Also let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see Duren (1983)).

The Koebe One Quarter Theorem (e.g.,see (Duren, 1983)) ensures that the image of $\mathbb{U}$ under every univalent function $f(z) \in \mathcal{A}$ contains the disk of radius $1/4$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z \qquad\qquad (z \in \mathbb{U})$$

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and

$$f\left(f^{-1}(w)\right)=w \qquad \left(|w|<r_0(f),\ r_0(f)\geqslant \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots.$$

A function $f\in\mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by the Taylor-Maclaurin series expansion (1.1).

For a brief history and examples of functions in the class Σ, see (Srivastava *et al.*, 2010) (see also (Brannan & Taha, 1988), (Lewin, 1967), (Taha, 1981)).

In fact, the aforecited work of Srivastava et al. (Srivastava *et al.*, 2010) essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Ali et al. (Ali *et al.*, 2012), Srivastava et al. (Srivastava *et al.*, 2015*b*)(see also (Akın & Sümer-Eker, 2014), (Deniz, 2013), (Frasin & Aouf, 2011), (Srivastava, 2012), Xu et al (Xu *et al.*, 2012*a*), (Xu *et al.*, 2012*b*) and the references cited in each of them).

Let $m\in\mathbb{N}=\{1,2,...\}$. A domain E is said to be *m-fold symmetric* if a rotation of E about the origin through an angle $2\pi/m$ carries E on itself (e.g.,see (Goodman, 1983)). It follows that, a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric in $\mathbb{U}$ if for every z in $\mathbb{U}$

$$f(e^{2\pi i/m}z)=e^{2\pi i/m}f(z).$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

A simple argument shows that $f\in\mathcal{S}_m$ is characterized by having a power series of the form

$$f(z)=z+\sum_{k=1}^{\infty}a_{mk+1}z^{mk+1} \qquad (z\in\mathbb{U},\ m\in\mathbb{N}). \tag{1.2}$$

Each bi-univalent function generates an m-fold symmetric bi-univalent function for each integer $m\in\mathbb{N}$. The normalized form of f is given as in (1.2) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al.(Srivastava *et al.*, 2014), is given as follows

$$g(w)=w-a_{m+1}w^{m+1}+\left[(m+1)a_{m+1}^2-a_{2m+1}\right]w^{2m+1} \tag{1.3}$$

$$-\left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3-(3m+2)a_{m+1}a_{2m+1}+a_{3m+1}\right]w^{3m+1}+\cdots$$

where $f^{-1}=g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$.

Recently, certain subclasses of m-fold bi-univalent functions class Σ_m similar to subclasses of Σ introduced and investigated by Sümer Eker (Sümer-Eker, 2016), Altınkaya and Yalçın (Altınkaya & Yalçın, 2015), Srivastava et al (Srivastava *et al.*, 2015*a*).

The aim of this paper is to introduce new subclasses of the function class bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and derive estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

2. Coefficient Estimates for the function class $\mathcal{A}_{\Sigma_m}(\lambda;\alpha)$

Definition 2.1. A function $f(z) \in \Sigma_m$ given by (1.2) is said to be in the class $\mathcal{A}_{\Sigma_m}(\lambda;\alpha)$ $(0 < \alpha \leqslant 1,\ 0 \leqslant \lambda \leqslant 1)$ if the following conditions are satisfied:

$$\left| arg\left(\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{2.1}$$

and

$$\left| arg\left(\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}) \tag{2.2}$$

where the function g is given by (1.3).

Theorem 2.1. *Let $f \in \mathcal{A}_{\Sigma_m}(\lambda;\alpha)$ $(0 < \alpha \leqslant 1,\ 0 \leqslant \lambda \leqslant 1)$ be given by (1.2). Then*

$$|a_{m+1}| \leqslant \frac{2\alpha}{m\sqrt{2\alpha[1+2\lambda(m+1)] + (1-\alpha)[1+\lambda(m+1)]^2}} \tag{2.3}$$

and

$$|a_{2m+1}| \leqslant \frac{\alpha(m+1)\left[1+|\alpha-1|\right]}{m^2[1+2\lambda(m+1)]}. \tag{2.4}$$

Proof. From (2.1) and (2.2) we have

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = [p(z)]^\alpha \tag{2.5}$$

and for its inverse map, $g = f^{-1}$, we have

$$\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)} = [q(w)]^\alpha \tag{2.6}$$

where $p(z)$ and $q(w)$ are in familiar Caratheodory Class $\mathcal{P}$ (see for details (Duren, 1983)) and have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$m[1+\lambda(m+1)]a_{m+1} = \alpha p_m, \tag{2.9}$$

$$2m[1+\lambda(2m+1)]a_{2m+1} - m[1+\lambda(m+1)]a_{m+1}^2 = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-m[1+\lambda(m+1)]a_{m+1} = \alpha q_m \tag{2.11}$$

and

$$m[(2m+1)+\lambda(m+1)(4m+1)]a_{m+1}^2 - 2m[1+\lambda(2m+1)]a_{2m+1} = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

From (2.9) and (2.11), we get

$$p_m = -q_m \tag{2.13}$$

and

$$2m^2[1+\lambda(m+1)]^2 a_{m+1}^2 = \alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also from (2.10), (2.12) and (2.14), we get

$$2m^2[1+2\lambda(m+1)]a_{m+1}^2 = \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2).$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{m^2\left[2\alpha[1+2\lambda(m+1)]+(1-\alpha)[1+\lambda(m+1)]^2\right]}. \tag{2.15}$$

Note that, according to the Caratheodory Lemma (see (Duren, 1983)), $|p_m| \leqslant 2$ and $|q_m| \leqslant 2$ for $m \in \mathbb{N}$. Now taking the absolute value of (2.15) and applying the Caratheodory Lemma for coefficients p_{2m} and q_{2m} we obtain

$$|a_{m+1}| \leqslant \frac{2\alpha}{m\sqrt{2\alpha[1+2\lambda(m+1)]+(1-\alpha)[1+\lambda(m+1)]^2}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted (2.3).

To find bounds on $|a_{2m+1}|$, we multiply $(2m+1)+\lambda(m+1)(4m+1)$ and $1+\lambda(m+1)$ to the relations (2.10) and (2.12) respectively and on adding them we obtain:

$$\begin{aligned}
&4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]a_{2m+1} \\
&= \alpha\left\{[(2m+1)+\lambda(m+1)(4m+1)]\,p_{2m} + [1+\lambda(m+1)]\,q_{2m}\right\} \\
&+ \frac{\alpha(\alpha-1)}{2}\left\{[(2m+1)+\lambda(m+1)(4m+1)]\,p_m^2 + [1+\lambda(m+1)]q_m^2\right\}.
\end{aligned}$$

Now using $p_m^2 = q_m^2$ and the Caratheodory Lemma again for coefficients p_m, p_{2m} and q_{2m} we obtain

$$|a_{2m+1}| \leqslant \frac{\alpha(m+1)[1+|\alpha-1|]}{m^2[1+2\lambda(m+1)]}.$$

This completes the proof of the Theorem 2.1.

3. Coefficient Estimates for the function class $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$

Definition 3.1. A function $f(z) \in \Sigma_m$ given by (1.2) is said to be in the class $\mathcal{A}_{\Sigma_m}(\lambda;\beta)$ $(0 \leqslant \lambda \leqslant 1, 0 \leqslant \beta < 1)$ if the following conditions are satisfied:

$$Re\left\{\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)}\right\} > \beta \qquad (z \in \mathbb{U}) \tag{3.1}$$

and

$$Re\left\{\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)}\right\} > \beta \qquad (w \in \mathbb{U}) \tag{3.2}$$

where the function $g(w)$ is given by (1.3).

Theorem 3.1. *Let $f \in \mathcal{A}_{\Sigma_m}(\lambda;\beta)$ $(0 \leqslant \lambda \leqslant 1,\ 0 \leqslant \beta < 1)$ be given by (1.2). Then*

$$|a_{m+1}| \leqslant \sqrt{\frac{2(1-\beta)}{m^2[1+2\lambda(m+1)]}} \tag{3.3}$$

and

$$|a_{2m+1}| \leqslant \frac{(1-\beta)(m+1)}{m^2[1+2\lambda(m+1)]}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\lambda w^2 g''(w)}{g(w)} = \beta + (1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$m[1+\lambda(m+1)]a_{m+1} = (1-\beta)p_m, \tag{3.7}$$

$$2m[1+\lambda(2m+1)]a_{2m+1} - m[1+\lambda(m+1)]a_{m+1}^2 = (1-\beta)p_{2m}, \tag{3.8}$$

$$-m[1+\lambda(m+1)]a_{m+1} = (1-\beta)q_m \tag{3.9}$$

and

$$m[(2m+1)+\lambda(m+1)(4m+1)]a_{m+1}^2 - 2m[1+\lambda(2m+1)]a_{2m+1} = (1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9) we get

$$p_m = -q_m \tag{3.11}$$

and

$$2m^2[1+\lambda(m+1)]^2 a_{m+1}^2 = (1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Also from (3.8) and (3.10), we obtain

$$2m^2[1+2\lambda(m+1)]a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Thus we have

$$\begin{aligned} |a_{m+1}^2| &\leqslant \frac{(1-\beta)}{2m^2[1+2\lambda(m+1)]}(|p_{2m}|+|q_{2m}|) \\ &\leqslant \frac{2(1-\beta)}{m^2[1+2\lambda(m+1)]}, \end{aligned}$$

which is the bound on $|a_{m+1}|$ as given in the Theorem 3.1.

In order to find the bound on $|a_{2m+1}|$, we multiply $(2m+1)+\lambda(m+1)(4m+1)$ and $1+\lambda(m+1)$ to the relations (3.8) and (3.10) respectively and on adding them we obtain:

$$\begin{aligned} &4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]a_{2m+1} \\ &= (1-\beta)\left\{[(2m+1)+\lambda(m+1)(4m+1)]p_{2m} + [1+\lambda(m+1)]q_{2m}\right\} \end{aligned}$$

or equivalently

$$a_{2m+1} = \frac{(1-\beta)[(2m+1)+\lambda(m+1)(4m+1)]p_{2m} + [1+\lambda(m+1)]q_{2m}}{4m^2[1+\lambda(2m+1)][1+2\lambda(m+1)]}$$

Applying the Caratheodory Lemma for the coefficients p_{2m} and q_{2m}, we find

$$|a_{2m+1}| \leqslant \frac{(1-\beta)(m+1)}{m^2[1+2\lambda(m+1)]},$$

which is the bound on $|a_{2m+1}|$ as asserted in Theorem 3.1.

Remark. For 1-fold symmetric bi-univalent functions, if we put $\lambda = 0$ in our Theorems, we obtain the Theorem 2.1 and the Theorem 3.1 which were given by Brannan and Taha (Brannan & Taha, 1988).

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Subordination Properties of Certain Subclasses of Multivalent Functions Defined By Srivastava-Wright Operator

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Abstract

Some subordination properties are investigated for functions belonging to each of the subclasses $\mathcal{V}(\lambda, A, B)$ and $\mathcal{W}(\lambda, A, B)$ of analytic p- valent functions involving the Srivastava-Wright operator in the open unit disk, $\mathbb{U}$ with suitable restrictions on the parameters λ, A and B. The authors also derive certain subordination results involving the Hadamard product (or convolution) of the associated functions. Relevant connections of the main results to various known results are established.

Keywords: Multivalent function, Srivastava-Wright Operator, Convex function, Differential subordination, Argument estimates.
2010 MSC: 30C45, 30C50, 30C55.

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1. Introduction

Let $\mathcal{A}_k(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k; p, k \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and $p-$valent in the unit disc, $\mathbb{U} := \mathbb{U}(1)$, where $\mathbb{U}(r) = \{z \in \mathbb{C} : |z| < r\}$. Also, let $\mathcal{A}(p) = \mathcal{A}_{p+1}(p)$ and $\mathcal{A} = \mathcal{A}(1)$. For the functions $f \in \mathcal{A}_k(p)$ of the form (1.1) and $g \in \mathcal{A}_k(p)$ given by $g(z) = z^p + \sum\limits_{n=k}^{\infty} b_n z^n$, the *Hadamard product (or convolution) of f and g* is defined by

$$(f * g)(z) := z^p + \sum_{n=k}^{\infty} a_n b_n z^n, \ z \in \mathbb{U}.$$

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If f and g are two analytic functions in $\mathbb{U}$, we say that *f is subordinate to g*, written symbolically as $f(z) \prec g(z)$, if there exists a *Schwarz function w*, which (by definition) is analytic in $\mathbb{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$.

If the function g is univalent in $\mathbb{U}$, then we have the following equivalence, (c.f (Miller & Mocanu, 1981, 2000)):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1, B_1, \ldots, \beta_s, B_s (q, s \in \mathbb{N})$ be positive and real parameters such that

$$1 + \sum_{i=1}^{s} B_i - \sum_{i=1}^{q} A_i > 0.$$

The *Wright generalized hypergeometric function*

$${}_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma\left(\alpha_i + nA_i\right)}{\prod_{i=1}^{s} \Gamma\left(\beta_i + nB_i\right)} \frac{z^n}{n!} \ (z \in \mathbb{U}).$$

If $A_i = 1 (i = 1, ..., q)$ and $B_i = 1 (i = 1, ..., s)$, we have the following relationship:

$$\Omega {}_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = {}_q F_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

where ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is the *generalized hypergeometric function* and

$$\Omega = \frac{\prod_{i=1}^{q} \Gamma\left(\beta_i\right)}{\prod_{i=1}^{s} \Gamma\left(\alpha_i\right)} \tag{1.2}$$

Now we define a function $\mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]$ by

$$\mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \Omega\, z^p\, {}_q\Psi_s[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]$$

and also consider the following linear operator

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] : \mathcal{A}_k(p) \to \mathcal{A}_k(p)$$

defined using the convolution

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}]f(z) = \mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] * f(z).$$

We note that, for a function f of the form (1.1), we have

$$\theta_p^{q,s}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}]f(z) = z^p + \sum_{n=k}^{\infty} \Omega\sigma_{n,p}(\alpha_1) a_n z^n, \tag{1.3}$$

where Ω is given by (1.2) and $\sigma_{n,p}(\alpha_1)$ is defined by

$$\sigma_{n,p}(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(n-p))\ldots\Gamma(\alpha_q + A_q(n-p))}{\Gamma(\beta_1 + B_1(n-p))\ldots\Gamma(\beta_s + B_s(n-p))(n-p)!}. \tag{1.4}$$

If for convenience, we write

$$\theta_p^{q,s}(\alpha_1)f(z) = \theta_p^{q,s}[(\alpha_1, A_1)\ldots(\alpha_q, A_q); (\beta_1, B_1)\ldots(\beta_s, B_s)]f(z)$$

then we can easily verify from (1.3) that

$$zA_1\left(\theta_p^{q,s}(\alpha_1)f(z)\right)' = \alpha_1\theta_p^{q,s}(\alpha_1+1)f(z) - (\alpha_1 - pA_1)\theta_p^{q,s}(\alpha_1)f(z)\ (A_1 > 0). \tag{1.5}$$

For $A_i = 1 (i = 1, ..., q)$ and $B_i = 1 (i = 1, ..., s)$, we obtain $\theta_p^{q,s}[\alpha_1]f(z) = H_{p,q,s}f(z)$, which is known as the Dziok-Srivastava operator; it was introduced and studied by Dziok and Srivastava (Dziok & Srivastava, 1999, 2003). Also, for $f(z) \in \mathcal{A}$, the linear operator $\theta_1^{q,s}[\alpha_1]f(z) = \theta[\alpha_1]$ is popularly known in the current literature as the Srivastava-Wright operator; it was systematically and firmly investigated by Srivastava (Srivastava, 2007).(see also (Kiryakova, 2011; Dziok & Raina, 2004) and (Aouf *et al.*, 2010)).

Remark. For $f \in \mathcal{A}(p), A_i = 1(1 = 1, 2, \ldots, q), B_i = 1(i = 1, 2, \ldots, s), q = 2$ *and* $s = 1$ by specializing the parameters α_1, α_2 *and* β_1 the operator $\theta_p^{q,s}(\alpha_1)$ gets reduced to the following familiar operators:

(i) $\theta_p^{2,1}[a, 1; c]f(z) = L_p(a, c)f(z)$[see Saitoh (Saitoh, 1996)];

(ii) $\theta_p^{2,1}[\mu + p, 1; 1]f(z) = D^{\mu+p-1}f(z)\ (\mu > -p)$, where $D^{\mu+p-1}$ is the $\mu + p - 1$- the order *Ruscheweyh derivative* of a function $f \in \mathcal{A}(p)$. [see Kumar and Shukla (Kumar & Shukla, 1984*a*,*b*)]

(iii) $\theta_p^{2,1}[1 + p, 1; 1 + p - \mu]f(z)$, where the operator $\Omega_z^{\mu,p}$ is defined by [see Srivastava and Aouf (Srivastava & Aouf, 1992)];

$$\Omega_z^{\mu,p}f(z) = \frac{\Gamma(1+p-\mu)}{\Gamma(1+p)}z^\mu D_z^\mu f(z)\ (0 \le \mu < 1; p \in \mathbb{N}),$$

where D_z^μ is the *fractional derivative operator*.

(iv) $\theta_p^{2,1}[\nu + p, 1; \nu + p + 1]f(z) = J_{\nu,p}(f)(z)$, where $J_{\nu,p}$ is the generalized *Bernadi-Libera-Livingston-integral operator* (see (Bernardi, 1996; Libera, 1969; Livingston, 1966));

(v) $\theta_p^{2,1}[\lambda + p, a; c]f(z) = I_p^\lambda(a, c)f(z)\ (a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \lambda > -p)$, where $I_p^\lambda(a, c)$ is the *Cho-Kwon-Srivastava* operator (Cho *et al.*, 2004);

Definition 1.1. For the fixed parameters A and B, with $0 \le B < 1, -1 \le A < B$ and $0 \le \lambda < p, p \in \mathbb{N}$ and for a analytic $p-$ valent function of the form (1.1) we define the following subclasses:

$$\mathcal{V}(\lambda, A, B) = \left\{ f \in \mathcal{A}_k(p) : \frac{1}{p-\lambda}\left(\frac{z\left[\theta_p^{q,s}(\alpha_1)f(z)\right]'}{\theta_p^{q,s}(\alpha_1)f(z)} - \lambda\right) \prec \frac{1+Az}{1+Bz} \right\} \tag{1.6}$$

and

$$\mathcal{W}(\lambda, A, B) = \left\{ f \in \mathcal{A}_k(p) : \frac{1}{p-\lambda}\left(1 + \frac{z[\theta_p^{q,s}(\alpha_1)f(z)]''}{[\theta_p^{q,s}(\alpha_1)f(z)]'} - \lambda\right) \prec \frac{1+Az}{1+Bz} \right\}. \tag{1.7}$$

The subclass $\mathcal{V}(\lambda, A, B)$ was discussed by Aouf et al., (Aouf *et al.*, 2010) for multivalent analytic functions with negative coefficients, also coefficients estimates, distortion theorem, the radii of $p-$valent starlikeness and $p-$valent convexity and modified Hadamard products were investigated. In (Murugusundaramoorthy & Aouf, 2013) Murugusundaramoorthy and Aouf obtained similar results for the meromorphic equivalent of the class $\mathcal{W}(\lambda, A, B)$. Sarkar et al., (Sarkar *et al.*, 2013) presented certain inclusion and convolution results involving the operator $\theta_p^{q,s}(\alpha_1)$ for functions belonging to certain favoured classes of analytic p-valent functions.
Motivated by the aforementioned works, in the present study we obtain certain strict subordination relationship involving the subclasses $\mathcal{V}(\lambda, A, B)$ and $\mathcal{W}(\lambda, A, B)$. Some subordination properties involving the linear operator defined in (1.3) are also considered. An argument estimate result is also obtained.

2. Preliminaries

Let $\mathcal{P}_m$ denote the class of function of the form

$$f(z) = 1 + a_m z^m + a_{m+1} z^{m+1} + \dots \tag{2.1}$$

that are analytic in the unit disc,$\mathbb{U}$. In proving our main results, we need each of the following definitions and lemmas.

Definition 2.1. (Wilf, 1961)
A sequence $\{b_n\}_{n\in\mathbb{N}}$ of complex numbers is said to be a *subordination factor sequence* if for each function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \mathbb{U}$, from the class of convex (univalent) functions in $\mathbb{U}$, denoted by S^c, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad (\text{where} \quad a_1 = 1).$$

Lemma 2.1. *(Wilf, 1961) A sequence $\{b_n\}$ is a subordinating factor sequence if and only if*

$$Re\left(1 + 2\sum_{n=1}^{\infty} b_n z^n\right) > 0,\ z \in \mathbb{U}. \tag{2.2}$$

Lemma 2.2. *(Miller & Mocanu, 1981, 2000) Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0) = 1$. Suppose also that the function ϕ given by (2.1). If*

$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \quad (Re\gamma \geq 0,\ \gamma \in \mathbb{C}^*), \tag{2.3}$$

then

$$\phi(z) \prec \psi(z) = \frac{\gamma}{m} z^{-\frac{\gamma}{m}} \int_0^z t^{\frac{\gamma}{m}-1} h(t)\, dt \prec h(z)$$

and ψ is the best dominant.

Lemma 2.3. *(Nunokawa, 1993)*

Let the function p be analytic in $\mathbb{U}$, such that $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi\delta}{2}, \quad \textit{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\delta}{2} \quad (\delta > 0),$$

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\delta,$$

where

$$k \geq \frac{1}{2}\left(c + \frac{1}{c}\right), \quad \textit{when} \quad \arg p(z_0) = \frac{\pi\delta}{2}$$

and

$$k \leq -\frac{1}{2}\left(c + \frac{1}{c}\right), \quad \textit{when} \quad \arg p(z_0) = -\frac{\pi\delta}{2},$$

where

$$p(z_0)^{1/\delta} = \pm ic, \quad \textit{and} \quad c > 0.$$

Lemma 2.4. *(Whittaker & Watson, 1927)*

For the complex numbers a, b and c, with $c \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$, the following identities hold:

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z), \; z \in \mathbb{U}, \tag{2.4}$$

$$\textit{for } Re c > Re b > 0, \tag{2.5}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \; z \in \mathbb{U}, \tag{2.6}$$

and

$$(b+1)\, {}_2F_1(1, b; b+1; z) = (b+1) + bz\, {}_2F_1(1, b+1; b+2; z), \; z \in \mathbb{U}. \tag{2.7}$$

3. Coefficient estimates and subordination results for the function classes $\mathcal{W}(\lambda, A, B)$ and $\mathcal{V}(\lambda, A, B)$

Unless otherwise mentioned, we shall assume throughout the sequel that $0 \leq \lambda < p, p \in \mathbb{N}$ *and* $0 \leq B < 1$. First, we will give sufficient conditions for a function to be in the classes $\mathcal{W}(\lambda, A, B)$.

Lemma 3.1. *A sufficient condition for an analytic p-valent function f of the form (1.1), to be in the class $\mathcal{W}(\lambda, A, B)$ is*

$$\sum_{n=k}^{\infty} \gamma_{n,p}|a_n| \leq p(B-A)(p-\lambda) \tag{3.1}$$

where

$$\gamma_{n,p} = \Omega\sigma_{n,p}(\alpha_1)n[(n-p)(1+B)-(A-B)(p-\lambda)], \ (n \geq k). \tag{3.2}$$

Proof. An analytic $p-$valent function f of the form (1.1) belongs to the class $\mathcal{W}(\lambda, A, B)$, if and only if there exists a *Schwarz function* w, such that

$$\frac{1}{p-\lambda}\left(1+\frac{z[\theta_p^{q,s}(\alpha_1)f(z)]''}{[\theta_p^{q,s}(\alpha_1)f(z)]'}-\lambda\right) = \frac{1+Aw(z)}{1+Bw(z)}, \ z \in \mathbb{U}.$$

Since $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$, the above relation is equivalent to

$$\left|\frac{[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'}{([\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]')B - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'}\right| < 1.$$

Thus it is sufficient to show that

$$\left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'\right|$$
$$-\left|([\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]')B - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'\right| < 0, \ z \in \mathbb{U}.$$

Indeed, letting $|z| = r$ $(0 < r < 1)$ and using (3.1), we have

$$\left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]'\right| -$$
$$\left|[\theta_p^{q,s}(\alpha_1)f(z)]' + z[\theta_p^{q,s}(\alpha_1)f(z)]'' - p[\theta_p^{q,s}(\alpha_1)f(z)]' - (p-\lambda)(A-B)[\theta_p^{q,s}(\alpha_1)f(z)]'\right|$$
$$\leq \sum_{n=k}^{\infty} n(n-p)\Omega\sigma_{n,p}(\alpha_1)|a_n|r^n - (B-A)p(p-\lambda)\, r^{p-1}$$
$$+\sum_{n=k}^{\infty} n[(n-p)B-(A-B)(p-\lambda)]\Omega\sigma_{n,p}(\alpha_1)|a_n|r^n = r^{p-1}\Big(\sum_{n=k}^{\infty}\gamma_{n,p}|a_n|r^{n-p+1} - (B-A)p(p-\lambda)\Big) < 0.$$

Hence $f \in \mathcal{W}(\lambda, A, B)$. □

Similarly, we have the following Lemma which gives sufficient condition for a function to be in the class $\mathcal{V}(\lambda, A, B)$.

Lemma 3.2. *A sufficient condition for an analytic p-valent function f of the form (1.1), to be in the class* $\mathcal{V}(\lambda, A, B)$ *is*

$$\sum_{n=k}^{\infty} \delta_{n,p}^{*}|a_n| \leq (B-A)(p-\lambda) \tag{3.3}$$

where

$$\delta_{n,p}^{*} = \Omega\sigma_{n,p}(\alpha_1)[(n-p)(1+B)-(A-B)(p-\lambda)],\ (n \geq k). \tag{3.4}$$

Our next result provides a sharp subordination result involving the functions of the class $\mathcal{W}(\lambda, A, B)$.

Theorem 3.1. *Let the sequence* $\{\gamma_{n,p}\}_{n\in\mathbb{N}}$ *defined in (3.2) be a nondecreasing sequence. If a function f of the form (1.1) belong to the class* $\mathcal{W}(\lambda, A, B)$. *and* $g \in \mathcal{S}^c$, *then*

$$\left(\epsilon\left(z^{1-p}\right) * g\right)(z) \prec g(z), \tag{3.5}$$

and

$$Re\left(z^{1-p}f(z)\right) > -\frac{1}{2\epsilon},\ z \in \mathbb{U}, \tag{3.6}$$

$$\textit{whenever} \quad \epsilon = \frac{\gamma_{k,p}}{2\left[(B-A)p(p-\lambda)\right]+\gamma_{k,p}}.$$

Moreover, if $(k-p)$ *is even, then the number* ϵ *cannot be replaced by a larger number.*

Proof. Supposing that the function $g \in \mathcal{S}^c$ is of the form

$$g(z) = \sum_{n=1}^{\infty} b_n z^n,\ z \in \mathbb{U} \quad (\text{where} \quad b_1 = 1),$$

then

$$\sum_{n=1}^{\infty} d_n b_n z^n = \left(\epsilon\left(z^{1-p}f\right) * g\right)(z) \prec g(z),$$

where

$$d_n = \begin{cases} \epsilon, & \text{if} \quad n = 1, \\ 0, & \text{if} \quad 2 \leq n \leq k-p, \\ \epsilon a_{n+p-1}, & \text{if} \quad n > k-p. \end{cases}$$

Now, using the Definition 2.1, the subordination result in (3.5) holds if $\{d_n\}$ is a subordinating factor sequence. Since $\{\gamma_{n,p}\}_{n\in\mathbb{N}}$ is a nondecreasing sequence we have,

$$\mathrm{Re}\left(1+2\sum_{n=1}^{\infty} d_n z^n\right) = \mathrm{Re}\left(1+\frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}} z + \sum_{n=k}^{\infty} \frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}} a_n z^{n-p}\right) \geq 1-\frac{\gamma_{k,p}}{p(p-\lambda)(B-A)+\gamma_{k,p}} r - \frac{r}{p(p-\lambda)(B-A)+\gamma_{k,p}} \sum_{n=k}^{\infty} \delta_{n,p}|a_n|, \ |z|=r<1. \tag{3.7}$$

Thus, by using Lemma 3.1 in (3.7) we obtain

$$\mathrm{Re}\left(1+2\sum_{n=1}^{\infty} c_n z^n\right) \geq 1-\frac{\gamma_{k,p}}{p(B-A)(p-\lambda)+\gamma_{k,p}} r - \frac{r}{p(B-A)(p-\lambda)+\gamma_{k,p}}(B-A)p(p-\lambda) > 0, \ z\in\mathbb{U},$$

which proves the inequality (2.2), hence also the subordination result asserted by (3.5). The inequality (3.6) asserted by Theorem 3.1 would follow from (3.5) upon setting

$$g(z)=\frac{z}{1-z}=\sum_{n=1}^{\infty} z^n, \ z\in\mathbb{U}.$$

We also observe that whenever the functions of the form

$$f_{n,p}(z)=z^p+\frac{(B-A)p(p-\lambda)}{\gamma_{n,p}} z^n, \ z\in\mathbb{U} \ (n\geq k),$$

belongs the class $\mathcal{W}(\lambda, A, B)$ and if $(k-p)$ is a even number, then

$$z^{1-p} f_{k,p}(z)\Big|_{z=-1} = -\frac{1}{2\epsilon},$$

and the constant ϵ is the best estimate. □

Using the same techniques as in the proof of Theorem 3.1, we have the following result.

Theorem 3.2. *Let the sequence* $\{\delta^*_{n,p}\}_{n\in\mathbb{N}}$ *defined by* (3.4) *be a nondecreasing sequence. If the function g of the form* (1.1) *belongs to the class* $\mathcal{V}(\lambda, A, B)$ *and* $h\in\mathcal{S}^c$, *then*

$$\left(\mu\left(z^{1-p} f\right) * h\right)(z) \prec h(z), \tag{3.8}$$

and

$$Re\left(z^{1-p}f(z)\right) > -\frac{1}{2\mu},\ z \in \mathbb{U}, \tag{3.9}$$

where

$$\mu = \frac{\delta^*_{k,p}}{2\left[(B-A)(p-\lambda)\right] + \delta^*_{k,p}}.$$

Moreover, if $(k-p)$ *is even, then the number* μ *cannot be replaced by a larger number.*

4. Subordination Properties of the operator $\theta_p^{q,s}(\alpha_1)$

In this section we obtain certain subordination properties involving the operator $\theta_p^{q,s}(\alpha_1)$.

Theorem 4.1. *For* $f \in \mathcal{A}_k(p)$ *let the operator* Q *be defined by*

$$Qf(z) := \left[1-\tau-\tau\frac{(\alpha_1 - pA_1)}{A_1}\theta_p^{q,s}(\alpha_1)f(z)\right] + \frac{\tau\alpha_1}{A_1}\left[\theta_p^{q,s}(\alpha_1+1)f(z)\right], \tag{4.1}$$

for $A_1 \neq 0$ *and* $\tau > 0$.

(i) If

$$\frac{Q^{(j)}f(z)(p-j)!}{z^{p-j}p!} \prec (1-\tau+\tau p)\frac{1+Az}{1+Bz} \quad (0 \le j \le p), \tag{4.2}$$

, then

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)(p-j)!\right]^{(j)}}{z^{p-j}p!} \prec \widetilde{g}(z) \prec \frac{1+Az}{1+Bz}, \tag{4.3}$$

where for m positive, $\widetilde{g}$ *is given by*

$$\widetilde{g}(z) = \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{1-\tau+\tau p}{\tau m}+1;\dfrac{Bz}{1+Bz}\right), & \text{if} \quad B \neq 0, \\ 1 + \dfrac{Az(1-\tau+\tau p)}{1-\tau+\tau(m+p)}, & \text{if} \quad B = 0, \end{cases}$$

and $\widetilde{g}$ *is the best dominant of* (4.3).

(ii)

$$Re\left(\frac{Q^{(j)}f(z)}{z^{p-j}}\right) > \frac{p!}{(p-j)!}\sigma,\ z \in \mathbb{U} \tag{4.4}$$

where

$$\sigma = \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\dfrac{1-\tau+\tau p}{\tau m}+1;\dfrac{B}{B-1}\right), & \text{if} \quad B \neq 0, \\ 1 - \dfrac{A(1-\tau+\tau p)}{1-\tau+\tau(p+m)}, & \text{if} \quad B = 0. \end{cases}$$

The inequality (4.4) *is the best possible.*

Proof. From (1.5) and (4.1) we easily obtain

$$Q^{(j)}f(z) = (1-\tau+\tau j)\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)} + \tau z\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j+1)},\ z \in \mathbb{U}. \tag{4.5}$$

Letting

$$g(z) := \frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}(p-j)!}{z^{p-j}p!}.$$

with $f \in \mathcal{A}_k(p)$, then g is analytic in $\mathbb{U}$ and has the form (2.1). Also, note that

$$(1-\tau+\tau p)\left[g(z) + \frac{\tau}{1-\tau+\tau p}zg'(z)\right] = \frac{Q^{(j)}f(z)(p-j)!}{z^{p-j}p!}. \tag{4.6}$$

Then, by (4.2) we have

$$g(z) + \frac{\tau}{1-\tau+\tau p}zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Now, by using Lemma 2.2 for $\gamma = \dfrac{1-\tau+\tau p}{\tau}$ and whenever $\gamma > 0$, by a changing of variables followed by the use of the identities (2.5), (2.6) and (2.7), we deduce that

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}(p-j)!}{z^{p-j}p!} \prec \widetilde{g}(z) = \frac{(1-\tau+\tau p)}{\tau m}z^{-\frac{(1-\tau+\tau p)}{\tau m}}\int_0^z t^{\frac{(1-\tau+\tau p)}{\tau m}-1}\frac{1+At}{1+Bt}\,dt$$

$$= \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;\dfrac{1-\tau+\tau p}{\tau m}+1;\dfrac{Bz}{1+Bz}\right), & \text{if} \quad B \neq 0, \\ 1 + \dfrac{A(1-\tau+\tau p)}{1-\tau+\tau(p+m)}z, & \text{if} \quad B = 0, \end{cases}$$

which proves the assertion (4.3) of our Theorem.

Next, in order to prove the assertion (4.4), it sufficies to show that

$$\inf\{\operatorname{Re}\widetilde{g}(z) : z \in \mathbb{U}\} = \widetilde{g}(-1). \tag{4.7}$$

Indeed, for $|z| \le r < 1$ we have

$$\operatorname{Re}\frac{1+Az}{1+Bz} \ge \frac{1-Ar}{1-Br},$$

and setting

$$\chi(s,z) = \frac{1+Asz}{1+Bsz} \quad \text{and} \quad d\mu(s) = \frac{1-\tau+\tau p}{\tau m}s^{\frac{1-\tau+\tau p}{\tau m}-1}\,ds \quad (0 \le s \le 1)$$

which is a positive measure on the closed interval [0, 1] whenever $\tau > 0$, we get

$$\widetilde{g}(z) = \int_0^1 \chi(s,z)\,d\mu(s),$$

and

$$\operatorname{Re}\widetilde{g}(z) \geq \int_0^1 \frac{1-Asr}{1-Bsr}\,d\mu(s) = \widetilde{g}(-r),\ |z| \leq r < 1.$$

Letting $r \to 1^-$ in the above inequality we obtain the assertion (4.7) of our Theorem. The estimate in (4.4) is the best possible since the function $\widetilde{g}$ is the best dominant of (4.3). □

Taking $q = 2$ and $s = 1$,for $A_i = B_i = 1, \alpha_1 = 1, \alpha_2 = \beta_1$ and $A = 1 - \dfrac{2\alpha(p-j)!}{(1-\tau+\tau p)p!}$ and $B = -1$ in Theorem 4.1 we get the following result:

Corollary 4.1. *Let $Qf(z) = (1-\tau)f(z) + \tau z f'(z)$, where $f \in \mathcal{A}_k(p)$. For $\tau > 0$*

$$Re\frac{Q^{(j)}f(z)(p-j)!}{z^{p-j}p!} > \alpha,\ z \in \mathbb{U} \quad \left(0 \leq \alpha < \frac{(1-\tau+\tau p)p!}{(p-j)!},\ 0 \leq j \leq p\right),$$

implies that

$$Re\frac{f^{(j)}(z)}{z^{p-j}} > \frac{\alpha}{1-\tau+\tau p} +$$
$$\left[\frac{p!}{(p-j)!} - \frac{\alpha}{1-\tau+\tau p}\right]\left[{}_2F_1\left(1, 1; \frac{1-\tau+\tau p}{\tau m} + 1; \frac{1}{2}\right) - 1\right],\ z \in \mathbb{U}.$$

The above inequality is the best possible.

Theorem 4.2. *For $f \in \mathcal{A}_k(p)$ let the operator Q be given by (4.1), and let $\tau > 0$.*

(i) If

$$Re\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} > \rho,\ z \in \mathbb{U} \quad \left(\rho < \frac{p!}{(p-j)!}\right),$$

then

$$Re\frac{Q^{(j)}f(z)}{z^{p-j}} > \rho(1-\tau+\tau p),\ |z| < R,$$

where

$$R = \left[\sqrt{1 + \left(\frac{\tau m}{1-\tau+\tau p}\right)^2} - \frac{\tau m}{1-\tau+\tau p}\right]^{\frac{1}{m}}. \tag{4.8}$$

(ii) If

$$Re\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{(-1)^j z^{-p-j}} < \rho,\ z \in \mathbb{U} \quad \left(\rho > \frac{p!}{(p-j)!},\ \right),$$

then

$$Re\frac{Q^{(j)}f(z)}{z^{p-j}} < \rho(1-\tau+\tau p),\ |z| < R.$$

The bound R is the best possible.

Proof. (i) Defining the function Φ by

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} =: \rho + \left[\frac{p!}{(p-j)!} - \rho\right]\Phi(z), \tag{4.9}$$

then Φ is an analytic function of the form (2.1) with positive real part in $\mathbb{U}$. Differentiating (4.9) with respect to z and using (4.5) we have

$$\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p) = \left[\frac{p!}{(p-j)!} - \rho\right]\left[(1-\tau+\tau p)\Phi(z) + \tau z\Phi'(z)\right]. \tag{4.10}$$

Now, by applying in (4.10) the following well-known estimate (MacGregor, 1963)

$$\frac{|z\Phi'(z)|}{\operatorname{Re}\Phi(z)} \leq \frac{2mr^m}{1-r^{2m}}, \quad |z| = r < 1, \tag{4.11}$$

we have

$$\operatorname{Re}\left[\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p)\right] \geq \tag{4.12}$$

$$\operatorname{Re}\Phi(z)\left[\frac{p!}{(p-j)!} - \rho\right]\left[(1-\tau+\tau p) - \frac{2\tau mr^m}{1-r^{2m}}\right], \quad |z| = r < 1.$$

Now, it is easy to see that the right hand side of (4.12) is positive whenever $r < R$, where R is given by (4.8). In order to show that the bound R is the best possible, we consider the function $f \in \mathcal{A}_k(p)$ defined by

$$\frac{\left[\theta_p^{q,s}(\alpha_1)f(z)\right]^{(j)}}{z^{p-j}} = \rho + \left[\frac{p!}{(p-j)!} - \rho\right]\frac{1+z^m}{1-z^m}.$$

Then,

$$\frac{Q^{(j)}f(z)}{z^{p-j}} - \rho(1-\tau+\tau p) =$$

$$\frac{\frac{p!}{(p-j)!} - \rho}{(1-z^m)^2}\left[(1-\tau+\tau p)\left(1-z^{2m}\right) + 2\tau mz^m\right] = 0,$$

for $z = R\exp^{\frac{i\pi}{m}}$, and the first part of the Theorem is proved.
Similarly, we can prove part (ii) of the Theorem. □

5. An argument estimate

In this section we obtain an argument estimate involving the operator $\theta_p^{q,s}(\alpha_1)$ and connected with the linear operator Q.

Theorem 5.1. *For* $f \in \mathcal{A}_k(p)$, *let the operator* Q *be defined by* (4.1), *and let* $0 \le \tau < \dfrac{1}{1-p}$. *If*

$$\left|\arg \frac{Q^{(j)} f(z)}{z^{p-j}}\right| < \frac{\pi\delta}{2},\ z \in \mathbb{U} \quad \big(\delta > 0,\ 0 \le j \le p\big), \tag{5.1}$$

then

$$\left|\arg \frac{\left[\theta_p^{q,s}(\alpha_1) f(z)\right]^{(j)}}{z^{p-j}}\right| < \frac{\pi\delta}{2},\ z \in \mathbb{U}.$$

Proof. For $f \in \mathcal{A}_k(p)$, if we let

$$q(z) := \frac{\left[\theta_p^{q,s}(\alpha_1) f(z)\right]^{(j)}}{z^{p-j}} \frac{(p-j)!}{p!},$$

then q is of the form (2.1) and it is analytic in $\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg q(z)| < \frac{\pi\delta}{2},\ |z| < |z_0| \quad \text{and} \quad |\arg q(z_0)| = \frac{\pi\delta}{2} \quad (\delta > 0)\,,$$

then, accorollaryding to Lemma 2.3 we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\delta \quad \text{and} \quad q(z_0)^{1/\delta} = \pm ic \quad (c > 0).$$

Also, from the equality (4.5) we get

$$\frac{Q^{(j)} f(z_0)}{z_0^{p-j}} = \frac{p!}{(p-j)!}(1-\tau+\tau p)q(z_0)\left[1 + \frac{\tau}{1-\tau+\tau p}\frac{z_0 q'(z_0)}{q(z_0)}\right].$$

If $\arg q(z_0) = \dfrac{\pi\delta}{2}$, then

$$\arg \frac{Q^{(j)} f(z_0)}{z_0^{p-j}} = \frac{\pi\delta}{2} + \arg\left(1 + \frac{\tau}{1-\tau+\tau p} ik\delta\right) = \frac{\pi\delta}{2} + \tan^{-1}\left(\frac{\tau}{1-\tau+\tau p} k\delta\right) \ge \frac{\pi\delta}{2},$$

whenever $k \ge \dfrac{1}{2}\left(c + \dfrac{1}{c}\right)$ and $0 \le \tau < \dfrac{1}{1-p}$, and this last inequality contradicts the assumption (5.1).

Similarly, if $\arg q(z_0) = -\dfrac{\pi\delta}{2}$, then we obtain

$$\arg \frac{Q^{(j)} f(z_0)}{z_0^{p-j}} \le -\frac{\pi\delta}{2},$$

which also contradicts the assumption (5.1).

Consequently, the function q need to satisfy the inequality $|\arg q(z)| < \dfrac{\pi\delta}{2}$, $z \in \mathbb{U}$, i.e. the conclusion of our theorem. □

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On a Class of Harmonic Univalent Functions Defined by Using a New Differential Operator

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Abstract

In this paper, a new class of complex-valued harmonic univalent functions defined by using a new differential operator is introduced. We investigate coefficient bounds, distortion inequalities, extreme points and inclusion results for this class.

Keywords: Harmonic functions, univalent functions, starlike and convex functions, differential operator.
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1. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics (e.g. see Choquet (Choquet, 1945), Dorff (Dorff, 2003), Duren (Duren, 2004)). A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense- preserving in D is that $\left|h'(z)\right| > \left|g'(z)\right|$ in D; see (Clunie & Sheil-Small, 1984).

Denote by SH the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in SH$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.1}$$

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Therefore

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad |b_1| < 1.$$

Note that SH reduces to the class S of normalized analytic univalent functions in U if the co-analytic part of f is identically zero.

In 1984 Clunie and Sheil-Small (Clunie & Sheil-Small, 1984) investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avcı and Zlotkiewicz (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999), Jahangiri (Jahangiri, 1999) studied the harmonic univalent functions.

The differential operator $D^n_{\alpha,\mu}(\lambda, w)$ $(n \in \mathbb{N}_0)$ was introduced by Bucur et al. (Bucur *et al.*, 2015). For $f = h + \overline{g}$ given by (1.1), we define the following differential operator:

$$D^n_{\alpha,\mu}(\lambda, w) f(z) = D^n_{\alpha,\mu}(\lambda, w) h(z) + (-1)^n \overline{D^n_{\alpha,\mu}(\lambda, w) g(z)},$$

where

$$D^n_{\alpha,\mu}(\lambda, w) h(z) = z + \sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n a_k z^k$$

and

$$D^n_{\alpha,\mu}(\lambda, w) g(z) = \sum_{k=1}^{\infty} \left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n b_k z^k,$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^\lambda$, with $D^n_{\alpha,\mu}(\lambda, w) f(0) = 0$.

Motivated by the differential operator $D^n_{\alpha,\mu}(\lambda, w)$, we define generalization of the differential operator for a function $f = h + \overline{g}$ given by (1.1).

$$D^0_{\alpha,\mu}(\lambda, w) f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$D^1_{\alpha,\mu}(\lambda, w) f(z) = (\alpha - \mu w^\lambda)(h(z) + \overline{g(z)}) + (\mu w^\lambda - \alpha + 1)(z h'(z) - \overline{z g'(z)}),$$

$$\vdots$$

$$D^n_{\alpha,\mu}(\lambda, w) f(z) = D\left(D^{n-1}_{\alpha,\mu}(\lambda, w) f(z)\right). \tag{1.2}$$

If f is given by (1.1), then from (1.2), we see that

$$D^n_{\alpha,\mu}(\lambda, w) f(z) = z + \sum_{k=2}^{\infty} \left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n \overline{b_k z^k}. \tag{1.3}$$

When , $w = \alpha = 0$, we get modified Salagean differential operator (Salagean, 1983).

Denote by $SH(\lambda, w, n, \alpha, \beta)$ the subclass of SH consisting of functions f of the form (1.1) that satisfy the condition

$$\Re\left(\frac{D^{n+1}_{\alpha,\mu}(\lambda, w) f(z)}{D^n_{\alpha,\mu}(\lambda, w) f(z)}\right) \geq \beta; \quad (0 \leq \beta < 1), \tag{1.4}$$

where $D^n_{\alpha,\mu}(\lambda, w)f(z)$ is defined by (1.3).

We let the subclass $\overline{SH}(\lambda, w, n, \alpha, \beta)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ in SH so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \; b_k \geq 0. \tag{1.5}$$

By suitably specializing the parameters, the classes $SH(\lambda, w, n, \alpha, \beta)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $SH(0, 0, 0, 0, 0) = SH^*(0)$ (Avcı (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999)),

(ii) $SH(0, 0, 0, 0, \beta) = SH^*(\beta)$ (Jahangiri (Jahangiri, 1999)),
$SH(0, 0, 0, 0, \beta) = \overline{S}_H(1, 0, \beta)$ (Yalçın (Yalçın, 2005)),

(iii) $SH(0, 0, 1, 0, 0) = KH(0)$ (Avcı (Avcı& Zlotkiewicz, 1990), Silverman (Silverman, 1998), Silverman and Silvia (Silverman, 1999)),

(iv) $SH(0, 0, 1, 0, \beta) = KH(\beta)$ (Jahangiri (Jahangiri, 1999)),
$SH(0, 0, 1, 0, \beta) = \overline{S}_H(2, 1, \beta)$ (Yalçın (Yalçın, 2005)),

(v) $SH(0, 0, n, 0, \beta) = H(n, \beta)$ (Jahangiri et al. (Jahangiri *et al.*, 2002)),
$SH(0, 0, n, 0, \beta) = \overline{S}_H(n + 1, n, \beta)$ (Yalçın (Yalçın, 2005)),

The object of the present paper is to give sufficient condition for functions $f = h+\overline{g}$ where h and g are given by (1.1) to be in the class $SH(\lambda, w, n, \alpha)$; and it is shown that this coefficient condition is also necessary for functions belonging to the subclass $\overline{SH}(\lambda, w, n, \alpha, \beta)$. Also, we obtain coefficient bounds, distortion inequalities, extreme points and inclusion results for this class.

2. Coefficient Bounds

Theorem 2.1. *Let $f = h + \overline{g}$ be so that h and g are given by (1.1). Furthermore, let*

$$\sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n |a_k| + \sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n |b_k| \leq 1-\beta, \tag{2.1}$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^{\lambda}$, $n \in \mathbb{N}_0$, $0 \leq \beta < 1$. Then f is sense-preserving, harmonic univalent in U and $f \in SH(\lambda, w, n, \alpha, \beta)$.

Proof. If $z_1 \neq z_2$,

$$\begin{aligned} \left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1 - \left|\frac{\sum\limits_{k=1}^{\infty} b_k\left(z_1^k - z_2^k\right)}{(z_1 - z_2) + \sum\limits_{k=2}^{\infty} a_k\left(z_1^k - z_2^k\right)}\right| \\ &> 1 - \frac{\sum\limits_{k=1}^{\infty} k\,|b_k|}{1 - \sum\limits_{k=2}^{\infty} k\,|a_k|} \end{aligned}$$

$$\geq 1-\frac{\sum_{k=1}^{\infty}\frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|b_k|}{1-\sum_{k=2}^{\infty}\frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|a_k|}$$

$$\geq 0,$$

which proves univalence. Note that f is sense preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty}\frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|a_k| \\
&\geq \sum_{k=1}^{\infty}\frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|b_k| > \sum_{k=1}^{\infty}k|b_k||z|^{k-1} \\
&\geq |g'(z)|.
\end{aligned}$$

Using the fact that $\Re(w) \geq \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$, it suffices to show that

$$\left|(1-\beta)D^{n}_{\alpha,\mu}(\lambda,w)+D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right|-\left|(1+\beta)D^{n}_{\alpha,\mu}(\lambda,w)-D^{n+1}_{\alpha,\mu}(\lambda,w)\right| \geq 0. \tag{2.2}$$

Substituting for $D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)$ and $D^{n}_{\alpha,\mu}(\lambda,w)f(z)$ in (2.2), we obtain

$$\begin{aligned}
&\left|(1-\beta)D^{n}_{\alpha,\mu}(\lambda,w)+D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right|-\left|(1+\beta)D^{n}_{\alpha,\mu}(\lambda,w)f(z)-D^{n+1}_{\alpha,\mu}(\lambda,w)f(z)\right| \\
\geq\ & 2(1-\beta)|z|-\sum_{k=2}^{\infty}\left[(k+1-\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}|a_k||z|^{k} \\
&-\sum_{k=1}^{\infty}\left[(k-1+\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}|b_k||z|^{k} \\
&-\sum_{k=2}^{\infty}\left[(k-1-\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}|a_k||z|^{k} \\
&-\sum_{k=1}^{\infty}\left[(k+1+\beta)+(k-1)(\mu w^{\lambda}-\alpha)\right]\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}|b_k||z|^{k} \\
\geq\ & 2(1-\beta)|z|\left(1-\sum_{k=2}^{\infty}\frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|a_k|\right. \\
&\left.-\sum_{k=1}^{\infty}\frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n}}{1-\beta}|b_k|\right).
\end{aligned}$$

This last expression is non-negative by (2.1), and so the proof is completed. □

Theorem 2.2. *Let* $f_n = h + \overline{g}_n$ *be given by (1.5). Then* $f_n \in \overline{SH}(\lambda, n, \alpha)$ *if and only if*

$$\sum_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k + \sum_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \leq 1-\beta, \quad (2.3)$$

where $\mu, \lambda, w \geq 0$, $0 \leq \alpha \leq \mu w^{\lambda}$, $n \in \mathbb{N}_0$, $0 \leq \beta < 1$.

Proof. The "if" part follows from Theorem 2.1 upon noting that $\overline{SH}(\lambda, w, n, \alpha, \beta) \subset SH(\lambda, w, n, \alpha, \beta)$. For the "only if" part, we show that $f \notin \overline{SH}(\lambda, w, n, \alpha, \beta)$ if the condition (2.3) does not hold. Note that a necessary and sufficient condition for $f_n = h + \overline{g}_n$ given by (1.5), to be in $\overline{SH}(\lambda, w, n, \alpha, \beta)$ is that the condition (1.4) to be satisfied. This is equivalent to

$$\Re\left\{\frac{(1-\beta)z - \sum\limits_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k}{z - \sum\limits_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k + \sum\limits_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}\right.$$

$$\left.\frac{-\sum\limits_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}{z - \sum\limits_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k z^k + \sum\limits_{k=1}^{\infty}(k+1)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k \overline{z}^k}\right\} \geq 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{(1-\beta) - \sum\limits_{k=2}^{\infty}(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1}}{1 - \sum\limits_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1} + \sum\limits_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}}$$

$$\frac{-\sum\limits_{k=1}^{\infty}(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}}{1 - \sum\limits_{k=2}^{\infty}\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^{n} a_k r^{k-1} + \sum\limits_{k=1}^{\infty}\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^{n} b_k r^{k-1}} \geq 0. \quad (2.4)$$

If the condition (2.3) does not hold, then the numerator in (2.4) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.4) is negative. This contradicts the required condition for $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ and so the proof is complete. □

3. Distortion Inequalities and Extreme Points

Theorem 3.1. *Let* $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. *Then for* $|z| = r < 1$ *we have*

$$|f_n(z)| \leq (1+b_1)\, r + \left(\frac{(1-\beta)}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} - \frac{(1+\beta)\left[2\left(\mu w^{\lambda}-\alpha\right)+1\right]^{n}}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} b_1\right) r^2,$$

and

$$|f_n(z)| \geq (1-b_1)\, r - \left(\frac{(1-\beta)}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} - \frac{(1+\beta)\left[2\left(\mu w^{\lambda}-\alpha\right)+1\right]^{n}}{(2-\beta)\left[\mu w^{\lambda}-\alpha+2\right]^{n}} b_1\right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Taking the absolute value of f_n we have

$$\begin{aligned}
|f_n(z)| &\leq (1+b_1)\, r + \sum_{k=2}^{\infty} (a_k + b_k)\, r^k \\
&\leq (1+b_1)\, r + \sum_{k=2}^{\infty} (a_k + b_k)\, r^2 \\
&= (1+b_1)\, r + \frac{(1-\beta)\, r^2}{(2-\beta)\,[\mu w^\lambda - \alpha + 2]^n} \sum_{k=2}^{\infty} \frac{(2-\beta)\left[\mu w^\lambda - \alpha + 2\right]^n}{(1-\beta)} [a_k + b_k] \\
&\leq (1+b_1)\, r + \frac{(1-\beta)\, r^2}{(2-\beta)\,[\mu w^\lambda - \alpha + 2]^n} \\
&\quad \times \sum_{k=2}^{\infty} \left(\frac{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n}{1-\beta} a_k \right. \\
&\quad \left. + \frac{(k+\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n}{1-\beta} b_k \right) \\
&\leq (1+b_1)\, r + \frac{(1-\beta)}{(2-\beta)\,[\mu w^\lambda - \alpha + 2]^n} \left(1 - \frac{(1+\beta)\left[2\left(\mu w^\lambda - \alpha\right) + 1\right]^n}{1-\beta} b_1 \right) r^2 \\
&\leq (1+b_1)\, r + \left(\frac{(1-\beta)}{(2-\beta)\,[\mu w^\lambda - \alpha + 2]^n} - \frac{(1+\beta)\left[2\left(\mu w^\lambda - \alpha\right) + 1\right]^n}{(2-\beta)\,[\mu w^\lambda - \alpha + 2]^n} b_1 \right) r^2.
\end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 3.1. □

Corollary 3.1. *Let f_n of the form (1.5) be so that $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. Then*

$$\left\{ w : |w| < \frac{(2-\beta)\left[\mu w^\lambda - \alpha + 2\right]^n - 1 + \beta}{(2-\beta)\left[\mu w^\lambda - \alpha + 2\right]^n} \right.$$

$$\left. - \frac{(2-\beta)\left[\mu w^\lambda - \alpha + 2\right]^n - (1+\beta)\left[2(\mu w^\lambda - \alpha) + 1\right]^n}{(2-\beta)\left[\mu w^\lambda - \alpha + 2\right]^n} \right\} \subset f_n(U).$$

Theorem 3.2. *Let f_n be given by (1.5). Then $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_{n_k}(z) \right),$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda-\alpha)+k\right]^n} z^k; \ \ (k \geq 2),$$

$$g_{n_k}(z) = z + (-1)^n \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda-\alpha)+k\right]^n} \overline{z}^k; \quad (k \geq 2),$$

$$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0.$$

In particular, the extreme points of $\overline{SH}(\lambda, w, n, \alpha, \beta)$ *are* $\{h_k\}$ *and* $\{g_{n_k}\}$.

Proof. For functions f_n of the form (1.5) we may write

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_{n_k}(z)) \\ &= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n} X_k z^k \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n} Y_k \overline{z}^k. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n}{1-\beta} \left(\frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n} X_k \right) \\ & + \sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n}{1-\beta} \left(\frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n} Y_k \right) \\ = & \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1, \text{ and so } f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta). \end{aligned}$$

Conversely, if $f_n \in \overline{SH}(\lambda, w, n, \alpha, \beta)$, then

$$a_k \leq \frac{1-\beta}{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n}$$

and

$$b_k \leq \frac{1-\beta}{(k+\beta)\left[(k+1)(\mu w^\lambda - \alpha) + k\right]^n}.$$

Setting

$$X_k = \frac{(k-\beta)\left[(k-1)(\mu w^\lambda - \alpha) + k\right]^n}{1-\beta} a_k; \ (k \geq 2),$$

$$Y_k = \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_k; \quad (k \geq 1),$$

and

$$X_1 = 1 - \left(\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k\right)$$

where $X_1 \geq 0$. Then

$$f_n(z) = X_1 z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{n_k}(z)$$

as required. □

4. Inclusion Results

Theorem 4.1. *The class $\overline{SH}(\lambda, w, n, \alpha, \beta)$ is closed under convex combinations.*

Proof. Let $f_{n_i} \in \overline{SH}(\lambda, w, n, \alpha, \beta)$ for $i = 1, 2, ...,$ where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k_i} \overline{z}^k.$$

Then by (2.3),

$$\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} a_{k_i} + \sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_{k_i} \leq 1. \tag{4.1}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{k_i}\right) \overline{z}^k.$$

Then by (4.1),

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} \left(\sum_{i=1}^{\infty} t_i a_{k_i}\right) \\
&+\sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-v} \left(\sum_{i=1}^{\infty} t_i b_{k_i}\right) \\
= &\sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{(k-\beta)\left[(k-1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} a_{k_i}\right. \\
&\left.+\sum_{k=1}^{\infty} \frac{(k+\beta)\left[(k+1)(\mu w^{\lambda}-\alpha)+k\right]^n}{1-\beta} b_{k_i}\right) \leq \sum_{i=1}^{\infty} t_i = 1.
\end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{SH}(\lambda, w, n, \alpha, \beta)$. □

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Results on Approximation Properties in Intuitionistic Fuzzy Normed Linear Spaces

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Abstract

In this paper we introduce the notions approximation properties (APs) and bounded approximation properties (BAPs) in the setting of intuitionistic fuzzy normed linear spaces (IFNLSs). Further, we define strong intuitionistic fuzzy continuous and strong intuitionistic fuzzy bounded operators and using them we prove the existence of an IFNLS which does not have the approximation property. In addition, we give example of an IFNLS with the AP which fails to have the BAP.

Keywords: Intuitionistic fuzzy normed linear space, approximation property, bounded approximation property.
2010 MSC: 55M20, 54H25, 47H09.

1. Introduction

In analysis many problems we study are concerned with large classes of objects most of which turn out to be vector spaces or linear spaces. Since limit process is indispensable in such problems, a metric or topology may be induced in those classes. If the induced metric satisfies the translation invariance property, a norm can be defined in that linear space and we get a structure of the space which is compatible with that metric or topology. The resulting structure is a normed linear space. There are situations where crisp norm can not measure the length of a vector accurately and in such cases the notion of fuzzy norm happens to be useful. There has been a systematic development of fuzzy normed linear spaces (FNLSs) and one of the important development over FNLS is the notion of intuitionistic fuzzy normed linear space (IFNLS). The study of analytic propertis of IFNLSs, their topological structure and generalizations, therefore, remain well motivated areas of research.

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The idea of a fuzzy norm on a linear space was introduced by Katsaras (Katsaras, 1984). Felbin (Felbin, 1992) introduced the idea of a fuzzy norm whose associated metric is of Kaleva and Seikkala (Kaleva & Seikkala, 1984) type. Cheng and Mordeson (Cheng & Mordeson, 1994) introduced another notion of fuzzy norm on a linear space whose associated metric is Kramosil and Michalek (Kramosil & Michalek, 1975) type. Again, following Cheng and Mordeson, one more notion of fuzzy normed linear space was given by Bag and Samanta (Bag & Samanta, 2003*a*).

The notion of intuitionistic fuzzy set (IFS) introduced by Atanassov (Atanassov, 1986) has triggered some debate (for details, see (Cattaneo & Ciucci, 2006; Dubois *et al.*, 2005; Grzegorzewski & Mrowka, 2005)) regarding the use of the terminology "intuitionistic" and the term is considered to be a misnomer on the following account:

- The algebraic structure of IFSs in not intuitionistic, since negation is involutive in IFS theory.
- Intuitionistic logic obeys the law of contradiction, IFSs do not.

Also IFSs are considered to be equivalent to interval-valued fuzzy sets and they are particular cases of L-fuzzy sets. In response to this debate, Atanassov justified the terminology in (Atanassov, 2005). Apart from the terminological issues, research in intuitionistic fuzzy setting remains well motivated as IFSs give us a very natural tool for modeling imprecision in real life situations which can not be handled with fuzzy set theory alone and also IFS found its application in various areas of science and engineering.

With the help of arbitrary continuous t-norm and continuous t-conorm, Saadati and Park (Saadati & Park, 2006) introduced the concept of IFNLS. There has been further development over IFNLS, e.g., the topological structure of an intuitionistic fuzzy 2-normed space has been studied by Mursaleen and Lohani in (Mursaleen & Lohani, 2009). Recently, a number of interesting properties of IFNLS have been studied by Mursaleen and Mohiuddine (Mursaleen & Mohiuddine, 2009*a*,*b*,*c*,*d*). Further, generalizing the idea of Saadati and Park, an intuitionistic fuzzy n-normed linear space (IFnNLS) has been defined by Vijayabalaji et al. (Vijayabalaji *et al.*, 2007*b*). More properties of IFnNLS have been studied by N. Thillaigovindan, S. Anita Shanti and Y. B. Jun in (Vijayabalaji *et al.*, 2007*a*). Some more recent work in similar context can be found in (Debnath, 2015; Debnath & Sen, 2014*a*,*b*; Esi & Hazarika, 2012; Mursaleen *et al.*, 2010*a*; Sen & Debnath, 2011).

In classical Banach space theory, some most important properties are "Approximation properties" which were investigated by Grothendick (Grothendiek, 1955). We say that a Banach space X has the approximation property (AP) if, for every compact K and $\epsilon > 0$, there is a bounded finite rank operator $T : X \longrightarrow X$ such that $||T(x) - x|| < \epsilon$, for all $x \in K$, i.e. $I(x)$-the identity operator on X- can be approximated by finite rank operators uniformly on compact sets. Also X has the bounded approximation property (BAP) if for every compact K and $\epsilon > 0$, there is a bounded finite rank operator $T : X \longrightarrow X$ with $||T|| \leqslant \lambda$ such that $||T(x) - x|| < \epsilon$ for all $x \in K$ for some $\lambda > 0$. The APs play very crucial role in the study of infinite dimensional Banach space theory and also in the investigation of Schauder bases. Some of the important references from related works being (Choi *et al.*, 2009; Enflo, 1973; Kim, 2008; Mursaleen *et al.*, 2010*b*; Szarek, 1987).

Yilmaz (Yilmaz, 2010*a*) introduced the notion of the AP in fuzzy normed spaces and established some interesting results on it. Very recently Keun Young Lee (Lee, 2015) identified some limitations in Yilmaz's definitions regarding the continuity of fuzzy operators. He modified Yilmaz's definitions and studied approximation property (AP) and bounded approximation property (BAP) on fuzzy normed spaces.

In this article we address the questions raised by Keun Young Lee (Lee, 2015) and also generalize the work of Figel and Johnson (Figel & Johnson, 1973) in the context of AP and BAP in the new setting of IFNLS.

First we recall some basic definitions and results which will be used subsequently.

Definition 1.1. (Saadati & Park, 2006) The 5-tuple $(X, \mu, \nu, *, \circ)$ is said to be an IFNLS if X is a linear space, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm, and μ, ν fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x, t) + \nu(x, t) \leqslant 1$,
(b) $\mu(x, t) > 0$,
(c) $\mu(x, t) = 1$ if and only if $x = 0$,
(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(e) $\mu(x, t) * \mu(y, s) \leqslant \mu(x + y, t + s)$,
(f) $\mu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(g) $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$,
(h) $\nu(x, t) < 1$,
(i) $\nu(x, t) = 0$ if and only if $x = 0$,
(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(k) $\nu(x, t) \circ \nu(y, s) \geqslant \nu(x + y, t + s)$,
(l) $\nu(x, t) : (0, \infty) \to [0, 1]$ is continuous in t,
(m) $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to 0} \nu(x, t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm. When no confusion arises, an IFNLS will be denoted simply by X.

Definition 1.2. (Debnath, 2012) Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > 1 - \varepsilon$ and $\nu(x_k - \xi, t) < \varepsilon$ for all $k \geqslant k_0$. It is denoted by $(\mu, \nu) - \lim x_k = \xi$.

Definition 1.3. (Saadati & Park, 2006) Let X be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\alpha \in (0, 1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > 1 - \alpha$ and $\nu(x_k - x_m, t) < \alpha$ for all $k, m \geqslant k_0$.

Definition 1.4. (Debnath & Sen, 2014a) Let X be an IFNLS. Then X is said to be complete if and only if every Cauchy sequence of X is convergent.

Definition 1.5. (Lael & Nourouzi, 2007) Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A subset S in X is said to be compact if each sequence of elements of S has a convergent subsequence.

Definition 1.6. (Debnath, 2012) Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. For $t > 0$, we define an open ball $B(x, r, t)$ with center at $x \in X$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in X : \mu(x - y, t) > 1 - r, \nu(x - y, t) < r\}.$$

Proof of the following lemma is similar to its analogue in case of fuzzy normed spaces (Bag & Samanta, 2003*b*).

Lemma 1.1. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS with the condition*

$$\mu(x, t) > 0 \text{ and } \nu(x, t) < 1 \text{ implies } x = 0, \text{ for all } t \in \mathbb{R}^+. \tag{1.1}$$

Let $\|x\|_\alpha = inf\{t \in \mathbb{R}^+ : \mu(x, t) > \alpha$ and $\nu(x, t) < 1 - \alpha\}$ for each $\alpha \in (0, 1)$. Then $\{\| \cdot \|_\alpha : \alpha \in (0, 1)\}$ is an ascending class of norms on X. These norms are called α- norms on the intuitionistic fuzzy norm (μ, ν).

Definition 1.7. (Mursaleen *et al.*, 2010*a*) Let (x_n) be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. It is said to be basis of X if for every $x \in X$ there exists a unique sequence (a_n) of scalars such that

$$(\mu, \nu) - \lim \textstyle\sum_{k=1}^{n} a_k x_k = x.$$

that is, for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists $n_0 = n_0(\alpha, \epsilon) \in N$ such that $n \geqslant n_0$ implies,

$$\mu(x - \textstyle\sum_{k=1}^{n} a_k x_k, \epsilon) > 1 - \alpha \text{ and } \nu(x - \sum_{k=1}^{n} a_k x_k, \epsilon) < \alpha, \text{ where } x = \sum_{k=1}^{\infty} a_k x_k.$$

2. Main Results

Now we are ready to discuss our main results. First we define some important notions in connection with approximation property in IFNLS.

Definition 2.1. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A complete IFNLS is said to have the approximation property, briefly AP, if for every compact set K in X and for each $\alpha > (0, 1)$ and $\epsilon > 0$, there exists an operator T of finite rank such that

$$\mu\left(T_\alpha(x) - x, \epsilon\right) > 1 - \alpha \text{ and } \nu\left(T_\alpha(x) - x, \epsilon\right) < \alpha$$

for every $x \in K$.

Definition 2.2. Let λ be a real number. An IFNLS $(X, \mu, \nu, *, \circ)$ is said to have the λ-bounded approximation property, briefly λ-BAP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in F(X, X, \lambda)$ such that

$$\mu\left(T(x) - x, \epsilon\right) > 1 - \alpha \text{ and } \nu\left(T(x) - x, \epsilon\right) < \alpha$$

for every $x \in K$.

Definition 2.3. Suppose that an IFNLS $(X, \mu, \nu, *, \circ)$ has a basis (x_n). For each positive integer m, the m^{th} natural projection P_m for x_m is the map

$$\sum_{n=1}^{\infty} a_n x_n \longrightarrow \sum_{n=1}^{m} a_n x_n \text{ from } (X, \mu, \nu, *, \circ) \text{ to } (X, \mu, \nu, *, \circ).$$

Definition 2.4. Let $(X, \mu, \nu, *, \circ)$ and $(Y, \mu', \nu', *, \circ)$ be two IFNLS and $T : X \longrightarrow Y$ be a linear operator where (μ, ν) and (μ', ν') are intuitionistic fuzzy normed. Then

1. The operator T is called strongly intuitionistic fuzzy (shortly sif) continuous at $a \in X$ if, for given $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x \in X$,
$$\mu'(T(x) - T(a), \epsilon) \geqslant \mu(x - a, \delta) \text{ and } \nu'(T(x) - T(a), \epsilon) \leqslant \nu(x - a, \delta).$$
If T is sif-continuous at each point of X, then T is said to be sif-continuous on X.
2. The operator T is called strongly intuitionistic fuzzy bounded on X if there exists a positive real number M such that $\mu'(T(x), t) \geqslant \mu(x, \frac{t}{M})$ and $\nu'(T(x), t) \leqslant \nu(x, \frac{t}{M})$ for all $x \in X$ and $t \in \mathbb{R}$. We will denote the set of all strongly intuitionistic fuzzy (shortly sif) bounded operators form X to Y by $F(X, Y)$. Then $F(X, Y)$ is a vector space. For all $M > 0$, $F(X, Y, M)$ is denoted by
$$\left\{T \in F(X, Y) : \mu'(T(x), t) \geqslant \mu(x, \tfrac{t}{M}), \nu'(T(x), t) \leqslant \nu(x, \tfrac{t}{M}), \forall x \in X, \forall t \in \mathbb{R}\right\},$$
where M is a positive real number.
For some $M > 0$ if $\mathbb{S} = F(X, Y, M)$ then $\mathbb{S}$ is called a bounded subset of $F(X, Y)$. Again the set of all finite rank sif-bounded operators from X to Y is denoted by $\bar{F}(X, Y)$. Then $\bar{F}(X, Y)$ is subspace of $F(X, Y)$. Similarly, we can say that $\bar{F}(X, Y, M)$ is also a subspace of $F(X, Y, M)$ for some $M > 0$.

Proof of the following is similar to its fuzzy analogue in (Bag & Samanta, 2005).

Lemma 2.1. *Let $(X, \mu, \nu, *, \circ)$ and $(Y, \mu', \nu', *, \circ)$ be two IFNLSs satisfying condition 1.1 and $T : X \longrightarrow Y$ be a linear operator. Then T is sif-bounded if and only if it is uniformly bounded with respect to α- norms of (μ, ν) and (μ', ν'). That is, there exists some $M > 0$, independent of α, such that $||T(x)||_\alpha \leqslant M||x||_\alpha$, for all $\alpha \in (0, 1)$.*

Remark. If $(X, \mu, \nu, *, \circ)$ and $(Y, \mu', \nu', *, \circ)$ be two IFNLSs satisfying the conditions:
$\mu(x, t) > 0$ and $\nu(x, t) < 1$ implies $x = 0$ for all $t \in \mathbb{R}^+$ and
for $x \neq 0$, $\mu(x, t)$ is continuous and strictly increasing on $\{t : 0 < \mu(x, t) < 1\}$, while $\nu(x, t)$ is continuous and strictly decreasing on $\{t : 0 < \mu(x, t) < 1\}$ and $M > 0$. Then we obtain

$$F(X, Y, M) = \{T \in F(X, Y) : ||T(x)||_\alpha \leqslant M||x||_\alpha, \forall x \in X, \forall \alpha \in (0, 1)\}.$$

Hence $F(X, Y, M)$ and $\bar{F}(X, Y, M)$ are bounded convex subsets of $F(X, Y)$.

Theorem 2.1. *Let X be a Banach space and (x_n) be a Schauder basis in X. Then (x_n) is a basis for an IFNLS $(X, \mu, \nu, *, \circ)$ where*

$$\mu(x, t) = \begin{cases} \frac{t - ||x||}{t + ||x||}, & \text{if } t > ||x|| \\ 0, & \text{if } t \leqslant ||x||, \end{cases}$$

$$\nu(x, t) = \begin{cases} 1 - \frac{t - ||x||}{t + ||x||}, & \text{if } t > ||x|| \\ 1, & \text{if } t \leqslant ||x||, \end{cases}$$

and every natural projection is sif-continuous.

Proof. Given that (x_n) is a basis for an IFNLS $(X, \mu, \nu, *, \circ)$.

It is enough to show that -

natural projection $P_n : (X, \mu, \nu, *, \circ) \to (X, \mu, \nu, *, \circ)$ is sif-bounded for each $x \in N$.

Let $n \in N$, $t \in R$, $x \in X$.

Consider $M = ||P_n||$.

If $t \leqslant 0$, the result is trivial.

Assume that $t > 0$. Then it is enough to show that

$$\mu\left(P_n\left(x\right), t\right) \geqslant \mu\left(x, \tfrac{t}{M}\right) \text{ and } \nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

The proof of $\mu\left(P_n\left(x\right), t\right) \geqslant \mu\left(x, \tfrac{t}{M}\right)$ can be established in a similar manner as in Proposition 3.4 of (Lee, 2015).

Now considering for ν, we have

$$t > M||x||,$$

then

$$\nu\left(x, \tfrac{t}{M}\right) = 1 - \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||}.$$

By the assumption,

$$t > M||x|| = ||P_n||||x|| \geqslant ||P_n\left(x\right)||$$

and

$$\frac{t - ||P_n(x)||}{t + ||P_n(x)||} \geqslant \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||}.$$

Therefore, we have

$$\nu\left(P_n\left(x\right), t\right) = 1 - \frac{t - ||P_n(x)||}{t + ||P_n(x)||} \leqslant 1 - \frac{\frac{t}{M} - ||x||}{\frac{t}{M} + ||x||} = \nu\left(x, \tfrac{t}{M}\right).$$

Hence

$$\nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

Secondly,

$$t \leqslant ||Mx||,$$

then

$$\nu\left(Mx, t\right) = 1.$$

Thus,

$$\nu\left(P_n\left(x\right), t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right).$$

□

So, we have the existence of an IFNLS having a basis such that every natural projection is sif-continuous. Now provide modified definitions of APs and BAPs in IFNLSs by incorporating the continuity of approximating operators.

Definition 2.5. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then X is said to be have the approximation property, briefly AP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}(X, X)$ such that

$$\mu(T(x) - x, \epsilon) > 1 - \alpha \text{ and } \nu(T(x) - x, \epsilon) < \alpha$$

for every $x \in K$.

Definition 2.6. Let $(X, \mu, \nu, *, \circ)$ be an IFNLS and λ be a positive real number. Then X is said to be have the λ- bounded approximation property, briefly λ - BAP, if for every compact set K in X and for each $\alpha \in (0, 1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}(X, X, \lambda)$ such that

$$\mu(T(x) - x, \epsilon) > 1 - \alpha \text{ and } \nu(T(x) - x, \epsilon) < \alpha$$

for every $x \in K$. We can also say that X has the BAP if X has the λ-BAP for some $\lambda > 0$.

Theorem 2.2. *Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. Then the following are equivalent.*

1. $(X, \mu, \nu, *, \circ)$ *has the AP.*
2. *If $(Y, \mu', \nu', *, \circ)$ is an IFNLS, then for every $T \in F(X, Y)$, every compact set K in $(X, \mu, \nu, *, \circ)$ and for each $\alpha \in (0, 1)$ and $t > 0$, there exists an operator $S \in \bar{F}(X, Y)$ such that*

$$\mu'(S(x) - T(x), t) > 1 - \alpha \text{ and } \nu'(S(x) - T(x), t) < \alpha$$

for each $x \in K$.

3. *If $(Y, \mu', \nu', *, \circ)$ is an IFNLS, then for every $T \in F(Y, X)$, every compact set K in $(Y, \mu', \nu', *, \circ)$ and for each $\alpha \in (0, 1)$ and $t > 0$, there exists an operator $S \in \bar{F}(Y, X)$ such that*

$$\mu(S(y) - T(y), t) > 1 - \alpha \text{ and } \nu(S(y) - T(y), t) < \alpha$$

for each $y \in K$.

Proof. $(i) \Rightarrow (ii)$

Let $T \in F(X, Y)$ and K be a compact set in $(X, \mu, \nu, *, \circ)$ and $\alpha \in (0, 1)$ and $t > 0$ and $t \in \mathbb{R}$. Then there exists a positive real number M such that

$$\mu'(T(x), t) \geqslant \mu\left(x, \tfrac{t}{M}\right) \text{ and } \nu'(T(x), t) \leqslant \nu\left(x, \tfrac{t}{M}\right)$$

for all $x \in X$.

Since $(X, \mu, \nu, *, \circ)$ has the AP, there exists an operator $R \in F(X, X)$ such that

$$\mu\left(R(x) - x, \tfrac{t}{M}\right) > 1 - \alpha \text{ and } \nu\left(R(x) - x, \tfrac{t}{M}\right) < \alpha$$

for every $x \in K$.

Now we put $S = TR$. Since T and R both are sif-bounded operators, therefore S is also a sif-bounded operator.

$$\begin{aligned}\mu'(S(x)-T(x),t) &= \mu'(TR(x)-T(x),t)\\ &\geqslant \mu\left(R(x)-x,\frac{t}{M}\right)\\ &> 1-\alpha.\end{aligned}$$

and

$$\begin{aligned}\nu'(S(x)-T(x),t) &= \nu'(TR(x)-T(x),t)\\ &\leqslant \nu\left(R(x)-x,\frac{t}{M}\right)\\ &< \alpha.\end{aligned}$$

for every $x \in K$.

$(i) \Rightarrow (iii)$

Let $T \in F(Y,X)$ and K be a compact set in $\left(Y,\mu',\nu',*,\circ\right)$ and $\alpha \in (0,1)$ and $t > 0$ and $t \in R$.

Since $(X,\mu,\nu,*,\circ)$ has the AP and $T(K)$ is compact set in $(X,\mu,\nu,*,\circ)$, there exists an operator $R \in \bar{F}(X,X)$ such that

$$\mu(R(x)-x,t) > 1-\alpha \text{ and } \nu(R(x)-x,t) < \alpha$$

for every $x \in T(K)$.

Now we put, $S = RT \in \bar{F}(Y,X)$. Then we have,

$$\begin{aligned}\mu(S(y)-T(y),t) &= \mu(RT(y)-T(y),t)\\ &> 1-\alpha.\end{aligned}$$

and

$$\begin{aligned}\nu(S(y)-T(y),t) &= \nu(RT(y)-T(y),t)\\ &< \alpha,\end{aligned}$$

for each $y \in K$.

Since (i) implies both (ii) and (iii), hence (i), (ii) and (iii) are equivalent.

Hence proposition is proved.

□

Proof of the following Lemma is similar to Lemma 4.2 of (Lee, 2015).

Lemma 2.2. *Let $(X,\mu,\nu,*,\circ)$ be an IFNLS and K be a subset in X. If K is a compact set in $(X,\mu,\nu,*,\circ)$, then for every $\alpha \in (0,1)$ and $t > 0$, there exists a finite set $\{x_1,x_2,\ldots,x_n\}$ in K such that for every $x \in K$ we have $x \in B(x_i,\alpha,t)$ for some x_i.*

Theorem 2.3. *Let $(X,\mu,\nu,*,\circ)$ be an IFNLS with intuitionistic fuzzy norm (μ,ν) and $M > 0$. Suppose that there exists a sequence $(T_n) \in \bar{F}(X,X,M)$ such that $T_n(x) \longrightarrow x$ for every $x \in X$, then $(X,\mu,\nu,*,\circ)$ has the AP.*

Proof. Let (T_n) be a sequence in $\bar{F}(X, X, M)$ such that

$$T_n(x) \longrightarrow x \text{ for every } x \in X.$$

Let $\alpha \in (0,1)$ and $t > 0$, and K be a compact set in $(X, \mu, \nu, *, \circ)$.

By the above Lemma, there exists a finite set $\{x_1, x_2, ..., x_n\} \subset K$ such that for $x \in K$ we have $x \in B(x_i, \alpha, t)$ for some x_i.

Then there exists $N_1, N_2 \in \mathbb{N}$ such that if $n \geqslant N_1, N_2$ we have,

$$\mu(T_n(x_i) - x_i, t) > 1 - \alpha \text{ and } \nu(T_n(x_i) - x_i, t) < \alpha$$

for each i.

Let $x \in K$ and choose i such that $x \in B(x_i, \alpha, t)$, that is,

$$\mu(x_i - x, t) > 1 - \alpha \text{ and } \nu(x_i - x, t) < \alpha.$$

Then for $n \geqslant N_1, N_2$,

$$\begin{aligned}
\mu(T_n(x) - x, t) &= \mu(T_n(x) + (-T_n(x_i)) + (T_n(x_i)) + (-x_i) + x_i + (-x), t) \\
&\geqslant min\left\{\mu\left(T_n(x - x_i), \frac{t}{3}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&\geqslant min\left\{\mu\left(x - x_i, \frac{t}{3M}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&> 1 - \alpha.
\end{aligned}$$

And

$$\begin{aligned}
\nu(T_n(x) - x, t) &= \nu(T_n(x) + (-T_n(x_i)) + (T_n(x_i)) + (-x_i) + x_i + (-x), t) \\
&\leqslant max\left\{\mu\left(T_n(x - x_i), \frac{t}{3}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&\leqslant max\left\{\mu\left(x - x_i, \frac{t}{3M}\right), \mu\left(T_n(x_i) - x_i, \frac{t}{3}\right), \mu\left(x_i - x, \frac{t}{3}\right)\right\} \\
&< \alpha.
\end{aligned}$$

Therefore, $\mu(T_n(x) - x, t) > 1 - \alpha$ and $\nu(T_n(x) - x, t) < \alpha$.

Hence $(X, \mu, \nu, *, \circ)$ has the AP. □

By using the above result we derive the following.

Theorem 2.4. *Suppose $(X, \mu, \nu, *, \circ)$ has a basis $\{x_n\}$ and every natural projection*

$$P_n : (X, (\mu, \nu)) \longrightarrow (X, (\mu, \nu))$$

is sif-continuous. Then $(X, \mu, \nu, *, \circ)$ has the AP but the converse is not necessarily true.

Theorem 2.5. *An IFNLS $(X, \mu, \nu, *, \circ)$ satisfying condition 1.1 has the AP if and only if for every compact set K in $(X, \mu, \nu, *, \circ)$ and for each $\alpha \in (0,1)$ and $\epsilon > 0$, there exists an operator $T \in \bar{F}(X, X)$ such that*

$$||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Theorem 2.6. *Let* $(X, \mu, \nu, *, \circ)$ *be an IFNLS satisfying condition 1.1 and* $\lambda > 0$. *Then* $(X, \mu, \nu, *, \circ)$ *has* λ*-BAP if and only if for every compact set* K *in* $(X, \mu, \nu, *, \circ)$ *and for each* $\alpha \in (0, 1)$ *and* $\epsilon > 0$, *there exists an operator* $T \in \bar{F}(X, X, \lambda)$ *such that*

$$||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Proof of the above two results follow from (Yilmaz, 2010*b*).

3. Examples

In this section, we give answers to the following interesting questions with proper examples:

1. Does every IFNLS have the AP?
2. Does in an IFNLSs the AP imply the BAP?

Now we are going to solve (in negative sense) the problem (i) and (ii) with the help of following two examples.

Example 3.1. As we know that there exists a Banach space $(X, ||\cdot||)$ which fails to have the approximation property, similarly there exists an IFNLS $(X, \mu, \nu, *, \circ)$ which fails to have the AP.

Let us define a function,

$$\mu, \nu : X \times \mathbb{R} \longrightarrow [0, 1] \text{ by}$$

$$\mu(x, t) = \begin{cases} 1, & \text{if } t > ||x|| \\ 0, & \text{if } t \leqslant ||x||. \end{cases}$$

and

$$\nu(x, t) = \begin{cases} 0, & \text{if } t > ||x|| \\ 1, & \text{if } t \leqslant ||x||. \end{cases}$$

where (μ, ν) is the intuitionistic fuzzy norm and $||x||_\alpha = ||x||$, for every $\alpha \in (0, 1)$.

Now suppose that $(X, \mu, \nu, *, \circ)$ has the AP.

Let $\alpha \in (0, 1)$ and $\epsilon > 0$ and K be a compact set in X. Since $||x||_\alpha = ||x||$ for each $\alpha \in (0, 1)$, K is compact in $(X, \mu, \nu, *, \circ)$. Then by Theorem 2.5, there exists an operator $T_\alpha \in \bar{F}(X, X)$ such that

$$||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$.

Hence we have,

$$||T(x) - x|| = ||T(x) - x||_\alpha < \epsilon$$

for every $x \in K$, which is a contradiction as $(X, ||\cdot||)$ fails to have the approximation property.

$(X, \mu, \nu, *, \circ)$ has fails to have the AP.

As in Example 4.9 of (Lee, 2015), we give below an example of the existence of an IFNLS which has the AP but fails to have BAP.

Example 3.2. Enflo and Lidenstrauss (Enflo, 1973; Lindenstrauss, 1971) has proved the existence of a Banach Space X_0 which has the metric approximation property but its dual space X_0^* fails to have the approximation property. There is a sequence $(||\cdot||_n)$ of equivalent norms on X_0 so that $(X_0, ||\cdot||_n)$ fails to have the n- BAP. Consider $X_n = (X_0, ||\cdot||_n)$. Thus $(\sum \oplus X_n)_{l_2}$ fails to have the BAP where $(\sum \oplus X_n)_{l_2}$ is a Banach space whose elements are sequence of the form $(x_1, x_2, ...)$, where $\sum_{n=1}^{\infty} ||x_n||_n^2 < \infty$ and $x_n \in X_n$.

Now we consider, $X = (\sum \oplus X_n)_{l_2}$, and define $||x|| = \left(\sum_{n=1}^{\infty} ||x_n||_n^2\right)^{\frac{1}{2}}$ and $||x||_1 = sup_n ||x_n||$ for all $x = (x_1, x_2, ...) \in X$.

Let us defined a function,

$$\mu, \nu : X \times \mathbb{R} \longrightarrow [0, 1] \text{ by}$$

$$\mu(x,t) = \begin{cases} 1, & \text{if } t > ||x|| \\ \frac{1}{2}, & \text{if } ||x||_1 < t \leqslant ||x|| \\ 0, & \text{if } t \leqslant ||x||_1, \end{cases}$$

and

$$\nu(x,t) = \begin{cases} 0, & \text{if } t > ||x|| \\ \frac{1}{2}, & \text{if } ||x||_1 < t \leqslant ||x|| \\ 1, & \text{if } t \leqslant ||x||_1, \end{cases}$$

where (μ, ν) is the intuitionistic fuzzy norm.

Consider the α-norms as-

$$||x||_\alpha = \begin{cases} ||x||, & \text{if } 1 > \alpha > \frac{1}{2} \\ ||x||_1, & \text{if } 0 < \alpha \leqslant \frac{1}{2}. \end{cases}$$

Suppose that $(X, \mu, \nu, *, \circ)$ has the BAP. Let us assume that K be a compact set in $(X, ||\cdot||)$. Then we have to show that K is a compact set in $(X, \mu, \nu, *, \circ)$.

Let $\epsilon > 0$ and (x_n) be a sequence in K. As K is compact subset in $(X, ||\cdot||)$, there exists subsequence (x_{n_k}) in $(X, ||\cdot||)$. Therefore there exists an $x \in X$ and integers $\mu, \nu > 0$ such that for $k \geqslant \mu, \nu$

$$||x_{n_k} - x|| < \epsilon.$$

Since $||x||_1 \leqslant ||x||$ for all $x \in X$, therefore for $k \geqslant \mu, \nu$

$$||x_{n_k} - x||_\alpha < \epsilon$$

for all $\alpha \in (0, 1)$.

Hence K is a compact set in $(X, \mu, \nu, *, \circ)$.

Next consider $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\epsilon > 0$. As K is a compact set in $(X, \mu, \nu, *, \circ)$ and using $\alpha \in \left(\frac{1}{2}, 1\right)$ and $\epsilon > 0$ we have $\lambda > 0$ and $T_{\alpha,\epsilon} \in \bar{F}(X, X, \lambda)$ such that

$$||T_{\alpha,\epsilon}(x) - x||_\alpha < \epsilon \text{ for every } x \in K.$$

Then we have $||T_{\alpha,\epsilon}(x) - x|| < \epsilon$ and $||T_{\alpha,\epsilon}(x)|| \leqslant \lambda ||x||$, which is a contradiction as $(X, ||\cdot||)$ fails to have the BAP.

Hence $(X, \mu, \nu, *, \circ)$ has fails to have the BAP.

Finally, we have to show that $(X, \mu, \nu, *, \circ)$ has the AP. Let $\epsilon > 0$ and K be a compact subset in $(X, \mu, \nu, *, \circ)$. Again let $P_j : X \longrightarrow \left(\sum_{n=1}^{j} \oplus X_n\right)_{l_2}$ be the projection given by

$$P((x)) = (x_1, x_2, ..., x_j).$$

Since K is a compact set in X, therefore by Theorem 2.4 of (Choi *et al.*, 2009) there exists a natural number $m \in N$ and a finite rank operator $T' : (\sum_{n=1}^{m} \oplus X_n)_{l_2} \longrightarrow (\sum_{n=1}^{m} \oplus X_n)_{l_2}$ such that

$$||kT' P_m(x) - x|| < \epsilon$$

for every $x \in K$, where k is the map defined as $k : (\sum_{n=1}^{m} \oplus X_n)_{l_2} \longrightarrow X$ such that

$$k(x_1, x_2, ..., x_m) = (x_1, x_2, ..., x_m, 0, ...).$$

Now we put $T = kT' P_m$. As T is a finite rank operator defined as $T : X \longrightarrow X$ and $||x||_1 \leqslant ||x||$ for all $x \in X$, we have

$$||T(x) - x||_1 < \epsilon$$

that is, for every $\alpha \in (0, 1)$, we have

$$||T(x) - x||_\alpha < \epsilon.$$

Next we have to show that T is sif-bounded on X. Since $(\sum_{n=1}^{m} \oplus X_n)_{l_2}$ and $(\sum_{n=1}^{m} \oplus X_n)_{l_\infty}$ are equivalent, there exists $M' > 1$ such that

$$(\sum_{n=1}^{m} ||x_n||_n^2)^{\frac{1}{2}} \leqslant M' sup_{1 \leqslant n \leqslant m} ||x_n||_n.$$

Then,

$$\begin{aligned} \|T(x)\|_1 \leqslant ||T(x)|| &= \|kT' P_m(x)\| \\ &\leqslant ||kT'||(\sum_{n=1}^{m} \|x_n\|_n^2)^{\frac{1}{2}} \\ &\leqslant ||kT'||M' \sup_{1 \leqslant n \leqslant m} ||x_n||_n \\ &\leqslant ||kT'||M'||x||_1. \end{aligned}$$

Taking $M = max\left\{||T||, ||kT'||, M'\right\}$, we have to show that

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(x, \tfrac{t}{M}\right) \text{ and } \nu\left(T\left(x\right),t\right) \leqslant \nu\left(x, \tfrac{t}{M}\right)$$

for all $x \in X$ and $t \in \mathbb{R}$.

If $t \leqslant 0$, the result is trivial.

Assume that $t > 0$. Then it is enough to show that

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right) \text{ and } \nu\left(T\left(x\right),t\right) \leqslant \nu\left(Mx,t\right), \text{ for all } x \in X \text{ and } t \in \mathbb{R}.$$

Now first consider for μ:

For the first condition:

$$t > M||x||$$

then

$$\mu\left(Mx,t\right) = 1$$

By the assumption,

$$t > M||x|| \geqslant ||T||||x|| \geqslant ||T\left(x\right)||$$

we have

$$\mu\left(T\left(x\right),t\right) = 1.$$

Hence

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right).$$

For the second condition:

$$||Mx||_1 < t \leqslant ||Mx||$$

then

$$\mu\left(Mx,t\right) = \tfrac{1}{2}.$$

By the assumption

$$t > M||x||_1 \geqslant ||kT^{'}||M^{'}||x||_1 \geqslant ||T\left(x\right)||_1$$

we have

$$\mu\left(T\left(x\right),t\right) \geqslant \tfrac{1}{2}.$$

Hence

$$\mu\left(T\left(x\right),t\right) \geqslant \mu\left(Mx,t\right).$$

For the third condition :

$$t \leqslant M||x||$$

we have

$$\mu(Mx, t) = 0.$$

Then by the assumption trivially we obtain,

$$\mu(T(x), t) \geqslant \mu(Mx, t).$$

Next considering for ν :
For the first condition :

$$t > M||x||$$

then

$$\nu(Mx, t) = 0.$$

By the assumption,

$$t > M||x|| \geqslant ||T||||x|| \geqslant ||T(x)||$$

we have

$$\nu(T(x), t) = 0.$$

Hence

$$\nu(T(x), t) \leqslant \nu(Mx, t).$$

For the second condition:

$$||Mx||_1 < t \leqslant ||Mx||$$

then

$$\nu(Mx, t) = \tfrac{1}{2}$$

By the assumption

$$t > M||x||_1 \geqslant ||kT'||M'||x||_1 \geqslant ||T(x)||_1,$$

thus

$$\nu(T(x), t) \leqslant \tfrac{1}{2}$$

Hence

$$\nu(T(x), t) \leqslant \nu(Mx, t)$$

For the third condition :

$$t \leqslant M||x||_1.$$

Then

$$\nu(Mx, t) = 1.$$

By the assumption trivially we have,

$$\nu(T(x), t) \leqslant \nu(Mx, t).$$

Hence $(X, \mu, \nu, *, \circ)$ has the AP.

4. Conclusion

In this paper we introduced and investigated the concepts of AP and BAP in the context of an IFNLS. We have shown that there are IFNLSs which fail to have the AP and also there are IFNLSs with AP but not the BAP. The current results give us a better understanding of the analytical structure of an IFNLS.

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Co-Universal Algebraic Extension with Hidden Parameters

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Abstract

In the research of underlying algebraic structures of real world phenomena, we can find some behavior anomalies that depend on external parameters that are not ruled by their axiom systems. These are not visible straightaway and we have to deduce their existence from the effects they cause. To add them in mathematical constructions, we introduce co-universal extensions of algebras and co-algebras based upon the dual construction of the Kleisli category associated to a monad.

To illustrate this topic we introduce two applications. The first one is an artificial example. In the second application we analyze language algebraic structures with a method that states a bridge between language and logic blindly, that is to say, handling statements through their expressions in those languages satisfying some adequate conditions, and disregarding their meanings.

Keywords: Algebraic extensions, hidden parameters, algebraic language structures, co-monad, Kleisli categories, blind logic.
2010 MSC: 18C99, 18C20, 68T50, 03B65.

1. Introduction

When we investigate the mathematical structures of real world phenomena, we can observe some anomalies that depend on parameters that are not ruled by those axioms that define their algebraic structures. For instance, the states of a Turing machine, contexts when we interpret sentences in any language, environments, positions, etc. Recall that only tape symbols are the visible part of Turing machines. By contrast, moves and states are not displayed in their tapes. They work in the background as hidden parameters, but we can deduce their existence from the behavior changes they cause.

In positional notations, the meaning of each word or symbol depends on their position. For instance, consider the following sentences: 1) "*Programmers know how to write code fast;*" and 2) "*Programmers know how to write fast code.*" Both consist of the same words, but their meanings

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are order-dependent. We can consider both orders as hidden parameters, and the meanings of the former sentences depend on them. Accordingly, to define a map μ sending each sentence in the English language E into its meaning in M, we have to add a parameter set $\mathcal{H}$ the members of which are associated to orders, contexts and styles. Thus, the domain of μ must be the Cartesian product $E \times \mathcal{H}$; where E denotes the set of all English sentences.

We can also find hidden parameters in psychology, physics, and random phenomena. For instance, the probability of remembering a name increases with the occurrence frequency, or when some noticeable fact is associated to it. Thus, frequency and remarkable facts can work as hidden parameters that can modify probabilities. In section 4, we analyze an artificial example of this kind.

To learn and interpret any language, we have to handle abstractions and inferences between the definitions of sentence meanings (Tudor-Răzvan & Manolescu, 2011). The topic goes as follows. If two words, say W_1 and W_2, have the same meaning, when we swap them in any sentence, we obtain an equivalent one. We introduce language structure conditions to build the inverse method. Thus, we can find logical relations and abstractions between the meanings of W_1 and W_2 when we observe that some set of proper sentences T_1 becomes T_2 when we swap W_1 and W_2 and each member of T_2 is a proper sentence too. To know that T_2 consists of right sentences, we need not know their meanings. It is sufficient to find them in any scholar paper. The method works as a blind logic and can give rise to many ambiguities, that we can avoid deducing the existence of hidden parameters. This topic is an enlargement of what Newell stated in (Newell & Simon, 1976). We do not dive in this topic deeply, because we only expose these ideas to illustrate applications of co-universal algebraic extensions that we introduce.

The main aim of this article consists of introducing an algebraic device to enrich categories with sets of external (hidden) parameters that are not ruled by the axioms defining them. We term these constructions co-universal because are based upon co-monads together with the associated dual constructions of Kleisli categories. Well-known universal extensions of **Set**, associated to monads, are categories of sets with fuzzy subsets (Mawanda, 1988). These extensions of **Set** arise from an endofunctor that sends each ordinary set X into $X \times M$, where M is a monoid of truth-values. We introduce co-universal extensions by a similar endofunctor $X \mapsto X \times M$ such that M is the set of hidden parameters.

2. Preliminaries

To simplify expressions, we state some auxiliary definitions and notations. We write in bold face font those symbols denoting categories. In particular, **Set** denotes the category of ordinary sets and maps. We use the symbol ◃ as an end-of-definition marker.

Notation. For each couple of sets X and Y, we denote by $X^Y_{\geq n}$ the subset of X^Y defined as follows.

$$X^Y_{\geq n} = \{f \in X^Y \mid \#(\mathrm{img}(f)) \geq n\}.$$

For instance, $X^Y_{\geq 2}$ consists of every non-constant map in X^Y.

For each subcategory $\mathbf{C}$ of **Set** and every non-empty set $\mathcal{H}$, we denote the members of the set $(\mathrm{Hom}_{\mathbf{C}}(X, Y))^{\mathcal{H}}$ by symbols with the accent ˇ to indicate that are maps from an arbitrary set $\mathcal{H}$ into a homset. For each member $\breve{f}$ of $(\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}$ we write the values of the independent variable $\mathcal{H}$ as subscripts. Thus, the expression $\breve{f}_\alpha \in \mathrm{Hom}_{\mathbf{C}}(X,Y)$ denotes the image of $\alpha \in \mathcal{H}$ under $\breve{f}$.

Definition 2.1. Let $\mathbf{C}$ be a subcategory of **Set**. For every set $\mathcal{H}$ with cardinality greater than 1, we term $\mathcal{H}$-extension of $\mathbf{C}$ the category $\mathbf{C}[\mathcal{H}]$, with the same object class as $\mathbf{C}$, such that, for every couple of sets X and Y,

$$\mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y) = \mathrm{Hom}_{\mathbf{C}}(X,Y) \bigcup \left\{ \coprod_{\alpha\in\mathcal{H}} \{\breve{h}_\alpha\} \middle| \breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2} \right\}. \tag{2.1}$$

Since $\mathbf{C}$ is a subcategory of $\mathbf{C}[\mathcal{H}]$, we only have to define those compositions involving morphisms in $\left\{ \coprod_{\alpha\in\mathcal{H}} \{\breve{h}_\alpha\} \middle| \breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2} \right\}$. We denote this composition by the infix symbol $\diamond$. For every couple of morphisms $f : X \to Y \in \mathrm{Hom}_{\mathbf{C}}(X,Y)$ and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(Y,Z)$ we define their composition as follows.

$$\left(\coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha\} \right) \diamond f = \coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha \circ f\} \tag{2.2}$$

Likewise, the composition of f and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(T,X)$ is

$$f \diamond \left(\coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha\} \right) = \coprod_{\alpha\in\mathcal{H}} \{f \circ \breve{g}_\alpha\} \tag{2.3}$$

Finally, we define the composition of two morphisms $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y)$ and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(Y,Z)$ by

$$\left(\coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha\} \right) \diamond \left(\coprod_{\alpha\in\mathcal{H}} \{\breve{f}_\alpha\} \right) = \coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha \circ \breve{f}_\alpha\}. \tag{2.4}$$

Since $\mathbf{C}$ is a subcategory of $\mathbf{C}[\mathcal{H}]$ with the same object class, identities are the same in both categories. ◁

Theorem 2.1. *Let* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *be two subcategories of* **Set**. *For every set* $\mathcal{H}$ *with cardinality greater than* 1*, and each functor* $T : \mathbf{C}_1 \to \mathbf{C}_2$*, the following statements hold.*

1) *There is an extension* $T^* : \mathbf{C}_1[\mathcal{H}] \to \mathbf{C}_2[\mathcal{H}]$ *of* T *with the same object-map.*

2) *If* $X_1 \xrightarrow{\sigma} T(X_2)$ *is a* T*-universal arrow, then* $X_1 \xrightarrow{\sigma} T^*(X_2)$ *is a* T^**-universal one.*

3) *If for every* $\alpha \in \mathcal{H}$*,* $X_1 \xrightarrow{\breve{\sigma}_\alpha} T(X_2)$ *is a* T*-universal arrow, then*

$$X_1 \xrightarrow{\coprod_{\alpha\in\mathcal{H}}\{\breve{\sigma}_\alpha\}} T^*(X_2)$$

is a T^**-universal one.*

Proof.

1) We define the extension T^* of T in the following terms. The object-maps of both T and T^* are the same. Recall that, by definition, $\mathrm{Obj}(\mathbf{C}_1) = \mathrm{Obj}(\mathbf{C}_1[\mathcal{H}])$. The images $T(f)$ and $T^*(f)$ of every morphism $f \in \mathrm{Mor}(\mathbf{C}_1)$ are the same. The image of each morphism $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\} \in \mathrm{Mor}(\mathbf{C}_1[\mathcal{H}]) \setminus \mathrm{Mor}(\mathbf{C}_1)$ is given by

$$T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}\Big) = \coprod_{\alpha\in\mathcal{H}}\{T(\breve{f}_\alpha)\} \tag{2.5}$$

The former definition is possible because, by equation (2.1), $\breve{f}_\alpha$ belongs to $\mathrm{Mor}(\mathbf{C}_1)$, for every $\alpha \in \mathcal{H}$.

It remains to be shown that T^* preserves morphism composition and identities. Since the restriction of T^* to $\mathrm{Mor}(\mathbf{C}_1)$ coincides with T, the extension T^* preserves identities and compositions between members of $\mathbf{C}_1$. We only have to show that T^* preserves morphism compositions involving some members of $\mathrm{Mor}(\mathbf{C}_1[\mathcal{H}]) \setminus \mathrm{Mor}(\mathbf{C}_1)$. For compositions like (2.2), taking into account (2.5),

$$\begin{aligned} T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{(\breve{g}_\alpha \circ f)\}\Big) &= \coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha \circ f)\} = \coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha) \circ T(f)\} = \\ &\Big(\coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha)\}\Big) \diamond T(f) = T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\Big) \diamond T^*(f) \end{aligned} \tag{2.6}$$

The proofs for compositions of the form (2.3) and (2.4) go as in the preceding case.

2) We have to show that, for every object Y and every morphism $f : X_1 \to T^*(Y)$ there is a unique $f^* : X_2 \to Y$ such that the following triangle commutes.

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & T^*(X_2) \\ & \underset{f}{\searrow} & \big\downarrow {\scriptstyle T^*(f^*)} \\ & & T^*(Y) \end{array} \tag{2.7}$$

If $f \in \mathrm{Mor}(\mathbf{C}_1)$, by hypothesis, this condition must be satisfied. Now, suppose that $f = \coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}$. Since for every α, the morphism $\breve{f}_\alpha : X \to T^*(Y)$ belongs to $\mathrm{Mor}(\mathbf{C}_1)$, there is a unique $\breve{f}^*_\alpha : X_2 \to T^*(Y) = T(Y)$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\sigma} & T^*(X_2) = T(X_2) \\ & \underset{\breve{f}_\alpha}{\searrow} & \big\downarrow {\scriptstyle T^*(\breve{f}^*)=T(\breve{f}_\alpha)} \\ & & T^*(Y) = T(Y) \end{array} \tag{2.8}$$

By virtue of (2.2) the following triangle is also commutative

$$\begin{array}{ccc} X_1 & \xrightarrow{\sigma} & T^*(X_2) \\ & \underset{\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}}{\searrow} & \big\downarrow {\scriptstyle T^*\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}^*\}\right)} \\ & & T^*(Y)) \end{array} \tag{2.9}$$

3) Let $X_1 \xrightarrow{\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}} T^*(Y)$ be a $\mathbf{C}_1$-morphism. By assumption, for every $\alpha \in \mathcal{H}$, there is a $\mathbf{C}_1$-morphism $\breve{f}^*_\alpha$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\breve{\sigma}_\alpha} & T(X_2) \\ & \searrow^{\breve{f}_\alpha} & \downarrow{\scriptstyle T(\breve{f}_\alpha)} \\ & & T(Y) \end{array} \tag{2.10}$$

hence, the following triangle is also commutative.

$$\begin{array}{ccc} X_1 & \xrightarrow{\breve{\sigma}_\alpha} & T(X_2) \\ & \searrow^{\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}} & \downarrow{\scriptstyle T\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}\right)} \\ & & T(Y) \end{array} \tag{2.11}$$

The uniqueness of $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}^*_\alpha\}$ is a consequence of being unique each $\breve{f}^*_\alpha$ that satisfies the commutativity of (2.10), for every $\alpha \in \mathcal{H}$.

□

3. Co-universal algebraic extensions with hidden parameters

For every subcategory $\mathbf{C}$ of $\mathbf{Set}$, being stable under Cartesian products, and each non-empty set $\mathcal{H}$ in Obj($\mathbf{C}$), we denote by $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$ the functor sending each set $X \in$ Obj($\mathbf{C}$) into $X \times \mathcal{H}$, and every map $f : X \to Y$ into

$$\mathcal{H}^\dagger(f) = f \times \mathrm{id}_\mathcal{H} : X \times \mathcal{H} \to Y \times \mathcal{H}. \tag{3.1}$$

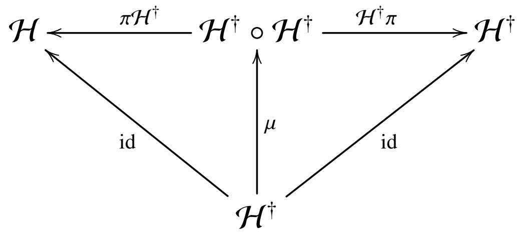
Notation. For every endofunctor $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$, we denote by π the natural transformation $\mathcal{H}^\dagger \xrightarrow{\pi} \mathrm{Id}$ such that, for each set X, the map $\pi_X : X \times \mathcal{H} \to X$ is the canonical projection; where $\mathrm{Id}_\mathbf{C} : \mathbf{C} \to \mathbf{C}$ denotes the identity endofunctor. Likewise, $\mathcal{H}^\dagger \xrightarrow{\mu} \mathcal{H}^\dagger \circ \mathcal{H}^\dagger$ is the natural transformation

$$\mu_X =: X \times \mathcal{H} \to X \times \mathcal{H} \times \mathcal{H} \tag{3.2}$$

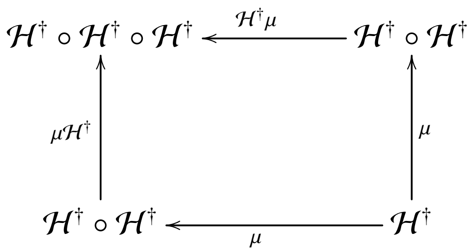
that sends each $(x, v) \in X \times \mathcal{H}$ into $(x, v, v) \in X \times \mathcal{H} \times \mathcal{H}$.

Proposition 3.1. *Let* $\mathbf{C}$ *be a subcategory of* $\mathbf{Set}$ *being stable under Cartesian products. For every nonempty set* $\mathcal{H} \in$ Obj($\mathbf{C}$), *the endofunctor* $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$ *together with both natural transformations* π *and* μ *form a comonad* $(\mathcal{H}^\dagger, \pi, \mu)$.

Proof. We show that the following diagrams commute.



(3.3)



(3.4)

Let X be a set and (x, v) any member of $\mathcal{H}^\dagger(X) = X \times \mathcal{H}$. By straightforward computations we obtain

$$\pi_{X\times\mathcal{H}}(\mu_X(x, v)) = \pi_{X\times\mathcal{H}}(x, v, v) = (x, v);$$

accordingly, $\pi\mathcal{H}^\dagger \circ \mu = \mathrm{id}$. The proofs for the right triangle and quadrangle (3.4) are similar. □

Definition 3.1. Let **C** be a subcategory of **Set** being stable under Cartesian products. For each set $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$ with cardinality greater than 1, a co-universal $\mathcal{H}$-extension of **C** with *hidden parameters* is the category $\mathbf{C}_\mathcal{H}$ defined as follows.

1. The object-classes of both $\mathbf{C}_\mathcal{H}$ and **C** are the same.
2. For each couple of objects X and Y, the set $\mathrm{Hom}_{\mathbf{C}_\mathcal{H}}(X, Y)$ consists of all maps from $\mathcal{H}^\dagger(X) = X \times \mathcal{H}$ into Y such that there is $\breve{f} \in (\mathrm{Hom}_\mathbf{C}(X, Y))^\mathcal{H}$ that satisfies the relation

$$\forall \alpha \in \mathcal{H} : \ f(x, \alpha) = \breve{f}_\alpha(x).$$

3. The composition $f \star g$ of two $\mathbf{C}_\mathcal{H}$-morphisms $g \in \mathrm{Hom}_{\mathbf{C}_\mathcal{H}}(X, Y)$ and $f \in \mathrm{Hom}_{\mathbf{C}_\mathcal{H}}(Y, Z)$ is given by

$$f \star g = f \circ \mathcal{H}^\dagger(g) \circ \mu_X \qquad (3.5)$$

4. The identity associated to each $\mathbf{C}_\mathcal{H}$-object X is the projection

$$\pi_X : \mathcal{H}^\dagger(X) = X \times \mathcal{H} \to X.$$

◁

Notation. As in the preceding definition, for every co-universal $\mathcal{H}$-extension $\mathbf{C}_\mathcal{H}$ of a subcategory **C** of **Set**, we denote the morphism composition by the infix symbol $\star$.

Definition 3.2. Let **C** be a subcategory of **Set** such that there is the co-universal $\mathcal{H}$-extension $\mathbf{C}_\mathcal{H}$. We say a $\mathbf{C}_\mathcal{H}$-morphism $f : X \times \mathcal{H} \to Y$ to be π-factorizable whenever there is $f^* \in \mathrm{Hom}_\mathbf{C}(X, Y)$ that satisfies the equation $f = f^* \circ \pi_X$. ◁

Lemma 3.1. *Let* $\mathbf{C}$ *be a subcategory of* **Set**, *being stable under Cartesian products. For every set* $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$ *and each* $\mathbf{C}_{\mathcal{H}}$*-object* X, *the associated* $\mathbf{C}_{\mathcal{H}}$*-identity* π_X *is* π*-factorizable. In addition,* $\pi_X^* = \mathrm{id}_X$.

Proof. Setting $\pi_X^* = \mathrm{id}_X$, the relation $\pi_X = \mathrm{id}_X \circ \pi_X$ leads to $\pi_X = \pi_X^* \circ \pi_X$. □

Lemma 3.2. *A* $\mathbf{C}_{\mathcal{H}}$*-morphism* $f(x, \alpha) = \breve{f}_\alpha(x)$ *is* π*-factorizable if and only if* $\breve{f} \in \mathrm{Hom}_{\mathbf{C}}(X, Y)^{\mathcal{H}}$ *is constant.*

Proof. Assume $\breve{f}$ to be a constant map, therefore the value of $f(x, \alpha)$ depends only on x. Thus, setting $f^*(x) = \breve{f}_\alpha(x)$, for every $(x, \alpha) \in X \times \mathcal{H}$, the relation $f = f^* \circ \pi_X$ holds. The proof for the converse implication is similar. □

Lemma 3.3. *The composition of* π*-factorizable morphisms is again* π*-factorizable.*

Proof. Let $f \circ \pi_X : X \times \mathcal{H} \to Y$ and $g \circ \pi_Y : Y \times \mathcal{H} \to Z$ be two π-factorizable morphisms. According to (3.5)

$$(g \circ \pi_Y) \star (f \circ \pi_X) = (g \circ \pi_Y) \circ \mathcal{H}^\dagger(f \circ \pi_X) \circ \mu_X = (g \circ \pi_Y) \circ ((f \circ \pi_X) \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X = g \circ f \circ \pi_X \quad (3.6)$$

therefore $(g \circ \pi_Y) \star (f \circ \pi_X) = (g \circ f) \circ \pi_X$ is π-factorizable. □

Definition 3.3. Let $\mathbf{C}$ be any subcategory of **Set**, being stable under Cartesian products. For each set $\mathcal{H}$ with cardinality greater than 1, and each $\alpha \in \mathcal{H}$, we define the map $\Gamma_{\alpha,A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C})$ as follows. For every couple $\mathbf{C}_{\mathcal{H}}$-objects X and Y, and each $f \in \mathrm{Hom}_{\mathbf{C}_{\mathcal{H}}}(X, Y)$:

$$\Gamma_{\alpha,A}(f) = \begin{cases} f^* \text{ if } f = f^* \circ \pi_X \text{ is } \pi\text{-factorizable} \\ \breve{f}_\alpha \text{ where } \breve{f} \in (\mathrm{Hom}_{\mathbf{C}}(X, Y))^{\mathcal{H}}_{\geq 2} \text{ otherwise.} \end{cases} \quad (3.7)$$

being $\breve{f}$ the map such that $\forall (x, \alpha) \in X \times \mathcal{H}$: $\breve{f}_\alpha(x) = f(x, \alpha)$. ◁

To agree with Lemma 3.2, in the former definition, when $f = f^* \circ \pi_X$ is π-factorizable, its image $\Gamma_{\alpha,A}(f)$ does not depend on the parameter α.

Proposition 3.2. *Let* $\mathbf{C}_{\mathcal{H}}$ *be a co-universal* $\mathcal{H}$*-extension of a subcategory* $\mathbf{C}$ *of* **Set** *with hidden parameters. For every* $\alpha \in \mathcal{H}$, *the map* $\Gamma_{\alpha,A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C})$ *preserves identities and morphism compositions.*

Proof. We have to show that, for every couple of morphisms $f : X \times \mathcal{H} \to Y$ and $g : Y \times \mathcal{H} \to Z$, and each $\alpha \in \mathcal{H}$, the map $\Gamma_{\alpha,A}$ satisfies the following relation.

$$\Gamma_{\alpha,A}(g \star f) = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f) \quad (3.8)$$

If both $f = f^* \circ \pi_X$ and $g = g^* \circ \pi_Y$ are π-factorizable, then

$$\begin{aligned}\Gamma_{\alpha,A}(g \star f) = \Gamma_{\alpha,A}((g^* \circ \pi_Y) \star (f^* \circ \pi_X)) = \\ \Gamma_{\alpha,A}((g^* \circ \pi_Y) \circ \mathcal{H}^\dagger(f^* \circ \pi_X) \circ \mu_X) = \\ \Gamma_{\alpha,A}((g^* \circ \pi_Y) \circ ((f^* \circ \pi_X) \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X) = \\ \Gamma_{\alpha,A}(g^* \circ f^* \circ \pi_X)) = g^* \circ f^* = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f)\end{aligned} \tag{3.9}$$

Thus, $\Gamma_{\alpha,A}$ preserves the composition of π-factorizable morphisms.

For non-π-factorizable morphisms, the expression $(g \star f)(x, \alpha)$ can be written explicitly as follows.

$$\begin{aligned}\forall (x, \alpha) \in X \times \mathcal{H} : \quad (g \star f)(x, \alpha) = \\ (g \circ \mathcal{H}^\dagger(f) \circ \mu_X)(x, \alpha) = (g \circ (f \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X)(x, \alpha) = \\ (g(f(x, \alpha), \alpha) = \breve{g}_\alpha(\breve{f}_\alpha(x)) = (\breve{g}_\alpha \circ \breve{f}_\alpha)(x);\end{aligned} \tag{3.10}$$

and by definition, $\Gamma_{\alpha,A}(f) = \breve{f}_\alpha$ and $\Gamma_{\alpha,A}(g) = \breve{g}_\alpha$; therefore

$$\Gamma_{\alpha,A}(g \star f) = (g \star f)\breve{}_\alpha = \breve{g}_\alpha \circ \breve{f}_\alpha = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f); \tag{3.11}$$

hence $\Gamma_{\alpha,A}$ also preserves the composition of non-π-factorizable morphisms.

If $g = g^* \circ \pi_Y$ is π-factorizable and f is not, the same procedure yields

$$\forall (x, \alpha) \in X \times \mathcal{H} : \quad (g \star f)(x, \alpha) = (g(f(x, \alpha), \alpha) = g^*(\breve{f}_\alpha(x)); \tag{3.12}$$

and this equation leads to (3.11). The proof when f is π-factorizable and g is not, is similar.

It remains to be shown that $\Gamma_{\alpha,A}$ preserves identities. According to Lemma 3.1 and equation (3.7), $\Gamma_{\alpha,A}(\pi_X) = \mathrm{id}_X$. □

Corollary 3.1. *With the same assumptions as in Proposition 3.2, for every fixed* $\alpha \in \mathcal{H}$, *the identity* $\mathrm{Id} : \mathrm{Obj}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Obj}(\mathbf{C})$ *and the map* $\Gamma_{\alpha,A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C})$ *form a functor* $\Gamma_\alpha = (\mathrm{Id}, \Gamma_{\alpha,A})$.

Proof. By definition, the object classes of $\mathbf{C}_{\mathcal{H}}$ and $\mathbf{C}$ are the same; hence the identity can be the object map of Γ_α. By Proposition 3.2 the map Γ_α, A preserves identities and morphism composition. □

Notation. For every subcategory $\mathbf{C}$ of **Set** being stable under Cartesian products, and each set $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$ with cardinality greater than 1, the expression

$$F_{\mathcal{H},A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C}[\mathcal{H}])$$

denotes the map such that, for each pair X and Y in $\mathrm{Obj}(\mathbf{C}_{\mathcal{H}})$ and every $f \in \mathrm{Hom}_{\mathbf{C}_{\mathcal{H}}}(X, Y)$:

$$F_{\mathcal{H},A}(f) = \begin{cases} \Gamma_\alpha(f) \text{ if } f \text{ is } \pi\text{-factorizable} \\ \coprod_{\alpha \in \mathcal{H}} \{\Gamma_\alpha(f)\} = \coprod_{\alpha \in \mathcal{H}} \{\breve{f}_\alpha\} \text{ otherwise.} \end{cases} \tag{3.13}$$

where Γ_α is the functor defined in Corollary 3.1.

Theorem 3.1 (Main)**.** *For every subcategory* **C** *of* **Set** *being stable under Cartesian products, and each set* $\mathcal{H} \in \text{Obj}(\mathbf{C})$ *with cardinality greater than* 1*, the following statements hold.*

1) *The identity* $\text{Id} : \text{Obj}(\mathbf{C}_{\mathcal{H}}) \to \text{Obj}(\mathbf{C}[\mathcal{H}])$ *together with the arrow map*

$$F_{\mathcal{H},A} : \text{Mor}(\mathbf{C}_{\mathcal{H}}) \to \text{Mor}(\mathbf{C}[\mathcal{H}])$$

form an isomorphism $F_{\mathcal{H}} = (\text{Id}, F_{\mathcal{H},A})$ *between both categories* $\mathbf{C}_{\mathcal{H}}$ *and* $\mathbf{C}[\mathcal{H}]$.

2) *If* **D** *is a subcategory of* **Set***, being stable under Cartesian products such that* $\mathcal{H}$ *belongs to* $\text{Obj}(\mathbf{D})$*, then every functor* $T : \mathbf{C} \to \mathbf{D}$ *gives rise to another one* $T^{\natural}_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{D}_{\mathcal{H}}$*, having the same object map as* T*, which satisfies the following relation.*

$$\forall f \in \text{Mor}(\mathbf{C}_{\mathcal{H}}) : \quad \Gamma_{\alpha} \circ T^{\natural}_{\mathcal{H}}(f) = T \circ \Gamma_{\alpha}(f) \tag{3.14}$$

3) *With the same conditions as in the preceding statement, if for every* $\alpha \in \mathcal{H}$*, the* **C***-morphism* $X_1 \xrightarrow{\breve{\sigma}_{\alpha}} T(X_2)$ *is a* T*-universal arrow, then* $X_1 \xrightarrow{\sigma} T^{\natural}_{\mathcal{H}}(X_2)$ *is a* $T^{\natural}_{\mathcal{H}}$*-universal one; where* σ *denotes the* $\mathbf{C}_{\mathcal{H}}$*-morphism* $\sigma : X_1 \times \mathcal{H} \to T^{\natural}_{\mathcal{H}}(X_2)$ *such that,* $\forall (x, \alpha) \in X_1 \times \mathcal{H}$*:* $\sigma(x, \alpha) = \breve{\sigma}_{\alpha}(x)$.

4) *With the same assumptions as in Statement 2), every* $T^{\natural}$*-algebra (co-algebra) is the extension with hidden parameters of an ordinary* $T^{\natural}_{\mathcal{H}}$*-algebra (co-algebra).*

Proof.

1) We have to show that $F_{\mathcal{H}}$ is a functor. For every object X, the $\mathbf{C}_{\mathcal{H}}$-identity is $\pi_X : X \times \mathcal{H} \to X$. According to Proposition 3.2, its image under $F_{\mathcal{H}}$ is $\Gamma_{\alpha}(\pi_X) = \text{id}_X$. Thus, $F_{\mathcal{H}}$ preserves identities.

To show that $F_{\mathcal{H}}$ preserves morphism composition, let $f = f^* \circ \pi_X : X \times \mathcal{H} \to Y$ and $g = g^* \circ \pi_Y : Y \times \mathcal{H} \to Z$ be two π-factorizable morphisms. By equation (3.13) and taking into account Lemma 3.3,

$$\begin{aligned} F_{\mathcal{H}}(g \star f) = \Gamma_{\alpha}(g \star f) = \\ \Gamma_{\alpha}(g^* \circ \pi_Y \circ \mathcal{H}^{\dagger}(f^* \circ \pi_X) \circ \mu_X) = \Gamma_{\alpha}(g^* \circ f^* \circ \pi_X) = \\ g^* \circ f^* = \Gamma_{\alpha}(g) \circ \Gamma_{\alpha}(f) = \Gamma_{\alpha}(g) \diamond \Gamma_{\alpha}(f); \end{aligned} \tag{3.15}$$

therefore

$$F_{\mathcal{H}}(g \star f) = \Gamma_{\alpha}(g \star f) = \Gamma_{\alpha}(g) \diamond \Gamma_{\alpha}(f) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.16}$$

If f and g are two non-π-factorizable morphisms, by definition,

$$\begin{aligned} (g \star f)(x, \alpha) = (g \circ \mathcal{H}^{\dagger}(f) \circ \mu_X)(x, \alpha) = \\ (g \circ (f \times \text{id}_{\mathcal{H}}) \circ \mu_X)(x, \alpha) = g(f(x, \alpha), \alpha) \end{aligned} \tag{3.17}$$

Let $\breve{f} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}$ and $\breve{g} \in (\mathrm{Hom}_{\mathbf{C}}(Y,Z))^{\mathcal{H}}$ be the maps such that

$$\begin{cases} \forall (x,\alpha) \in X \times \mathcal{H}: & \breve{f}_\alpha(x) = f(x,\alpha) \\ \forall (y,\alpha) \in Y \times \mathcal{H}: & \breve{g}_\alpha(x) = g(y,\alpha). \end{cases} \tag{3.18}$$

These relations together with (3.17) lead to

$$\forall x \in X : (g \star f)(x) = g((f(x,\alpha),\alpha) = \breve{g}_\alpha(\breve{f}_\alpha(x)) = (\breve{g}_\alpha \circ \breve{f}_\alpha)(x), \tag{3.19}$$

for every fixed $\alpha \in \mathcal{H}$. Consequently, by virtue of (2.4) and (3.13),

$$F_{\mathcal{H}}(g \star f) = \coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha \circ \breve{f}_\alpha\} = \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha\}\Big) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{f}_\alpha\}\Big) = \\ \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(g)\}\Big) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(f)\}\Big) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.20}$$

If $g = g^* \circ \pi_Y$ is π-factorizable and f is not, the same procedure yields,

$$F_{\mathcal{H}}(g \star f) = \coprod_{\alpha\in\mathcal{H}} \{g^* \circ \breve{f}_\alpha\} = g^* \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{f}_\alpha\}\Big) = \\ \Gamma_\alpha(g) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(f)\}\Big) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.21}$$

By the same method, we can build the proof when f is π-factorizable and g is not.

Since $F_{\mathcal{H}}$ preserves identities and morphism composition, it is a functor.

To be an isomorphism, $F_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{C}[\mathcal{H}]$ must be full, faithful, and bijective on objects. By definition, the object-classes of $\mathbf{C}$, $\mathbf{C}_{\mathcal{H}}$, and $\mathbf{C}[\mathcal{H}]$ are the same. Because the object map Id of $F_{\mathcal{H}}$ is the identity, $F_{\mathcal{H}}$ is bijective on objects.

It remains to be shown that $F_{\mathcal{H}}$ is full and faithful. The class $\mathrm{Mor}(\mathbf{C}[\mathcal{H}])$ consists of the ordinary maps in $\mathrm{Mor}(\mathbf{C})$ together with the coproduct class

$$\mathrm{Cprd}(\mathbf{C},\mathcal{H}) = \Big\{ \coprod_{\alpha\in\mathcal{H}} \{\breve{h}_\alpha\} \Big| \breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2}) \wedge (X,Y) \in \mathrm{Obj}(\mathbf{C}) \times \mathrm{Obj}(\mathbf{C}) \Big\}$$

For every map $f : X \to Y$ in $\mathrm{Mor}(\mathbf{C})$ there is the preimage $F^{-1}_{\mathcal{H}}(f) = f \circ \pi_X$, because $F_{\mathcal{H}}(f \circ \pi_X) = \Gamma_\alpha(f \circ \pi_x) = f$. Likewise, for each $\mathbf{C}[\mathcal{H}]$-morphism $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}$ lying in $\mathrm{Cprd}(\mathbf{C},\mathcal{H}) \subseteq \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y)$, the preimage is the morphism $f : X \times \mathcal{H} \to Y$ that satisfies the relation $f(x,\alpha) = \breve{f}_\alpha(x)$, for each fixed $\alpha \in \mathcal{H}$ and every $x \in X$; hence $F_{\mathcal{H}}$ is full.

To see that $F_{\mathcal{H}}$ is faithful, we split the class $\mathrm{Mor}(\mathbf{C}_{\mathcal{H}})$ into the subclass $\mathbf{C}_{\mathcal{H},\pi}$ of π-factorizable morphisms and its complement $\complement_{\mathrm{Mor}(\mathbf{C}_{\mathcal{H}})}\mathbf{C}_{\mathcal{H},\pi}$. If the images of two π-factorizable morphisms $f : X \times \mathcal{H} \to Y$ and $f : X \times \mathcal{H} \to Y$ are the same, then $\Gamma_\alpha(f) = \Gamma_\alpha(g)$; so then $f = \Gamma_\alpha(f) \circ \pi_X = \Gamma_\alpha(g) \circ \pi_X = g$.

Since the image of every π-factorizable morphisms belongs to **C**, we only have to show that, the restriction of $F_{\mathcal{H}}$ to each homset in $\complement_{\text{Mor}(\mathbf{C}_{\mathcal{H}})}\mathbf{C}_{\mathcal{H},\pi}$ is also injective. Let $f : X \times \mathcal{H} \to Y$ and $g : X \times \mathcal{H} \to Y$ be two morphisms with the same image $\coprod_{\alpha\in\mathcal{H}}\{\breve{h}_\alpha\}$. By definition, for every $(x, \alpha) \in X \times \mathcal{H}$: $f(x, \alpha) = \Gamma_\alpha(f)(x) = \breve{h}_\alpha(x) = \Gamma_\alpha(g)(x) = g(x, \alpha)$; therefore $f = g$. Finally, the image under $F_{\mathcal{H}}$ of each π-factorizable morphism f belongs to Mor(**C**), while the image of every non-π-factorizable one g lies in Cprd(**C**, $\mathcal{H}$). Since both sets are disjoint, $F_{\mathcal{H}}(f) \neq F_{\mathcal{H}}(g)$.

2) According to the preceding statement, there is the isomorphism $F_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{C}[\mathcal{H}]$; hence we can define $T^\natural_{\mathcal{H}}$ by

$$T^\natural_{\mathcal{H}} = F^{-1}_{\mathcal{H}} \circ T^* \circ F_{\mathcal{H}} \tag{3.22}$$

where $T^* : \mathbf{C}[\mathcal{H}] \to \mathbf{D}[\mathcal{H}]$ is the extension of T defined in Theorem 2.1. Taking into account (3.13), every π-factorizable morphism $f \in \text{Mor}(\mathbf{C}_{\mathcal{H}})$ satisfies the equation,

$$\Gamma_\alpha \circ T^\natural_{\mathcal{H}}(f) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^* \circ F_{\mathcal{H}}(f) = T^* \circ \Gamma_\alpha(f) \tag{3.23}$$

because $\Gamma_\alpha = F_{\mathcal{H}}$. Since f is π-factorizable, $\Gamma_\alpha(f) \in \text{Mor}(\mathbf{C})$, hence $T^* \circ \Gamma_\alpha(f) = T \circ \Gamma_\alpha(f)$. Thus, the former equation leads to

$$\Gamma_\alpha \circ T^\natural_{\mathcal{H}}(f) = T \circ \Gamma_\alpha(f) \tag{3.24}$$

For each non-π-factorizable morphism $f : X \times \mathcal{H} \to Y$,

$$\Gamma_\alpha \circ T^\natural_{\mathcal{H}}(f) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^* \circ F_{\mathcal{H}}(f) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^*\Big(\coprod_{\beta\in\mathcal{H}}\{\breve{f}_\beta\}\Big) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}}\Bigg(\coprod_{\beta\in\mathcal{H}}\{T(\breve{f}_\beta)\}\Bigg) = \Gamma_\alpha(h) \tag{3.25}$$

where $h : T(X) \times \mathcal{H} \to T(Y)$ is the map defined by

$$\forall (x, \alpha) \in T(X) \times \mathcal{H} : \quad h(x, \alpha) = T(\breve{f}_\alpha)(x).$$

Thus, $\Gamma_\alpha(h) = \breve{h}_\alpha = T(\breve{f}_\alpha) = T(\Gamma_\alpha(f))$. This relation and equation (3.25) lead to equation (3.14).

3) The image of σ under $F_{\mathcal{H}}$ is $\coprod_{\alpha\in\mathcal{H}}\{\sigma_\alpha\}$. Since $T^\natural_{\mathcal{H}} = F^{-1}_{\mathcal{H}} \circ T^* \circ F_{\mathcal{H}}$ and $F_{\mathcal{H}}$ is a category isomorphism, statement **3)** is a consequence of Theorem 2.1.

4) If (X, σ_X) is a $T^\natural_{\mathcal{H}}$-algebra, for every $\alpha \in \mathcal{H}$, its image $\Gamma_\alpha(X, \sigma_X) = (\Gamma_\alpha(X), \Gamma_\alpha(\sigma_X))$ under Γ_α is a T-algebra. By definition, every set $X \in \text{Obj}(\mathbf{C})$ remains unaltered under Γ_α. Accordingly, $(\Gamma_\alpha(X), \Gamma_\alpha(\sigma_X)) = (X, \Gamma_\alpha(\sigma_X))$. In addition, although $\sigma_X : T^\natural_{\mathcal{H}}(X) \times \mathcal{H} \to X$ is a $\mathbf{C}_{\mathcal{H}}$-morphism, its image under Γ_α is an ordinary map. According to statement 2), and taking into account (3.7),

$$\Gamma_\alpha\left(T^\natural_{\mathcal{H}}(X) \xrightarrow{\sigma_X} X\right) = \Gamma_\alpha \circ T^\natural_{\mathcal{H}}(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} \Gamma_\alpha(X) =$$
$$T \circ \Gamma_\alpha(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} \Gamma_\alpha(X) = T(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} X \quad (3.26)$$

therefore, $(X, \Gamma_\alpha(\sigma_X))$ is a T-algebra, where $\Gamma_\alpha(\sigma_X)$ is either the image of α under the map $\breve{\sigma}_X \in (\mathrm{Hom}_{\mathbf{C}}(T(X), X))^{\mathcal{H}}$ whenever σ_X is not π-factorizable, or the map σ^*_X such that $\sigma_X = \sigma^*_X \circ \pi_X$ otherwise. Likewise, if $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is a morphism between two $T^\natural_{\mathcal{H}}$-algebras, the following quadrangle commutes.

$$\begin{array}{ccc} T^\natural_{\mathcal{H}}(X) & \xrightarrow{\sigma_X} & X \\ {\scriptstyle T^\natural_{\mathcal{H}}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ T^\natural_{\mathcal{H}}(Y) & \xrightarrow[\sigma_Y]{} & Y \end{array} \quad (3.27)$$

Consequently, taking into account Statement 2), its image under Γ_α

$$\begin{array}{ccc} T(X) & \xrightarrow{\Gamma_\alpha(\sigma_X)} & X \\ {\scriptstyle T(\Gamma_\alpha(f))}\downarrow & & \downarrow{\scriptstyle \Gamma_\alpha(f)} \\ T(Y) & \xrightarrow[\Gamma_\alpha(\sigma_Y)]{} & Y \end{array} \quad (3.28)$$

is also commutative, and both $(X, \Gamma_\alpha(\sigma_X))$ and $(Y, \Gamma_\alpha(\sigma_Y))$ are ordinary T-algebras. The proof for co-algebras is the dual one.

□

Remark. The main application of the former result consists of considering most T-algebras (co-algebras) as restrictions or particular cases of $T^\natural_{\mathcal{H}}$-algebras (co-algebras) when we observe behavior changes. The members of $\mathcal{H}$ that work as parameters need not be ruled by the axioms of the extended constructs, and remain hidden until we observe either any anomalous event, or some behavior changes. In the following sections we expose two illustrative applications.

4. Bernoulli distribution with hidden parameters.

Probability spaces can be formalized as co-algebras. For instance, let $(\Omega, \mathcal{E}, P)$ be a probability space; where Ω is the set of outcomes, $\mathcal{E}$ the set of events, and $P : \mathcal{E} \to [0, 1]$ the probability assignation. If T : **Set** $\to$ **Set** is the endofunctor sending each set into $[0, 1]$, and every map $f : X \to Y$ into the identity id : $[0, 1] \to [0, 1]$, then $P : \mathcal{E} \to T(\mathcal{E}) = [0, 1]$ gives rise to a co-algebra. A map $f : \mathcal{E}_1 \to \mathcal{E}_2$ is a morphism whenever the following quadrangle commutes.

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{P_1} & T(\mathcal{E}_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle T(f)=\mathrm{id}} \\ \mathcal{E}_2 & \xrightarrow[P_2]{} & T(\mathcal{E}_2) \end{array}$$

We can interpret these co-algebras as restrictions of those with hidden parameters, such that the probability assignations P_1 and P_2 depend on some parameter set $\mathcal{H}$. The following paragraphs illustrate these ideas.

Let X be a random variable, with Bernoulli distribution, like tossing a coin n-times. Let (T, P) be the associated co-algebra, where P denotes the probability assignation. Let $S = f_1, f_2, f_3 \dots f_n$ be the observed relative frequency sequence of the event $X = 1$ (success) in some experiment. Suppose that the sequence S converges in probability to $\frac{1}{2}$, and the relative frequencies satisfy the relation $\forall n \in \mathbb{N} : \ f_n \leq \frac{1}{2}$. By the *weak law of large numbers* we know that $p = q = \frac{1}{2}$ and both events (success and failure) are equiprobable. Nevertheless, the relation $\forall n \in \mathbb{N} : \ f_n \leq \frac{1}{2}$ leads to $P(f_n \leq \frac{1}{2}) = 1$. This relation is not a consequence of probability laws. By contrast, it does not satisfy the expected symmetry in equiprobable situations. We can interpret this fact introducing hidden parameters as follows.

We can consider (T, P) as a particular case of an extension $(T^{\natural}_{\mathcal{H}}, \tilde{P})$ with a hidden parameter set $\mathcal{H} = \{\tau, \omega\}$, where the probability assignation is a $\mathbf{Set}_{\mathcal{H}}$-morphism $\tilde{P} : X \times \mathcal{H} \to T(X) = [0, 1]$ defined as follows.

$$\tilde{P}(X, \alpha) = \begin{cases} \frac{1}{2} \text{ if } (X, \alpha) = (0, \tau) \\ \frac{1}{2} \text{ if } (X, \alpha) = (1, \tau) \\ 1 \text{ if } (X, \alpha) = (0, \omega) \\ 0 \text{ if } (X, \alpha) = (1, \omega) \end{cases} \tag{4.1}$$

Now, suppose that

$$\forall n \in \mathbb{N} : \quad \alpha = \begin{cases} \omega \text{ if } n = 1 \\ \tau \text{ if } n > 1 \text{ and } f_{n-1} < \frac{1}{2} \\ \omega \text{ if } n > 1 \text{ and } f_{n-1} = \frac{1}{2} \end{cases} \tag{4.2}$$

With these conditions the relative frequency sequence of the event $X = 1$ converges in probability to $\frac{1}{2}$ and keeps always less than or equal to $\frac{1}{2}$. Notice that the parameter α takes the value ω whenever the event $f_n = \frac{1}{2}$ occurs; otherwise keeps equal to τ.

In the former example, we can see that hidden parameters correspond to "events" or "situations" that can occur in real world phenomena. This example is artificial, but there are natural random phenomena whose probability assignation can be modified by hidden parameters. For instance, the frequency under which a word "w" occurs increases its probability occurrence. However, in smart text, under excessive repetition the probability occurrence of "w" can vanish. Academic style, smartness, and word repetition can be regarded as hidden parameters that modify the occurrence probability of any word.

5. Structured Languages

As in (Palomar Tarancón, 2011), for each nonempty object-class $\mathbf{C}$, we denote by $\mathbf{C}^{\curlyvee}$ the generic object of $\mathbf{C}$. For instance, if $\mathbf{C}$ is the set $\{n \in \mathbb{N} \mid n \equiv 1 \,(\mathrm{mod}\ 2)\}$, then $\mathbf{C}^{\curlyvee}$ denotes the concept of odd positive integer. To avoid any exception, we apply the same operator to singletons or one-member classes. The generic object of any singleton $\{O\}$ coincides with its unique member; hence

$$\{O\}^{\curlyvee} = O. \tag{5.1}$$

Definition 5.1. A predicate $P(X) \in \mathbf{Pr}$ is self-contradictory provided that $\neg P(X)$ is a tautology. ◃

It is straightforward consequence of the preceding definition that if $P(X)$ is a tautology, its negation $\neg P(X)$ is self-contradictory. If $P(X)$ is not self-contradictory there is at least one object O such that $P(O)$ is true; otherwise $\neg P(X)$ would be true for every value of X, hence a tautological predicate.

In this section, **Pr** denotes a predicate class of higher-order logic, being stable under conjunctions, disjunctions and negations. Likewise, **Mc**(**Pr**) denotes an object class satisfying the following axioms.

Axiom 5.1. *If a predicate* $P(X) \in \mathbf{Pr}$ *is neither self-contradictory nor tautological, the class* **Mc**(**Pr**) *contains the generic object* $\{O \mid P(O)\}^{\curlyvee}$.

Axiom 5.2. *For every* $O \in \mathbf{Mc}(\mathbf{Pr})$ *there is* $P(X) \in \mathbf{Pr}$ *such that*

$$\{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\} = \{O\}.$$

Definition 5.2. An attributive definition for a member O of **Mc**(**Pr**) is any predicate $P(X) \in \mathbf{Pr}$ such that $O = \{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\}^{\curlyvee}$. If the class $\{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\}$ is a singleton, we say $P(X)$ to be a strictly attributive definition of O. ◃

Remark. In natural languages, most words denote generic objects of equivalence classes. For instance, the word "polygon" denotes a class that contains "triangles" and "quadrangles among others. Each of these words again denotes some object class. Attributive definitions consist of an attribute or property that is stated by a predicate $P(X)$. The defined object O is the generic one of the class that satisfies $P(X)$. Thus, if O_1 is a concretion of O obtained by adding another property $Q(X)$, that is, if O_1 is the generic object of the class $\{R \mid P(R) \wedge Q(R)\}$, then $P(X) \wedge Q(X) \Rightarrow P(X)$.

Lemma 5.1. *Each predicate* $P(X) \in \mathbf{Pr}$ *that is neither tautological nor self-contradictory, gives rise to a strictly attributive definition for some object* $O \in \mathbf{Mc}(\mathbf{Pr})$.

Proof. Let $P^*(Y, P(X))$ denote the predicate

$$\text{" } Y \text{ is the generic object of the class } \quad \mathbf{C} = \{O \in \mathbf{Mc}(\mathbf{Pr}) \mid P(O)\}."$$

The class **C** is nonempty because, by hypothesis, $P(X)$ is not self-contradictory (see Definition 5.1). According to Axiom 5.1 there is the generic object $\mathbf{C}^{\curlyvee}$ in **Mc**(**Pr**), besides, taking into account (5.1),

$$\{O \in \mathbf{Mc}(\mathbf{Pr}) \mid P^*(O, P(X))\}^{\curlyvee} = \{\mathbf{C}^{\curlyvee}\}^{\curlyvee} = \mathbf{C}^{\curlyvee}.$$

Consequently, it is a strictly definition. □

Definition 5.3. The class **Mc**(**Pr**) can be enriched with an order relation $\preceq$ such that, between every couple of objects O_1 and O_2, the relation $O_1 \preceq O_2$ holds whenever there are two attributive definitions $P_{O_1}(X)$ and $P_{O_2}(X)$ for O_1 and O_2, respectively, such that $P_{O_1}(X) \Rightarrow P_{O_2}(X)$. ◃

Enriched with the relation $\preceq$, the class **Mc**(**Pr**) satisfies the structure of a category **Mc**(**Pr**, $\preceq$) such that, for every couple of objects O_1 and O_2, the set $\mathrm{Hom}_{\mathbf{Mc}(\mathbf{Pr},\preceq)}(O_1, O_2)$ either is empty or it is the singleton $\{O_1 \preceq O_2\}$. From now on, we assume that the category **Mc**(**Pr**, $\preceq$) satisfies the following axiom.

Axiom 5.3. *The object-class of* **Mc**(**Pr**, $\preceq$) *contains with each subset* $\{O_i \mid i \in I\}$ *its coproduct* $\coprod_{i\in I} O_i$*; where* I *is any nonempty index set.*

Notation. For each phrase W in any meaningful language, we denote by $||W||$ the meaning associated to W.

Remark. Let $P_1(X)$, $P_2(X)$, and $P(X)$ be attributive definitions for O_1, O_2, and $O_1 \coprod O_2$, respectively. According to the definition of $\preceq$, the following relations are true: $P_1(X) \Rightarrow P(X)$ and $P_2(X) \Rightarrow P(X)$. Thus, $P(X)$ is the more restrictive definition that both objects O_1 and O_2 satisfy. In other words, $O_1 \coprod O_2$ is the more concrete abstraction of both objects O_1 and O_2. For instance,

$$||\textit{Large positive integer}|| \coprod ||\textit{small positive integer}|| = ||\textit{positive integer}||.$$

Notation. For every object O in **Mc**(**Pr**) the expression $|O|$ denotes the predicate class $\{P(X) \in \mathbf{Pr} \mid P(O)\}$.

Lemma 5.2. *For every object* $O \in \mathrm{Ob}\,(\mathbf{Mc}(\mathbf{Pr}, \preceq))$ *and each predicate* $Q(X) \in |O|$*, the statement*

$$\forall P(X) \in |O| : \quad Q(X) \Rightarrow P(X) \tag{5.2}$$

is true if and only if $Q(X)$ *is a strictly attributive definition for* O*.*

Proof. First assume $Q(X)$ to be a strictly attributive definition for O, and let $P(X)$ be a member of $|O|$. Suppose that (5.2) is false; hence there is O_1 such that the conjunction $Q(O_1) \wedge (\neg P(O_1))$ is true. Since $Q(X)$ is a strictly attributive definition for O, this relation leads to $O = O_1$ because, by Definition 5.2, the set $\{X \mid Q(X)\}$ must be a singleton. Consequently, these relations lead to $\neg P(O)$, which contradicts the initial assumption $P(X) \in |O|$.

Now suppose that (5.2) holds, and let $Q_1(X)$ be a strictly attributive definition for O. As we have just seen, $Q_1(X) \Rightarrow Q(X)$. Since O must satisfy its own definition $Q_1(X) \in |O|$. As a consequence of (5.2) this membership relation leads to $Q(X) \Rightarrow Q_1(X)$; consequently $Q_1(X) \Leftrightarrow Q(X)$, and $Q(X)$ is also an attributive definition for O. □

Definition 5.4. From now on, we term structured language on a category **Mc**(**Pr**, $\prec$) each 4–tuple $\mathfrak{L} = (A, A^*, A^{**}, M)$ such that,

1. The set A is a finite collection of symbols (alphabet).
2. The set A^* is a partial (syntactic) free-monoid generated by A. We term "word" each member of A^*.
3. The set A^{**} is a partial free-monoid generated by A^*. We say each member of A^{**} to be a phrase.

4. The symbol M denotes a nonempty subset of A^{**} each of its members has a meaning lying in $\mathbf{Mc}(\mathbf{Pr})$. The set A^* contains words denoting the concepts of conjunction, disjunction, and negation. In addition, M is stable under conjunctions, disjunctions and negations.
5. The set M contains with each subset $\{W_i \mid i \in I\}$ a phrase the meaning of which is the coproduct $\coprod_{i \in I} \|W_i\|$, (see Axiom 5.3). ◁

The members of M can be also single words because each meaningful word can be regarded as a one-word phrase. As usual, we term sentence each meaningful phrase. Likewise, statements are truth-valued sentences.

Notation. For each structured language $\mathfrak{L} = (A, A^*, A^{**}, M)$, we denote by $\perp^{\curlyvee}$ a variable ranging over all phrases in A^{**}. This notation allows us to write patterns obtained from any phrase. For instance, consider a phrase $W = w_1 w_2 \dots w_i w_{i+1} \dots w_{i+j} \dots w_n$, where the w_i are the involved words. Substituting the sub-phrase $V = w_i w_{i+1} \dots w_{i+j}$ by $\perp^{\curlyvee}$, we obtain the pattern $W_V(\perp^{\curlyvee})$ that sends each phrase $U = u_1, u_2 \dots u_k \in A^{**}$ into

$$W_V(U) = w_1 w_2 \dots u_1 u_2 \dots u_k w_{j+1} \dots w_n.$$

For instance, let W be the phrase

We can evaluate the area of every polygon.

If we substitute the one-word phrase "*polygon*" by $\perp^{\curlyvee}$, we obtain the pattern

$$W_V(\perp^{\curlyvee}) = \textit{We can evaluate the area of every } \perp^{\curlyvee}.$$

The subscript V in the expression W_V denotes V to be the sub-phrase that we substitute by the variable $\perp^{\curlyvee}$. If U = "*regular triangle,*" then

$$W_V(\textit{regular triangle}) = \textit{We can evaluate the area of every regular triangle.}$$

Definition 5.5. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. A pattern $W_V(\perp^{\curlyvee})$ is continuous provided that for every couple U_1 and U_2 of phrases in M the following conditions hold.

1. If both relations $W_V(U_1) \in M$ and $\|U_2\| \preceq \|U_1\|$ are true, then $W_V(U_2) \in M$.
2. Let $\mathbf{D} = \{U_i \mid i \in I\} \subseteq M$ be a subset with cardinality greater than 1. If a phrase $R \in M$ denotes the object $\coprod_{i \in I} \|U_i\|$, and for every $i \in I$: $W_V(U_i) \in M$, then $W_V(R) \in M$. ◁

Example 5.1. Let $W_V(\perp^{\curlyvee})$ be the English pattern "*The area of every* $\perp^{\curlyvee}$ *is finite.*" Let U_1 denote the word "*triangle*" and U_2 the phrase "*regular triangle.*" If M denotes the class of meaningful English sentences, then the phrase $W_V(U_1)$= *"The area of every triangle is finite"* belongs to M. Likewise, the relation $\|U_2\| \preceq \|U_1\|$ holds because if $\|U_2\|$ is a regular triangle, it is also a triangle. Indeed, $W_V(U_2) \in M$. Finally, $\|U_1\| \coprod \|U_2\| = \|U_1\|$, and by assumption, $W_V(U_1) \in M$.

Since the conjunction of a set of phrases is again a phrase, it is a straightforward consequence that the conjunction of a set of patterns is again a pattern. By definition, there is some symbol or word in each structured language that denotes conjunction. From now on, we denote by the symbol $\mathring{\wedge}$ the conjunction in any structured language. Thus, if the considered language is the English one, $\mathring{\wedge}$ stands for the word "and".

Proposition 5.1. *The conjunction of a set of continuous patterns is again continuous.*

Proof. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. Let $\mathbf{P} = \{W_{V_i}(\perp^{\curlyvee}) \mid i \in I\}$ a set of patterns in $\mathfrak{L}$ and $\mathbf{P}(\perp^{\curlyvee}) = \mathring{\bigwedge}_{i\in I} W_{V_i}(\perp^{\curlyvee})$ the conjunction of all members of $\mathbf{P}$. Let $U_0 \in M$ and $U_1 \in M$ be two phrases such that $||U_1|| \preceq ||U_0||$ and

$$\forall i \in I : \quad W_{V_i}(U_0) \in M \tag{5.3}$$

By continuity,

$$\forall i \in I : \quad W_{V_i}(U_1) \in M \tag{5.4}$$

hence, taking into account Definition 5.4, $\mathbf{P}(U_0) \in M$ and $\mathbf{P}(U_1) \in M$. □

Theorem 5.1. *For every continuous pattern $W_V(\perp^{\curlyvee})$ the following statements are true.*

1. *There is a $\preceq$-maximum element in the class*

$$\mathbf{W} = \{||U|| \mid W_V(U) \in M)\}.$$

2. *If $||U||$ is the $\preceq$-maximum element of* $\mathbf{W}$*, the predicate*

$$P(X) = X \textit{ is the maximum element of } \mathbf{W}$$

is a strictly attributive definition of $||U||$, whenever $P(X) \in$ **Pr**.

Proof.

1. If every element in a chain $||U_1|| \preceq ||U_2|| \preceq \cdots \preceq ||U_n||$ lies in $\mathbf{W}$, by Definition 5.5, so does its upper bound $\coprod_{0<i\leq n} ||U_i||$. Thus, $\mathbf{W}$ satisfies the conditions of Zorn's Lemma. Accordingly, there is, at least, one $\preceq$-maximal element $||U_1||$ in $\mathbf{W}$.
   To see that $||U_1||$ is the maximum element of $\mathbf{W}$, let $||U|| \in \mathbf{W}$ be any member. By virtue of both Definition 5.4 and Definition 5.5, there is a phrase R in M such that $||R|| = ||U|| \coprod ||U_1||$; hence there are the **Mc**(**Pr**)-morphisms $||U_1|| \preceq ||R||$ and $||U|| \preceq ||R||$. Since $||U_1||$ is maximal these relations lead to $||R|| = ||U_1||$ and $||U|| \preceq ||R|| = ||U_1||$. Accordingly, $||U_1||$ is comparable with every member of $\mathbf{W}$.
2. It is a straightforward consequence of the maximum-element uniqueness.

□

Definition 5.6. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. A pattern class $\mathbf{Pt} = \{W_{i,V_i}(\perp^{\curlyvee}) \mid i \in I\}$ is compatible provided that there is at least one phrase U in M such that, for every $i \in I$: $W_{i,V_i}(U) \in M$. ◁

Recall that, by virtue of statement 4) in Definition 5.4, the conjunction of all phrases in **Pt** again belongs to M.

Notation. By $\in^{\partial}$ we denote the "sub-phrase/phrase" relationship. For instance, if

$$W = w_1 w_2 \ldots w_i w_{i+1} \ldots w_{i+j} \ldots w_n$$

is a phrase, the following expression denotes the word sequence $w_i w_{i+1} \ldots w_{i+j}$ to be a sub-phrase.

$$w_i w_{i+1} \ldots w_{i+j} \in^{\partial} w_1 w_2 \ldots w_i w_{i+1} \ldots w_{i+j} \ldots w_n.$$

From now on, for each phrase set A^{**} and every $V \in A^{**}$, the expression A^{**}_V denotes the subset $A^{**}_V = \{W \in A^{**} \mid V \in^{\partial} W\}$. Likewise, $[A^{**}, M]$ denotes the phrase-set collection

$$[A^{**}, M] = \bigcup_{V \in M} \{X \subseteq A^{**}_V \mid V \in X\} \tag{5.5}$$

Finally, for every couple of phrases V_1 and V_2, the expression $\langle V_1 \leftrightarrows V_2 \rangle : M \to M$ denotes the result of substituting each occurrence of the sub-phrase V_1 in W by one of V_2. If W does not contain any occurrence of V_1, then $\langle V_1 \leftrightarrows V_2 \rangle W = W$. Likewise, the infix operator $\leftrightarrows$ can be used to obtain patterns; for instance $\langle V_1 \leftrightarrows \perp^{\curlyvee} \rangle W = W_{V_1}(\perp^{\curlyvee})$.

Notation. From now on, for each $V \in M$ and every $X \subseteq A^{**}_V$, the expression $\mathrm{Pat}(X)$ denotes the pattern class defined as follows.

$$\mathrm{Pat}(V, X) = \{\langle V \leftrightarrows \perp^{\curlyvee} \rangle W \mid W \in X\}$$

Proposition 5.2. *If $\mathfrak{L} = (A, A^*, A^{**}, M)$ is a structured language, for every $V \in M$, each subset E of A^{**}_V satisfies the following statements.*

1. *The pattern class $\mathrm{Pat}(V, E) = \{\langle V \leftrightarrows \perp^{\curlyvee} \rangle W \mid W \in E\}$ is compatible.*
2. *Let E_0 a nonempty subset of E. Let $\mathbf{U}_V(\perp^{\curlyvee})$ and $\mathbf{V}_V(\perp^{\curlyvee})$ be the conjunctions of the pattern classes $\mathrm{Pat}(V, E)$ and $\mathrm{Pat}(V, E_0)$, respectively. If both patterns $\mathbf{U}_V(\perp^{\curlyvee})$ and $\mathbf{V}_V(\perp^{\curlyvee})$ are continuous, the maximum elements $\|U\|$ and $\|U_0\|$ of the classes $\mathbf{W} = \{\|X\| \mid \mathbf{U}_V(X) \in M\}$ and $\mathbf{W}_0 = \{\|X\| \mid \mathbf{V}_V(X) \in M\}$ respectively, satisfy the relation $\|U\| \preceq \|U_0\|$.*

Proof. 1. By definition, for each $W \in E$: $W_V(V) = W$; hence

$$\forall W_V \in \mathrm{Pat}(V, E) : \; W_V(V) \in M.$$

2. Since $\mathrm{Pat}(V, E_0)$ is a subset of $\mathrm{Pat}(V, E)$, for each phrase P the relation $\mathbf{U}_V(P) \in M$ leads to $\mathbf{V}_V(P) \in M$; therefore $\|U\|$ belongs to $\mathbf{W}_0$. By assumption, $\|U_0\|$ is the maximum element of the class $\mathbf{W}_0$, then $\|U\| \preceq \|U_0\|$.

□

Lemma 5.3. *For every $E \in [A^{**}, M]$, there is a unique $V \in M$ such that $E \subseteq A^{**}_V$ and $V \in E$.*

Proof. It is a straightforward consequence of (5.5). □

Definition 5.7. For each structured language $\mathfrak{L} = (A, A^*, A^{**}, M)$, the expression $\mathbf{Ph}(\mathfrak{L})$ denotes the small category the object class of which is

$$\mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L})) = [A^{**}, M]$$

For every pair of objects E_1 and E_2, the homset $\mathrm{Hom}_{\mathbf{Ph}(\mathfrak{L})}(E_1, E_2)$ consists of each map $f : E_1 \to E_2$ that satisfies the following condition.

$$\forall W \in E_1 : \quad \langle V_1 \leftrightarrows V_2 \rangle W = f(W) \tag{5.6}$$

where V_1 and V_2 are members of M such that $E_1 \subseteq A^{**}_{V_1}$ and $E_2 \subseteq A^{**}_{V_2}$ ◁

Recall that, by virtue of Lemma 5.3, for every $\mathbf{Ph}(\mathfrak{L})$-object E, there is $V \in M$ such that $E \subseteq A^{**}_V$ and $V \in E$.

The map $\mathfrak{T}_O : \mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L})) \to \mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L}))$ sending each set $E \in A^{**}_V$ into the singleton $\mathfrak{T}_O(E) = \{V\}$ is the object-map for an endofunctor $\mathfrak{T} : \mathbf{Ph}(\mathfrak{L}) \to \mathbf{Ph}(\mathfrak{L})$ that sends each morphism $f \in \mathrm{Hom}(E_1, E_2)$ into the map $\mathfrak{T}(f) : \{V_1\} \to \{V_2\}$ such that $V_1 \mapsto V_2$. Indeed, this map definition satisfies the condition $\langle V_1 \leftrightarrows V_2 \rangle V_1 = V_2$. We denote this endofunctor by $\mathfrak{T}$.

Proposition 5.3. *Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ a structured language. Let V_1 and V_2 two members of M. If two $\mathfrak{T}$-algebras (E_1, σ_1) and (E_2, σ_2) satisfy the following hypotheses*

1. *There is a morphism $f : (E_1, \sigma_1) \to (E_2, \sigma_2)$.*
2. *The sets E_1 and E_2 are subsets of $A^{**}_{V_1}$ and $A^{**}_{V_2}$, respectively. In addition, all members of both pattern classes*
$$\mathrm{Pat}(V_1, \sigma_1(V_1)) = \{\langle V_1 \leftrightarrows \perp^{\curlyvee} \rangle W \mid W \in \sigma_1(V_1)\}$$
and
$$\mathrm{Pat}(V_2, \sigma_2(V_2)) = \{\langle V_2 \leftrightarrows \perp^{\curlyvee} \rangle W \mid W \in \sigma_2(V_2)\}$$
are continuous.
3. *The objects $\|V_1\|$ and $\|V_2\|$ are the $\preceq$-maximum elements of the object classes $\mathbf{W}_1 = \{\|X\| \mid \mathbf{P}_1(X) \in M\}$ and $\mathbf{W}_2 = \{\|X\| \mid \mathbf{P}_2(X) \in M\}$, respectively; where*
$$\mathbf{P}_1(\perp^{\curlyvee}) = \mathring{\bigwedge}_{W(\perp^{\curlyvee}) \in \mathrm{Pat}\left(V_1, \sigma_1(V_1)\right)} W(\perp^{\curlyvee}),$$
and
$$\mathbf{P}_2(\perp^{\curlyvee}) = \mathring{\bigwedge}_{W(\perp^{\curlyvee}) \in \mathrm{Pat}\left(V_2, \sigma_2(V_2)\right)} W(\perp^{\curlyvee}),$$
respectively.

then the phrases V_1 and V_2 satisfy the relation $\|V_2\| \preceq \|V_1\|$.

Proof. By the definition of $\mathfrak{T}$, and taking into account hypothesis 2), the following relations are true.

$$\begin{cases} \mathfrak{T}(E_1) = \{V_1\} \\ \mathfrak{T}(E_2) = \{V_2\} \\ \sigma_1(V_1) \subseteq E_1 \\ \sigma_2(V_2) \subseteq E_2 \end{cases} \tag{5.7}$$

The existence of the morphism f leads to the relation

$$\forall W \in \sigma_1(V_1): \quad \langle V_1 \leftrightarrows V_2 \rangle W = f(W) \in \sigma_2(V_2) \tag{5.8}$$

therefore $\mathrm{Pat}(V_1, \sigma_1(V_1)) \subseteq \mathrm{Pat}(V_2, \sigma_2(V_2))$. By virtue of Proposition 5.2, this relation leads to $\|V_2\| \preceq \|V_1\|$. □

Remark. By the former proposition we know that $\|U_2\| \preceq \|U_1\|$; accordingly if $P_1(X)$ and $P_2(X)$ are attributive definitions of $\|U_1\|$ and $\|U_2\|$, respectively, the relation $P_2(X) \Rightarrow P_1(X)$ holds (see Definition 5.3). We can deduce this relation, simply, by knowing that substituting V_1 by V_2 in every member of the phrase set $\sigma_1(V_1)$ we obtain a subset of $\sigma_1(V_2)$. This property is a straightforward consequence of the $\mathbf{Ph}(\mathfrak{L})$-morphism definition. Thus, observing occurrences of some sub-phrases in two phrase sets we can find logical implications between attributive definitions of their meanings blindly, that is, without knowing what they mean. Nevertheless, several meanings can be assigned to the same phrase in natural languages or artificial ones, depending on the context, state, style, among other circumstances. Accordingly, contexts, states, styles work as hidden parameters in a set $\mathcal{H}$. Consequently, to apply the method arising from the preceding result, and to interpret sentences in a language properly, we must consider that each $\mathfrak{T}$-algebra (E, σ) is a particular case of a $\mathfrak{T}^{\natural}_{\mathcal{H}}$-algebra; where the members of $\mathcal{H}$ denote states, contexts, styles, frequencies and any other event modifying the meaning of any phrase.

6. Conclusion

Hidden parameters are handled implicitly in Computer Science and Linguistics. We can find noticeable instances almost in each subject. This is a very exciting research field. Theorem 3.1 is the bridge between structured sets, namely, algebras (co-algebras), and any set of hidden parameters that modify their behavior. For instance, the research of those relative frequency anomalies that can be interpreted as the action of hidden parameters is an open problem.

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Modeling Evolution by Evolutionary Machines: A New Perspective on Computational Theory and Practice

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Abstract

The main goal of this paper is the further development of the foundations of evolutionary computations, connecting classical ideas in the theory of algorithms and the contemporary state of art in evolutionary computations. To achieve this goal, we develop a general approach to evolutionary processes in the computational context, building mathematical models of computational systems, called evolutionary machines or automata. We introduce two classes of evolutionary automata: basic evolutionary automata and general evolutionary automata. Relations between computing power of these classes are explored. Additionally, several other classes of evolutionary machines are investigated, such as bounded, periodic and recursively generated evolutionary machines. Different properties of these evolutionary machines are obtained.

Keywords: Turing unorganized machines, evolutionary computation, evolutionary automata, evolutionary Turing machines, evolutionary finite automata, evolutionary inductive Turing machines.
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1. Introduction

The classical theory of algorithms has been developed under the influence of Alan Turing, who was one of the founders of theoretical computer science and whose model of computation, which is now called Turing machine, is the most popular in computer science. He also had many other ideas. In this report the National Physical Laboratory in 1948 (Turing, 1992), Turing proposed a new model of computation, which he called unorganized machines (u-machines). There were two types of u-machines: based on Boolean networks and based on finite state machines.

- A-type and B-type u-machines were Boolean networks made up of a fixed number of two-input NAND gates (neurons) and synchronized by a global clock. While in A-type u-machines the connections between neurons were fixed, B-type u-machines had modifiable

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switch type interconnections. Starting from the initial random configuration and applying a kind of genetic algorithm, B-type u-machines were supposed to learn which of their connections should be on and which off.

- P-type u-machines were tapeless Turing machines reduced to their finite state machine control, with an incomplete transition table, and two input lines for interaction: the pleasure and the pain signals.

Although Turing never formally defined a genetic algorithm or evolutionary computation, in his B-type u-machines, he predicted two areas at the same time: neural networks and evolutionary computation (more precisely, evolutionary artificial neural networks), while his P-type u-machines represent reinforcement learning. However, this work had no impact on these fields (Eberbach *et al.*, 2004), although these ideas are one of the (almost forgotten) roots of evolutionary computation.

Evolutionary computation theory is still very young and incomplete. Until recently, evolutionary computation did not have a theoretical model that represented practice in this domain. Even though there are many results on the theory of evolutionary algorithms (see, e.g., (Michalewicz & Fogel, 2004), (He & Yao, 2004), (Holland, 1975), (Rudolph, 1994), (Wolpert & Macready, 1997), (Koza, 1992, 1994; Koza *et al.*, 1999), (Michalewicz, 1996), very little has been known about expressiveness, or computational power, of evolutionary computation (EC) and its scalability. Of course, there are many results on the theory of evolutionary algorithms (again see, for instance, (Michalewicz & Fogel, 2004), (He & Yao, 2004), (Holland, 1975), (Rudolph, 1994), (Wolpert & Macready, 1997), (Koza, 1992, 1994; Koza *et al.*, 1999), (Michalewicz, 1996)). Studied in EC theoretical topics include convergence in the limit (elitist selection, Michalewiczs contractive mapping GAs, (1+1)-ES), convergence rate (Rechenbergs 1/5 rule), the Building Block analysis (Schema Theorems for GA and GP), best variation operators (No Free Lunch Theorem). However, these authors do not introduce automata models - rather they apply a high-quality mathematical apparatus to existing process models, such as Markov chains, etc. They also cover only some aspects of evolutionary computation like convergence or convergence rate, neglecting for example EC expressiveness, self-adaptation, or scalability. In other words, EC is not treated as a distinct and complete area with its own distinct model situated in the context of general computational models. This means that in spite of intensive usage of mathematical techniques, EC lacks more complete theoretical foundations. As a result, many properties of evolutionary processes could not be precisely studied or even found by researchers. Our research is aimed at filling this gap to define more precisely conditions under which evolutionary algorithms will work and will be superior compared to other optimization methods.

In 2005, the evolutionary Turing machine model was proposed to provide more rigorous foundations for EC (Eberbach, 2005). An evolutionary Turing machine is an extension of the conventional Turing machine, which goes beyond the Turing machine and belongs to the class of super-recursive algorithms (Burgin, 2005). In several papers, the authors studied and extended the ETM (evolutionary Turing machine) model to reflect cooperation and competition (Burgin & Eberbach, 2008), universality (Burgin & Eberbach, 2009*b*), self-evolution (Eberbach & Burgin, 2007), and expressiveness of evolutionary finite automata (Burgin & Eberbach, 2009*a*), (Burgin & Eberbach, 2012).

In this paper, we continue developing a general approach to evolutionary processes in the computational context constructing mathematical models of the systems, functioning of which is based on evolutionary processes, and study properties of such systems with the emphasis on their generative power. Two classes are introduced in Section 2: basic evolutionary automata and general evolutionary automata. Relations between computing power of these classes are explored in Section 3. Additionally, several other classes of evolutionary machines are investigated, such as bounded, periodic and recursively generated evolutionary machines. Different properties of these evolutionary machines are obtained. Section 4 contains conclusions and some open problems.

2. Modeling Evolution by Evolutionary Machines

Evolutionary algorithms describe artificial intelligence processes based on the theory of natural selection and evolution. Evolutionary computation is directed by evolutionary algorithms. In technical terms, an evolutionary algorithm is a probabilistic beam hill climbing search algorithm directed by the chosen fitness function. It means that the beam (population size) maintains multiple search points, hill climbing implies that only a current search point from the search tree is remembered and used for optimization (going to the top of the hill), and the termination condition very often is set to the optimum of the fitness function.

Let X be the representation space, also called the optimization space, for species (systems) used in the process of optimization and a fitness function $f : X \to \mathbb{R}^+$ is chosen, where $\mathbb{R}^+$ is the set of nonnegative real numbers.

Definition 2.1. *A generic evolutionary algorithm EA* can be represented as the collection $EA = (X, s, v, f, R, X[0], F)$ and described in the form of the functional equation (recurrence relation) R working in a simple iterative loop in discrete time t, defining generations $X[t]$, $(t = 0, 1, 2, 3, ...)$ with $X[t + 1] = s(v(X[t]))$, where

- $X[t] \subseteq X$ is a population under a representation consisting of one or more individuals from the set X (e.g., fixed binary strings for genetic algorithms (GAs), finite state machines for evolutionary programming (EP), parse trees for genetic programming (GP), vectors of reals for evolution strategies (ES)),

- s is a selection operator (e.g., truncation, proportional, tournament),

- v is a variation operator (e.g., variants and compositions of mutation and crossover),

- $X[0]$ is an initial population,

- $F \subseteq X$ is the set of final populations satisfying the termination condition (goal of evolution). The desirable termination condition is the optimum in X of the fitness function $f(x)$, which is extended to the fitness function $f(X[t])$ of the best individual in the population $X[t] \subseteq F$, where $f(x)$ typically takes values in the domain of nonnegative real numbers. In many cases, it is impossible to achieve or verify this optimum. Thus, another stopping criterion is used (e.g., the maximum number of generations, the lack of progress through several generations.).

The above definition is applicable to all typical EAs, including GA, EP, ES, GP. It is possible to use it to describe other emerging subareas like ant colony optimization, or particle swarm optimization. Of course, it is possible to think and implement more complex variants of evolutionary algorithms.

Evolutionary algorithms evolve population of solutions x, but they may be the subject of self-adaptation (like in ES) as well. For sure, evolution in nature is not static, the rate of evolution fluctuates, their variation operators are subject to slow or fast changes, and its goal (if it exists at all) can be a subject of modifications as well.

Formally, an evolutionary algorithm looking for the optimum of the fitness function violates some classical requirements of recursive algorithms. If its termination condition is set to the optimum of the fitness function, it may not terminate after a finite number of steps. To fit it to the conventional algorithmic approach, an artificial (or somebody can call it pragmatic) stop criterion has had to be added (see e.g., (Michalewicz, 1996), (Michalewicz & Fogel, 2004), (Koza, 1992, 1994; Koza *et al.*, 1999)). To remain recursive, i.e., to give some result after a finite number of steps, the evolutionary algorithm has to reach the set F of final populations satisfying the termination condition after a finite number of generations or to halt when no visible progress is observable. Usually this is a too restrictive condition, and naturally, in a general case, evolutionary algorithms form a special class of super-recursive algorithms.

To formalize the concept of an evolutionary algorithm in mathematically rigorous terms, we define a formal algorithmic model of evolutionary computation - an evolutionary automaton also called an evolutionary machine.

Let K be a class of automata working with words in an alphabet E. It means that the representation or optimization space X is the set E^* of all words in an alphabet E.

Definition 2.2. *A basic evolutionary K-machine* (BEM), also called *basic evolutionary K-automaton*, is a (possibly infinite) sequence $E = \{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ from K each working on the population $X[t] \subseteq X (t = 0, 1, 2, 3, ...)$ where:

- the automaton $A[t]$ called a component, or more exactly, a level automaton, of E represents (encodes) a one-level evolutionary algorithm that works with the generation $X[t]$ of the population by applying the *variation operators* v and *selection operator* s;
- the zero generation $X[0]$ is given as input to E and is processed by the automaton $A[0]$, so that either $X[0]$ is the result of the whole computation by E when it satisfies the search condition or $A[0]$ generates/produces the first generation $X[1]$ as its output, which goes to the automaton $A[1]$;
- for all $t = 1, 2, 3, ...$, the generation $X[t + 1]$ is obtained by applying the variation operator v and selection operator s to the generation $X[t]$ and these operations are performed by the automaton $A[t]$, which receives $X[t]$ as its input; the generation $X[t+1]$ either is the result of the whole computation by E when it satisfies the search condition or it goes to the automaton $A[t + 1]$;
- the goal of the BEM E is to build a population Z satisfying the search condition.

The desirable search condition is the optimum of the fitness performance measure $f(x[t])$ of the best individual from the population $X[t]$. There are different modes of the EM functioning and different termination strategies. When the search condition is satisfied, then working in the recursive mode, the EM E halts (t stops to be incremented), otherwise a new input population $X[t+1]$ is generated by $A[t]$. In the inductive mode, it is not necessary to halt to give the result (cf. (Burgin, 2005)). When the search condition is satisfied and E is working in the inductive mode, the EM E stabilizes (the population $X[t]$ stops changing), otherwise a new input population $X[t+1]$ is generated by $A[t]$.

We denote the class of all basic evolutionary machines with level automata from K by BEAK.

Definition 2.3. *A general evolutionary K-machine* (GEM), also called *general evolutionary K-automaton*, is a (possibly infinite) sequence $E = \{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ from K each working on generations $X[i] \subseteq X$ where:

- the automaton $A[t]$ called a component, or more exactly, a level automaton, of E represents (encodes) a one-level evolutionary algorithm that works with generations $X[i]$ of the population by applying the variation operators v and selection operator s;

- the zero generation $X[0] \subseteq X$ is given as input to E and is processed by the automaton $A[0]$, which generates/produces the first generation $X[1]$ as its output, which either is the result of the whole computation by E when it satisfies the search condition or it goes to the automaton $A[1]$;

- for all $t = 1, 2, 3, ...$, the automaton $A[t]$, which receives $X[i]$ as its input either from $A[t+1]$ or from $A[t-1]$, then $A[t]$ applies the variation operator v and selection operator s to the generation $X[t]$, producing the generation $X[t+1]$ as its output, which either is the result of the whole computation by E when it satisfies the search condition or it goes either to $A[t+1]$ or to $A[t-1]$. To perform such a transmission, the automaton A[t] uses one of the two techniques: transmission by the output and transmission by the state. In transmission by the output, the automaton $A[t]$ uses two more symbols u_{up} and u_{dw} in its output alphabet, giving one of these symbols as a part of its output in addition to the regular output $X[t+1]$. If this part of the output is u_{up}, then $A[t]$ sends the output generation $X[t+1]$ to $A[t+1]$. If the additional part of the output is u_{dw}, then $A[t]$ sends the output generation $X[t+1]$ to $A[t-1]$. In transmission by the state, the automaton $A[t]$ uses two more symbols u_{up} and u_{dw} as its final-transmission states. In these states the automaton $A[t]$ stops computing and performs the necessary transmission of the output - to the automaton $A[t+1]$ when the state is u_{up} and to the automaton $A[t-1]$ when the state is u_{dw}.

- the goal of the GEM E is to build a population Z satisfying the search condition.

We denote the class of all general evolutionary K-machines GEAK. As any basic evolutionary K-machine is also a general evolutionary K-machine, we have inclusion of classes $BEAK \subseteq GEAK$.

Let us consider some examples of evolutionary K-machines. An important class of evolutionary machines are evolutionary finite automata (Burgin & Eberbach, 2009*a*), (Burgin & Eberbach, 2012). Here K consists of finite automata.

Definition 2.4. *A basic (general) evolutionary finite automaton* (EFA) is a basic (general) evolutionary machine E in which all automata $A[t]$ are finite automata $G[t]$ each working on the population $X[t]$ in generations $t = 0, 1, 2, 3, ...$

We denote the class of all general evolutionary finite automata by GEFA. It is possible to take as K deterministic finite automata, which form the class DFA, or nondeterministic finite automata, which form the class NFA. This gives us four classes of evolutionary finite automata: BEDFA (GEDFA) of all deterministic basic (general) evolutionary finite automata and BENFA (GENFA) of all nondeterministic basic (general) evolutionary finite automata.

Evolutionary Turing machines (Burgin & Eberbach, 2008), (Eberbach, 2005) are form another important class of evolutionary machines.

Definition 2.5. *A basic (general) evolutionary Turing machine* (ETM) $E = \{T[t]; t = 0, 1, 2, 3, ...\}$ is a basic (general) evolutionary machine E in which all automata $A[t]$ are Turing machines $T[t]$ each working on population $X[t]$ in generations $t = 0, 1, 2, 3, ...$

Turing machines $T[t]$ as components of E perform multiple computations (Burgin, 1983). Variation and selection operators are recursive to allow performing level computation on Turing machines.

Definition 2.6. *A basic (general) evolutionary inductive Turing machine* (EITM) $EI = \{M[t]; t = 0, 1, 2, ...\}$ is a basic (general) evolutionary machine E in which all automata $A[t]$ are inductive Turing machines $M[t]$ (Burgin, 2005) each working on the population $X[t]$ in generations $t = 0, 1, 2, ...$

Simple inductive Turing machines are abstract automata (models of algorithms) closest to Turing machines. The difference between them is that a Turing machine always gives the final result after a finite number of steps and after this it stops or, at least, informs when the result is obtained. Inductive Turing machines also give the final result after a finite number of steps, but in contrast to Turing machines, inductive Turing machines do not always stop the process of computation or inform when the final result is obtained. In some cases, they do this, while in other cases they continue their computation and give the final result. Namely, when the content of the output tape of a simple inductive Turing machine forever stops changing, it is the final result.

Definition 2.7. *A basic (general) evolutionary inductive Turing machine* (EITM) $EI = \{M[t]; t = 0, 1, 2, ...\}$ has *order n* if all inductive Turing machines $M[t]$ have order less than or equal to n and at least, one inductive Turing machine $M[t]$ has order n.

We remind that inductive Turing machines with recursive memory are called inductive Turing machines of the first order (Burgin, 2005). The memory E is called n-inductive if its structure is constructed by an inductive Turing machine of the order n. Inductive Turing machines with n-inductive memory are called inductive Turing machines of the order $n + 1$. We denote the class of all evolutionary inductive Turing machines of the order n by $EITM_n$.

Definition 2.8. *A basic (general) evolutionary limit Turing machine* (ELTM) $EI = \{LTM[t]; t = 0, 1, 2, ...\}$ is a basic (general) evolutionary machine E in which all automata $A[t]$ are limit Turing machines $LTM[t]$ (cf. (Burgin, 2005)) each working on the population $X[t]$ in generations $t = 0, 1, 2, ...$

When the search condition is satisfied, then the ELTM EI stabilizes (the population $X[t]$ stops changing), otherwise a new input population $X[t+1]$ is generated by LT $M[t]$. We denote the class of all evolutionary limit Turing machines of the first order by ELTM.

Basic and general evolutionary K-machines from BEAK and GEAK are called unrestricted because sequences of the level automata $A[t]$ and the mode of the evolutionary machines functioning are arbitrary. For instance, there are unrestricted evolutionary Turing machines when K is equal to T and unrestricted evolutionary finite automata when K is equal to FA. However it is possible to consider only basic (general) evolutionary K-machines from BEAK (GEAK) in which sequences of the level automata have some definite type Q. Such machines are called Q-formed basic (general) evolutionary K-machines and their class is denoted by $BEAK^Q$ for basic machines and $GEAKQ$ for general machines. When the type Q contains all finite sequences, we have bounded basic (general) evolutionary K-machines. Some classes of bounded basic evolutionary K-machines are studied in (Burgin & Eberbach, 2010) for such classes K as finite automata, pushdown automata, Turing machines, or inductive Turing machines, i.e., such classes as bounded basic evolutionary Turing machines or bounded basic evolutionary finite automata. When the type Q contains all periodic sequences, we have periodic basic (general) evolutionary K-machines. Some classes of periodic basic evolutionary K-machines are studied in (Burgin & Eberbach, 2010) for such classes K as finite automata, push down automata, Turing machines, inductive Turing machines and limit Turing machines. Note that while in a general case, evolutionary automata cannot be codified by finite words, periodic evolutionary automata can be codified by finite words.

Another condition on evolutionary machines determines their mode of functioning or computation. Here we consider the following modes of functioning/computation.

1. The finite-state mode: any computation is going by state transition where states belong to a fixed finite set.
2. The bounded mode: the number of generations produced in all computations is bounded by the same number.
3. The terminal or finite mode: the number of generations produced in any computation is finite.
4. The recursive mode: in the process of computation, it is possible to reverse the direction of computation, i.e., it is possible to go from higher levels to lower levels of the automaton, and the result is defined after finite number of steps.
5. The inductive mode: the computation goes in one direction, i.e., without reversions, and if for some t, the generation $X[t]$ stops changing, i.e., $X[t] = X[q]$ for all $q > t$, then $X[t]$ is the result of computation.
6. The inductive mode with recursion: recursion (reversion) is permissible and if for some t, the generation $X[t]$ stops changing, i.e., $X[t] = X[q]$ for all $q > t$, then $X[t]$ is the result of computation.
7. The limit mode: the computation goes in one direction and the result of computation is the limit of the generations $X[t]$.
8. The limit mode with recursion: recursion (reversion) is permissible and the result of computation is the limit of the generations $X[t]$.

These modes are complementary to the three traditional modes of computing automata: computation, acceptation and decision/selection (Burgin, 2010). Existence of different modes of computation shows that the same algorithmic structure of an evolutionary automaton/machine E provides for different types of evolutionary computations. We see that only general evolutionary machines allow recursion. In basic evolutionary machines, the process of evolution (computation) goes strictly in one direction. Thus, general evolutionary machines have more possibilities than basic evolutionary machines and it is interesting to relations between these types of evolutionary machines. This is done in the next section. Note that utilization of recursive steps in evolutionary machines provides means for modeling reversible evolution, as well as evolution that includes periods of decline and regression.

3. Computing and Accepting Power of Evolutionary Machines

As we know from the theory of automata and computation, it is proved that different automata or different classes of automata are equivalent. However there are different kinds of equivalence. Here we consider two of them: functional equivalence and linguistic equivalence.

Definition 3.1. (Burgin, 2010)

a. Two automata A and B are functionally equivalent if given the same input, they give the same output.
b. Two classes of automata A and B are functionally equivalent if for any automaton from A, there is a functionally equivalent automaton from B and vice versa.

For instance, it is proved that deterministic and nondeterministic Turing machines are functionally equivalent (cf., for example, (Hopcroft *et al.*, 2001)). Similar results are true for evolutionary automata.

Theorem 3.1. *(Burgin & Eberbach, 2010) For any basic n-level evolutionary finite automaton E, there is a finite automaton AE functionally equivalent to E.*

Here we study relations between basic and general evolutionary machines, assuming that all these machines work in the terminal mode.

Let $P : X \times U \to N$ be a function such that

$$U = \{u_{1,up}, u_{1,dw}, u_{2,up}, u_{2,dw}, ..., u_{k,up}, u_{k,dw}, ...\},$$

$$P(x, u_{k,up}) = k + 1$$

and

$$P(x, u_{k,dw}) = k - 1$$

for any x from optimization space $X = E^*$.

Definition 3.2. (Burgin, 2010) The P-conjunctive parallel composition $\wedge_P A_i$ of the algorithms/automata A_i $(i = 1, 2, 3, ..., n)$ is an algorithm/automaton D such that the result of application of D to any input u is equal to $A_i(u)$ when $P(u) = i$.

This concept allows us to show in a general case of the terminal mode that basic and general evolutionary machines are equivalent.

Theorem 3.2. *If a class K is closed with respect to P-conjunctive parallel composition, then for any general evolutionary K-machine, there is a functionally equivalent basic evolutionary K-machine.*

Proof. Let us consider an arbitrary general evolutionary K-machine $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. We correspond the evolutionary system $H = \{C[t]; t = 0, 1, 2, 3, ...\}$ to the K-machine E. Each component $C[t]$ in H is a system that consists of the automata $C_0[t], C_1[t], C_2[t], C_3[t], ..., C_t[t]$ such that for all $k = 0, 1, 2, 3, ..., t$, the automaton $C_k[t]$ is a copy of the automaton $A[k]$ and it uses the elements $u_{k,up}$ and $u_{k,dw}$ instead of the elements u_{up} and u_{dw} employed by $A[k]$.

The system H has the same search condition as the evolutionary K-machine E and functions in the following way. The zero generation $X[0] \subseteq X$ is given as input to the automaton $C_0[0]$, which is a copy of the automaton $A[0]$ and is processed by the automaton $C_0[0]$, which generates/produces the first generation $X[1]$ as its output. Then $X[1]$ either is the result of the whole computation by H when it satisfies the search condition or it goes to the automaton $C_1[1]$ as its input. In the general case, for all $t = 1, 2, 3, ...$ and $k = 1, 2, 3, ..., t$, the automaton $C_k[t]$ receives $X[t]$ as its input either from $C_{k+1}[t-1]$ when the automaton $A[k]$ receives its input from $A[k+1]$ or from $C_{k-1}[t-1]$ when the automaton $A[k]$ receives its input from $A[k-1]$. Then $C_k[t]$ applies the variation operator v and selection operator s to the generation $X[t]$ and producing the generation $X[t+1]$. Then either this generation is the result of the whole computation by H when it satisfies the search condition or $C_k[t]$ sends this generation either to $C_{k+1}[t+1]$ when the automaton $A[k]$ sends its output to $A[k+1]$ or to $C_{k-1}[t+1]$ when the automaton $A[k]$ sends its output to $A[k-1]$.

In such a way, the system H simulates functioning of the general evolutionary K-machine $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. Let us prove this by induction on the number of steps that the K-machine E is making.

The base of induction:

Making the first step, the K-machine E receives is the zero generation $X[0] \subseteq X$ as its input, processes it by the first automaton $A[0]$ producing the first generation $X[1]$, which either is the result of the whole computation by E when it satisfies the search condition or it goes to the automaton $A[1]$.

Making the first step, the system H receives the zero generation $X[0] \subseteq X$ as its input, processes it by the first automaton $C_0[0]$ producing the first generation $Z[1]$, which either is the result of the whole computation by H when it satisfies the search condition or it goes to the automaton $C_1[1]$. Because the system H has the same search condition as the evolutionary K-machine E, $C_0[0]$ is a copy of the automaton $A[0]$, while $C_1[1]$ is a copy of the automaton $A[1]$, we have the equality $Z[1] = X[1]$ and the first step of the system H exactly simulates the first step of the K-machine E.

The general step of induction:

We suppose that making $n-1$ steps the system H exactly simulates $n-1$ steps of the K-machine E. It means that making $n-1$ steps, both systems E and H produce the same n-th generation $X[n]$ using automata $A[r]$ $(r \leqslant n-1)$ and $C_r[n-1]$, correspondingly, and this output either is the result of the whole computation by E and by H when it satisfies the search condition or it goes either to the automaton $A[r+1]$ or to the automaton $A[r-1]$ in E and either to the automaton $C_{r+1}[n]$ or to the automaton $C_{r-1}[n]$ in H.

Then the automaton $A[r+1]$ (or $A[r-1]$) in E produces the next generation $X[n+1]$, applying the variation operator v and selection operator s to the generation $X[n]$ and producing the next generation $X[n+1]$. When the automaton $A[r+1]$ in E produces the next generation $X[n+1]$, then either this generation is the result of the whole computation by E when it satisfies the search condition or $A[r+1]$ sends this generation either to $A[r+2]$ or to $A[r]$. When the automaton $A[r-1]$ in E produces the next generation $X[n+1]$, then either this generation is the result of the whole computation by E when it satisfies the search condition or $A[r-1]$ sends this generation either to $A[r-2]$ or to $A[r]$.

At the same time, the automaton $C_{r+1}[n]$ (or $C_{r-1}[n]$) applies the variation operator v and selection operator s to the generation $X[n]$ and producing the generation $Z[n+1]$. When the automaton $C_{r+1}[n]$ in H produces the next generation $Z[n+1]$, then either this generation is the result of the whole computation by H when it satisfies the search condition or $C_{r+1}[n]$ sends this generation either to $C_{r+2}[n+1]$ or to $C_r[n+1]$. When the automaton $C_{r-1}[n]$ in H produces the next generation $Z[n+1]$, then either this generation is the result of the whole computation by H when it satisfies the search condition or $C_{r-1}[n]$ sends this generation either to $C_{r-2}[n]$ or to $C_r[n]$.

Because system H has the same search condition as the evolutionary K-machine E, $C_{r+1}[n]$ is a copy of the automaton $A[r+1]$, while $C_{r-1}[n]$ is a copy of the automaton $A[r-1]$, we have the equality $Z[n+1] = X[n+1]$ and the n-th step of the system H exactly simulates the n-th step of the K-machine E.

Now it is possible to conclude that the system H exactly simulates functioning of the K-machine E. However, the system H is not an evolutionary K-machine. So we need to build a basic evolutionary K-machine B equivalent to H. We can do this using P -conjunctive parallel composition. This composition allows us for all $t = 0, 1, 2, 3, ...$, to substitute each system $\{C_0[t], C_1[t], C_2[t], C_3[t], ..., C_t[t]\}$ by an automaton $B[t]$ from K, which by the definition of function P and P-conjunctive parallel composition, works exactly as this system. Then by construction of the system H, $B = \{B[t]; t = 0, 1, 2, 3, ...\}$ is a basic evolutionary K-machine B equivalent to H. Theorem is proved. □

Corollary 3.1. *If a class K is closed with respect to P-conjunctive parallel composition, then classes GEAK and BEAK are functionally equivalent.*

The class T of all Turing machines is closed with respect to P-conjunctive parallel composition (Burgin, 2010). Thus, Theorem 3.2 implies the following result.

Corollary 3.2. *Classes GEAT of all general evolutionary Turing machines and BEAT of all basic evolutionary Turing machines are functionally equivalent.*

The class IT of all inductive Turing machines is closed with respect to P-conjunctive parallel composition (Burgin, 2010). Thus, Theorem 3.2 implies the following result.

Corollary 3.3. *Classes GEAIT of all general evolutionary inductive Turing machines and BEAIT of all basic evolutionary inductive Turing machines are functionally equivalent.*

Corollary 3.4. *Classes $GEAIT_n$ of all general evolutionary inductive Turing machines of order n and $BEAIT_n$ of all basic evolutionary inductive Turing machines of order n are functionally equivalent.*

The same is true for evolutionary limit Turing machines.

Corollary 3.5. *Classes GEALT of all general evolutionary limit Turing machines and BEALT of all basic evolutionary limit Turing machines are functionally equivalent.*

Definition 3.3. (Burgin, 2010)

a. Two automata A and B are linguistically equivalent if they accept (generate) the same language.
b. Two classes of automata A and B are linguistically equivalent if they accept (generate) the same class of languages.

For instance, it is proved that deterministic and nondeterministic finite automata are linguistically equivalent (cf,. for example (Hopcroft *et al.*, 2001)). It is proved that functional equivalence is stronger than linguistic equivalence (Burgin, 2010).

Because P-conjunctive parallel composition of the level automata in an evolutionary automaton allows the basic evolutionary K-machine to choose automata for data transmission, it is possible to prove the following results.

Theorem 3.3. *If a class K is closed with respect to P-conjunctive parallel composition, then for any general evolutionary K-machine, there is a linguistically equivalent basic evolutionary K-machine.*

Proof. Let us consider an arbitrary general evolutionary K-machine $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. Then by Theorem 3.2, there is a basic evolutionary K-machine L that is functionally equivalent to E. As it is proved in (Burgin, 2010), functional equivalence implies linguistic equivalence. So, the K-machine L is linguistically equivalent to the K-machine E. Theorem is proved. □

Corollary 3.6. *If a class K is closed with respect to P-conjunctive parallel composition, then classes GEAK and BEAK are linguistically equivalent.*

The class T of all Turing machines is closed with respect to P-conjunctive parallel composition (Burgin, 2010). Thus, Theorem 3.3 implies the following result.

Corollary 3.7. *Classes GEAT of all general evolutionary Turing machines and BEAT of all basic evolutionary Turing machines are linguistically equivalent.*

The class IT of all inductive Turing machines is closed with respect to P-conjunctive parallel composition (Burgin, 2010). Thus, Theorem 3.3 implies the following results.

Corollary 3.8. *Classes GEAIT of all general evolutionary inductive Turing machines and BEAIT of all basic evolutionary inductive Turing machines are linguistically equivalent.*

Corollary 3.9. *Classes $GEAIT_n$ of all general evolutionary inductive Turing machines of order n and $BEAIT_n$ of all basic evolutionary inductive Turing machines of order n are linguistically equivalent.*

The same is true for evolutionary limit Turing machines.

Corollary 3.10. *Classes GEALT of all general evolutionary limit Turing machines and BEALT of all basic evolutionary limit Turing machines are linguistically equivalent.*

Obtained results allow us to solve the following problem formulated in (Burgin & Eberbach, 2010).

Problem 3.1. *Are periodic evolutionary finite automata more powerful than finite automata?*

To solve it, we need additional properties of periodic evolutionary finite automata.

Theorem 3.4. *Any general (basic) periodic evolutionary finite automaton F with the period $k > 1$ is functionally equivalent to a periodic evolutionary finite automaton E with the period* 1.

Proof. Let us consider an arbitrary basic periodic evolutionary finite automaton $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. By the definition of basic periodic evolutionary automata (cf. Section 2), the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of finite automata $A[t]$ is either finite or periodic, i.e., there is a finite initial segment of this sequence such that the whole sequence is formed by infinite repetition of this segment. Note that finite sequences are also treated as periodic (Burgin & Eberbach, 2010). When the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ from K is finite, then by Theorem 3.2, the evolutionary machine E is functionally equivalent to a finite automaton AE. By the definition of periodic evolutionary automata, AE is a periodic evolutionary finite automaton with the period 1. Thus, in this case, theorem is proved.

Now let us assume that the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ is infinite. In this case, there is a finite initial segment $H = \{A[t]; t = 0, 1, 2, 3, ..., n\}$ of this sequence such that the whole sequence is formed by infinite repetition of this segment H. By the definition of bounded basic evolutionary automata (cf. Section 2), H is a basic n-level evolutionary finite automaton. Then by Theorem 3.1 from (Burgin & Eberbach, 2010), there is a finite automaton AH functionally equivalent to H. Thus, the evolutionary machine E is functionally equivalent to the basic periodic evolutionary finite automaton $B = \{B[t]; t = 0, 1, 2, 3, ...\}$ in which all automata $B[t] = AH$ for all $t = 0, 1, 2, 3, ...$ Thus, B is a basic periodic evolutionary finite automaton with the period 1. This concludes the proof for basic periodic evolutionary finite automata.

Now let us consider an arbitrary general periodic evolutionary finite automaton $E = \{A[t]; t = 0, 1, 2, 3, ...\}$. By the definition of general periodic evolutionary automata (cf. Section 2), the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of finite automata $A[t]$ is either finite or periodic, i.e., there is a finite initial segment of this sequence such that the whole sequence is formed by infinite repetition of this segment.

At first, we show that when the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ from K is finite, i.e., $E = \{A[t]; t = 0, 1, 2, 3, ..., n\}$, then the evolutionary machine E is functionally equivalent to a finite automaton AE . It is possible to assume that the automata $A[t]$ use transmission by the output when the automaton $A[t]$ uses two more symbols u_{up} and u_{dw} in its output alphabet, giving one of these symbols in its output in addition to the regular output $X[t+1]$, i.e., the output has the form (w, u_{up}) or (w, u_{dw}). If the second part of the output is u_{up} , then $A[t+1]$ sends the output generation $X[t+1]$ to $A[t+1]$. If the second part of the output is u_{dw}, then $A[t+1]$ sends the output generation $X[t+1]$ to $A[t-1]$.

We change all automata $A[t]$ to the automata $C[t]$ in the following way. If $\{q_0, q_1, q_2, ..., q_k\}$ is the set of all states of the automaton $A[t]$, then we take $\{q_{t,0}, q_{t,1}, q_{t,2}, ..., q_{t,k}\}$ as the set of all states of the automaton $C[t]$ $(t = 0, 1, 2, ..., n)$ and in the transition rules of $C[t]$, we change each q_l to $q_{t,l}$. In addition, we change the symbols u_{up} and u_{dw} to the symbols $u_{t,up}$ and $u_{t,dw}$ in the alphabet and in the transition rules of $C[t]$.

By construction, the new system $AE = \{C[t]; t = 0, 1, 2, 3, ..., n\}$ is a finite automaton functionally equivalent to the general periodic evolutionary finite automaton $E = \{A[t]; t = 0, 1, 2, 3, ..., n\}$. Then by the definition of periodic evolutionary automata (cf. Section 2), the automaton AE is a general periodic evolutionary finite automaton with the period 1. Thus, in the finite case, theorem is proved.

Now let us assume that the sequence $\{A[t]; t = 0, 1, 2, 3, ...\}$ of automata $A[t]$ is infinite. In this case, there is a finite initial segment $H = \{A[t]; t = 0, 1, 2, 3, ..., n\}$ of this sequence such that the whole sequence is formed by infinite repetition of this segment H. By the definition of bounded general evolutionary automata (cf. Section 2), H is a general n-level evolutionary finite automaton. Then as we have already proved, there is a finite automaton AH functionally equivalent to H. Thus, the evolutionary machine E is functionally equivalent to the general periodic evolutionary finite automaton $B = \{B[t]; t = 0, 1, 2, 3, ...\}$ in which all automata $B[t] = AH$ for all $t = 0, 1, 2, 3, ...$ Thus, B is a general periodic evolutionary finite automaton with the period 1. This concludes the proof for general periodic evolutionary finite automata. Theorem is proved. □

Functional equivalence implies linguistic equivalence (Burgin, 2010). Thus, Theorem 3.4 implies the following result.

Corollary 3.11. *Any general (basic) periodic evolutionary finite automaton F with the period* $k > 1$ *is linguistically equivalent to a periodic evolutionary finite automaton E with the period* 1.

As a periodic evolutionary finite automaton F with the period 1 consists of multiple copies of the same finite automaton, we have the following results.

Theorem 3.5. *Any basic periodic evolutionary finite automaton F is linguistically equivalent to a finite automaton.*

Proof. By Theorem 3.4, any basic periodic evolutionary finite automaton F with the period $k > 1$ is functionally equivalent to a basic periodic evolutionary finite automaton E with the period 1. It means that all levels in the evolutionary finite automaton E are copies of the same finite automaton. As a finite automaton accepts (computes) a regular language (Hopcroft *et al.*, 2001), the language of the evolutionary finite automaton E is also regular. As the evolutionary finite automaton F is linguistically equivalent to the automaton E, the language L of the evolutionary finite automaton F is also regular. Then there is a finite automaton D that accepts (computes) L (Hopcroft *et al.*, 2001). Thus, the evolutionary finite automaton F is linguistically equivalent to the finite automaton D. Theorem is proved. □

Corollary 3.12. *Basic periodic evolutionary finite automata have the same accepting power as finite automata.*

Theorem 3.6. *Any general periodic evolutionary finite automaton E is equivalent to a one-dimensional cellular automaton.*

Proof. By Theorem 3.4, any general periodic evolutionary finite automaton G with the period $k > 1$ is functionally equivalent to a general periodic evolutionary finite automaton E with the period 1. By definition, E is a sequence of copies of the same finite automaton, which each of them is connected with two its neighbors, and this is exactly a one-dimensional cellular automaton (Trahtenbrot, 1974).

At the same time, taking a finite automaton A with a feedback that connects the automaton output with the automaton input, we see that A can simulate a periodic evolutionary finite automaton E with the period 1 because in E all level automata are copies of the same finite automaton. □

In the theory of cellular automata, it is proved that for any Turing machine T, there is a cellular automaton functionally equivalent to T (Trahtenbrot, 1974). Thus, Theorem 3.6 implies the following result.

Corollary 3.13. *General periodic evolutionary finite automata have the same accepting power as Turing machines.*

Consequently, we have the following result.

Corollary 3.14. *General periodic evolutionary finite automata have more accepting power than basic periodic evolutionary finite automata and than finite automata.*

Note that we cannot apply Theorem 3.2 to periodic evolutionary finite automata because the general evolutionary machine constructed in the proof of this theorem is not periodic.

These results also allow us to solve Problem 4 from (Burgin & Eberbach, 2010).

Problem 3.2. *What class of languages is generated/accepted by periodic evolutionary finite automata?*

Namely, we have the following results.

Corollary 3.15. *The class of languages generated/accepted by basic periodic evolutionary finite automata coincides with regular languages.*

Corollary 3.16. *The class of languages generated/accepted by general periodic evolutionary finite automata coincides with recursively enumerable languages.*

Note that for unrestricted evolutionary finite automata results of Theorems 3.5, 3.6 and their corollaries are not true. Namely, we have the following result.

Theorem 3.7. *The class GEAFA of general unrestricted evolutionary finite automata and the class BEAFA of basic unrestricted evolutionary finite automata have the same accepting power.*

Proof. Indeed, as it is demonstrated in (Eberbach & Burgin, 2007), basic unrestricted evolutionary finite automata can accept any formal language. In particular, they accept any language that general unrestricted evolutionary finite automata accept. As general unrestricted evolutionary finite automata are more general than basic unrestricted evolutionary finite automata, the class of the languages accepted by the former automata is, at least, as big as the class of the languages accepted by the latter automata. Thus, these classes coincide, which means that the class of all general unrestricted evolutionary finite automata and the class of all basic unrestricted evolutionary finite automata have the same accepting power. □

The results from this paper show that in some cases, general evolutionary machines are more powerful than basic evolutionary machines, e.g., for all periodic evolutionary finite automata, while in other cases, it is not true, e.g., for all evolutionary finite automata, general and basic evolutionary finite automata have the same computing power. There are similar results in the theory of classical automata and algorithms. For instance, deterministic and nondeterministic finite automata have the same accepting power. Deterministic and nondeterministic Turing machines have the same accepting power. However, nondeterministic pushdown automata have more accepting power than deterministic pushdown automata.

4. Conclusion

We started our paper with a description of Turings unorganized machines (u-machines) that were supposed to work under the control of some kind of genetic algorithms (note that Turing never formally defined a genetic algorithm or evolutionary computation). This was our inspiration. However, our evolutionary machines are closely related to conventional Turing machines, as well as to the subsequent definitions of genetic algorithms from 1960-80s. This means that level automata of evolutionary machines are finite automata, pushdown automata or Turing machines rather than more primitive NAND logic gates of u-machines. We have introduced several classes of evolutionary machines, such as bounded, periodic and recursively generated evolutionary machines, and studied relations between these classes, giving an interpretation of how modern u-machines could be formalized and how plentiful their computations and types are. Of course, we will never know whether Turing would accept our definitions of evolutionary automata and formalization of evolutionary computation.

In this paper, we introduced two fundamental classes of evolutionary machines/automata: general evolutionary machines and basic evolutionary machines, exploring relations between these classes. Problems of generation of evolutionary machines/automata by automata from a given class are also studied. Examples of such evolutionary machines are evolutionary Turing machines generated by Turing machines and evolutionary inductive Turing machines generated by inductive Turing machines.

There are open problems important for the development of EC foundations.

Problem 4.1. *Can an inductive Turing machine of the first order simulate an arbitrary periodic evolutionary inductive Turing machine of the first order?*

Problem 4.2. *Are there necessary and sufficient conditions for general evolutionary machines to be more powerful than basic evolutionary machines?*

In (Burgin, 2001), topological computations are introduced and studied. This brings us to the following problem.

Problem 4.3. *Study topological computations for evolutionary machines.*

As we can see from results of this paper, in some cases general evolutionary machines are more powerful than basic evolutionary machines, e.g., for all evolutionary finite automata, while in other cases, it is not true, e.g., for all periodic evolutionary machines.

Note that the approach presented in this paper has an enormous space to grow. First of all, similar to natural evolution, our evolutionary automata/machines are not static, i.e., we cover the case of evolution of evolution (currently explored in a very limited way in evolution strategies by changing the σ parameter in mutation). Secondly, our evolutionary finite automata cover already both evolutionary algorithms (i.e., genetic algorithms, evolutionary programming, evolution strategies and genetic programming)and swarm intelligence algorithms, being simple iterative algorithms of the class or regular languages/finite automata. In the evolutionary automata approach, there is a room to grow to invent new types of evolutionary and swarm intelligence algorithms of the class of evolutionary pushdown automata, evolutionary Turing machines or evolutionary inductive Turing machines.

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On a New BV_σ I-Convergent Double Sequence Spaces

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Abstract

In this article we study ${}_2({}_0BV^I_\sigma(M))$, ${}_2BV^I_\sigma(M)$, ${}_2({}_\infty BV^I_\sigma(M))$ double sequence spaces with the help of BV_σ space and an Orlicz function M. The BV_σ space was introduced and studied by (Mursaleen, 1983). We study some of its properties and prove some inclusion relations.

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1. Introduction

Let $\mathbb{N}, \mathbb{R}, \mathbb{C}$ be the sets of all natural, real, and complex numbers respectively. We denote

$$ {}_2\omega = \{x = (x_{ij}) : x_{ij} \in \mathbb{R} \; or \; \mathbb{C}\}, $$

showing the space of all real or complex sequences.

Definition 1.1. A double sequence of complex numbers is defined as a function $X : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}) where the two subscripts run through the sequence of natural numbers independent of each other. A number a$\in \mathbb{C}$ is called double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that,

$$ |(x_{ij}) - a| < \epsilon, \text{ for all } i, j \geqslant N, \tag{1.1} $$

(see (Habil, 2006)). Let l_∞ and c denote the Banach space bounded and convergent sequences, respectively, with norm $\|x\|_\infty = \sup_k |x_k|$. Let v be denote the space of sequences of bounded variation. That is,

$$ v = \{x = (x_k) : \sum_{k=0}^{\infty} |x_k - x_{k-1}| < \infty, x_{-1} = 0\} \tag{1.2} $$

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where v is a Banach space normed by $\|x\| = \sum_{k=0}^{\infty} |x_k - x_{k-1}|$ (see (Mursaleen, 1983)). Let σ be an injective mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or σ-mean if and only if:
1. $\phi(x) \geqslant 0$ where the sequence $x = (x_k)$ has $x_k \geqslant 0$ for all k,
2. $\phi(e) = 1$ where e={1,1,1,1,......},
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \{x = (x_k) : \lim_{m\to\infty} t_{m,k}(x) = L \text{ uniformly in } k, L = \sigma - \lim x\} \tag{1.3}$$

where m$\geqslant 0, k > 0$.

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} + ... + x_\sigma^m(k)}{m+1} \text{ and } t_{-1,k} = 0, \tag{1.4}$$

where $\sigma^m(k)$ denote the m^{th}-iterate of $\sigma(k)$ at k . In this case σ is the translation mapping,that is, $\sigma(k) = k + 1, \sigma$-mean is called a Banach limit and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences. The special case of (1.4) in which $\sigma(k) = k + 1$ was given by (Lorentz, 1948), and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c in the sense that

$$\phi(x) = \lim x, \text{ for all } x \in c. \tag{1.5}$$

Theorem 1.1. *A σ-mean extends the limit functional on* c *in the sense that $\phi(x) = \lim x$ for all $x \in$ c if and only if σ has no finite orbits. That is, if and only if for all $k \geqslant 0, j \geqslant 1, \sigma^j(k) \neq k$, (see (Khan, 2008))*

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x), \tag{1.6}$$

assuming that $t_{-1,k}(x) = 0$. A straight forward calculation shows that (Mursaleen, 1983),

$$\phi_{m,k}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^{m} J(x_\sigma^j(k) - x_\sigma^{j-1}(k)), & \text{if } m \geqslant 1 \\ x_k, \quad \text{if } m = 0. \end{cases}$$

For any sequence x,y and scalar λ, we have $\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y)$ and $\phi_{m,k}(\lambda x) = \lambda\phi_{m,k}(x)$.

Definition 1.2. A sequence $x \in l_\infty$ is of σ-bounded variation if and only if:
(i)$\sum |\phi_{m,k}(x)|$ converges uniformly in k,
(ii) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

We denote by BV_σ, the space of all sequences of σ-bounded variation (see (Khan, 2008)):

$$BV_\sigma = \{x \in l_\infty : \sum_m |\phi_{m,k}(x)| < \infty, \text{ uniformly in } k\}.$$

Theorem 1.2. *BV_σ is a Banach space normed by*

$$\|x\| = \sup_k \sum_{m=0}^{\infty} |\phi_{m,k}(x)|, \tag{1.7}$$

(see (Khan & Ebadullah, 2012)).
Subsequently invariant mean studied by (Mursaleen, 1983), (Ahmad & Mursaleen, 1988), (Raimi & A., 1963), (Khan & Ebadullah, 2011), (Khan & Ebadullah, 2012), (Schaefer, 1972) and many others.

Definition 1.3. A function $M : [0, \infty) \longrightarrow [0, \infty)$ is said to be an Orlicz function if it satisfies the following conditions;
(i)M is continuous ,convex and non-decreasing,
(ii)$M(0) = 0, M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Remark. (see (Tripathy & Hazarika, 2011)). **(i)** If the convexity of an Orlicz function is replaced by $M(x + y) \leqslant M(x) + M(y)$, then this function is called Modulus function.
(ii) If M is an Orlicz function , then $M(\lambda X) \leqslant \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy $\triangle_2$-condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \leqslant KLM(u)$ for all values of L> 1(see (Tripathy & Hazarika, 2011)). (Lindenstrauss & Tzafriri, 1971) used the idea of an Orlicz function to construct the sequence space $l_M = \{x \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty$ for some $\rho > 0\}$. The space l_∞ becomes a Banach space with the norm

$$\|x\| = \inf \Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leqslant 1\Big\}, \tag{1.8}$$

which is called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(t) = t^p$ for $1 \leqslant p < \infty$. Later on, some Orlicz sequence spaces were investigated by (Hazarika & Esi, 2013), (Maddox, 1970), (Parshar & Choudhary, 1994), (Bhardwaj & Singh, 2000), (Et, 2001), (Tripathy & Hazarika, 2011) and many others. Initially, as a generalization of statistical convergence, the notation of I-convergence was introduced and studied by P. Kostyrko and Wilczynski(Kostyrko *et al.*, 2000). Later on, it was studied by Hazarika and Esi (Hazarika & Esi, 2013) and many others.

Definition 1.4. A double sequence $x = x_{ij} \in {}_2\omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geqslant \epsilon\} \in I. \tag{1.9}$$

In this case, we write $I - \lim x_{ij} = L$.

Definition 1.5. Let X be a non empty set. Then, a family of sets $I \subseteq 2^X$ is said to be an Ideal in X if
(i)$\phi \in I$;
(ii)I is additive; that is,$A, B \in I \Rightarrow A \cup B \in I$;
(iii)I is hereditary that is,$A \in I, B \subseteq A \Rightarrow B \in I$.

An Ideal $I \subseteq 2^X$ is called non trivial if $I \neq 2^X$. A non trivial ideal $I \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq I$.
A non trivial ideal I is maximal if there cannot exist any non trivial ideal $J \neq I$ containing I as a subset.

Definition 1.6. A non empty family of sets $\mathcal{F} \subseteq 2^X$ is said to be filter on X if and only if
(i) $\emptyset \notin \mathcal{F}$;
(ii) for, A, B $\in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$;
(iii) for each A$\in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$. For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I. That is,

$$\mathcal{F}(I) = \{K \subseteq N : K^c \in I\}, \quad \text{where} K^c = N - K. \tag{1.10}$$

Definition 1.7. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I - null if L=0. In this case, we write

$$I - \lim x_{ij} = 0. \tag{1.11}$$

Definition 1.8. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists numbers $m = m(\epsilon), n = n(\epsilon)$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geqslant \epsilon\} \in I. \tag{1.12}$$

Definition 1.9. A double sequence $(x_{ij}) \in {}_2\omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\}. \tag{1.13}$$

Definition 1.10. A double sequence space E is said to be solid or normal if $x_{ij} \in E$ implies that $(\alpha_{ij}x_{ij}) \in E$ for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.11. A double sequence space E is said to be symmetric if $(x_{\pi(i)\pi(j)}) \in E$ whenever $(x_{ij}) \in E$, where $\pi(i)$ and $\pi(j)$ is a permutation on $\mathbb{N}$.

Definition 1.12. A double sequence space E is said to be sequence algebra if $(x_{ij}y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 1.13. A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$, for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

Definition 1.14. Let K=$\{(n_i, k_j) : i, j \in \mathbb{N}; n_1 < n_2 < n_3 < ....and\ \ k_1 < k_2 < k_3 < ....\} \subseteq \mathbb{N} \times \mathbb{N}$ and E be a double sequence space.A K-step space of E is a sequence space

$$\lambda_k^E = \{(\alpha_{ij}x_{ij}) : (x_{ij}) \in E\}.$$

Definition 1.15. A cannonical preimage of a sequence $(x_{n_ik_j}) \in E$ is a sequence $(b_{nk}) \in E$ defined as follows

$$b_{n,k} = \begin{cases} a_{n,k}, & \text{for n , k } \in K \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.16. A sequence space E is said to be monotone if it contains the cannonical preimages of all its stepspaces.

Remark. If $I = I_f$, the class of all finite subsets of $\mathbb{N}$. Then I is an admissible ideal in $\mathbb{N}$ and I_f convergence coincides with the usual convergence.

Definition 1.17. If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then I is an admissible ideal in $\mathbb{N}$ and we call the I_δ -convergence as the logarithmic statistical convergence.

Definition 1.18. If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then,I is an admissible ideal in $\mathbb{N}$ and we call the I_d-convergence as asymptotic statistical covergence.

Lemma 1.1. *((Tripathy & Hazarika, 2011)). Every solid space is monotone.*

Lemma 1.2. *Let $\mathcal{F}(I)$ and $M \subseteq N$. If $M \notin I$, then $M \cap K \notin I$.*

Lemma 1.3. *If $I \subseteq 2^N$ and $M \subseteq N$.If $M \notin I$, then $M \cap N \notin I$.*

2. Main Results

Recently (Khan & Khan, 2013) introduced and studied the following sequence space. For m,n$\geqslant$ 0

$$ {}_2BV_\sigma^I = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N} : |\phi_{mnij}(x) - L| \geqslant \epsilon\} \in I, for\ some\ L \in \mathbb{C}\}. \quad (2.1)$$

In this article we introduce the following double sequence spaces. For m,n$\geqslant$ 0

$$ {}_2BV_\sigma^I(M) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|\phi_{mnij}(x) - L|}{\rho}) = 0,\ for\ some\ L \in \mathbb{C}, \rho > 0\} \quad (2.2)$$

$$ {}_2({}_0BV_\sigma^I(M)) = \{x = (x_{ij}) \in {}_2\omega : I - \lim M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0, \rho > 0\}, \quad (2.3)$$

$$ {}_2({}_\infty BV_\sigma^I(M)) = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N}\times\mathbb{N}\ \ \exists k > 0\ s.t\ M(\frac{|\phi_{mnij}(x)|}{\rho}) \geqslant k\} \in I, \rho > 0\} \quad (2.4)$$

$$ {}_2({}_\infty BV_\sigma(M)) = \{x = (x_{ij}) \in {}_2\omega : \sup M(\frac{|\phi_{mnij}(x)|}{\rho}) < \infty, \rho > 0\}. \quad (2.5)$$

We also denote

$$ {}_2M_{BV_\sigma}^I(M) = {}_2BV_\sigma^I(M) \cap {}_2({}_\infty BV_\sigma(M))$$

and

$$ {}_2({}_0M_{BV_\sigma}^I(M)) = {}_2({}_0BV_\sigma^I(M)) \cap {}_2({}_\infty BV_\sigma(M)).$$

Theorem 2.1. *For any Orlicz function M, the classes of double sequence ${}_2({}_0BV_\sigma^I(M))$,${}_2BV_\sigma^I(M)$, ${}_2({}_0M_{BV_\sigma}^I(M))$, and ${}_2M_{BV_\sigma}^I(M)$ are linear spaces.*

Proof. Let $x = (x_{ij}), (y_{ij}) \in {}_2BV^I_\sigma(M)$ be any two arbitrary elements, and let α, β are scalars. Now, since $(x_{ij}), (y_{ij}) \in {}_2BV^I_\sigma(M)$. Then this implies that $\exists$ some positive numbers $L_1, L_2 \in \mathbb{C}$ and $\rho_1, \rho_2 > 0$ such that,

$$I - \lim_{i,j} M\Big(\frac{|\phi_{mnij}(x) - L_1|}{\rho_1}\Big) = 0, \tag{2.6}$$

$$I - \lim_{i,j} M\Big(\frac{|\phi_{mnij}(y) - L_2|}{\rho_2}\Big) = 0. \tag{2.7}$$

$\Rightarrow$ for any given $\epsilon > 0$, the sets

$$\Rightarrow \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{mnij}(x) - L_1|}{\rho_1}\Big) \geqslant \frac{\epsilon}{2}\} \in I, \tag{2.8}$$

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{mnij}(y) - L_2|}{\rho_2}\Big) \geqslant \frac{\epsilon}{2}\} \in I. \tag{2.9}$$

Now let

$$A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{ij}(x) - L_1|}{\rho_1}\Big) < \frac{\epsilon}{2}\} \in I, \tag{2.10}$$

$$A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{|\phi_{ij}(y) - L_2|}{\rho_2}\Big) < \frac{\epsilon}{2}\} \in I. \tag{2.11}$$

be such that $A_1^c, A_2^c \in I$. Let $\rho_3 = max\{2|\alpha|\rho_1\ ,\ 2|\beta|\rho_2\}$
Since M is non decreasing and convex function,we have

$$M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) = M(\frac{|(\alpha\phi_{mnij}(x) + \beta\phi_{mnij}(y)) - (\alpha L_1 + \beta L_2)|}{\rho_3})$$

$$\begin{aligned} &= M(\tfrac{|\alpha(\phi_{mnij}(x) - L_1) + \beta(\phi_{mnij}(y) - L_2)|}{\rho_3}) \\ &\leqslant M(\tfrac{|\alpha||\phi_{mnij}(x) - L_1|}{\rho_3}) + M(\tfrac{|\beta||\phi_{mnij}(y) - L_2|}{\rho_3}) \\ &\leqslant M(\tfrac{|\alpha||\phi_{mnij}(x) - L_1|}{\rho_1}) + M(\tfrac{|\beta||\phi_{mnij}(y) - L_2|}{\rho_2}) \\ &\leqslant \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

$\Rightarrow \{(i,j) \in \mathbb{N} \times \mathbb{N} : M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) > \epsilon\} \in I$
implies that , $I - \lim_{i,j} M(\frac{|\phi_{mnij}(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3}) = 0$.
Thus $\alpha(x_{ij}) + \beta(y_{ij}) \in {}_2BV^I_\sigma(M)$. As (x_{ij}) and (y_{ij}) are two arbitrary element then $\alpha x_{ij} + \beta y_{ij} \in {}_2BV^I_\sigma(M)$ for all $x_{ij},\ y_{ij} \in {}_2BV^I_\sigma(M)$, for all scalars α, β . Hence ${}_2BV^I_\sigma(M)$ is linear space. The proof for other spaces will follow similarly. □

Theorem 2.2. *Let M_1, M_2 be two Orlicz functions and statisfying $\triangle_2$ condition ,then*
(a)$X(M_2) \subseteq X(M_1 M_2)$
(b)$X(M_1) \cap X(M_2) \subseteq X(M_1 + M_2)$ for $X = {}_2BV^I_\sigma,\ {}_2({}_0BV^I_\sigma),\ {}_2M^I_{BV_\sigma},\ {}_2({}_0M^I_{BV_\sigma})$.

Proof. (a)Let $x = (x_{ij}) \in \ {}_2({}_0BV^I_\sigma(M_2))$ be an arbitrary element
$\Rightarrow \rho > 0$ such that

$$I - \lim M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0. \tag{2.12}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon$ for $0 < t \leqslant \delta$.
Write $y_{ij} = M_2(\frac{|\phi_{mnij}(x)|}{\rho})$ and
consider,

$$\lim_{ij} M_1(y_{ij}) = \lim_{y_{ij} \leqslant \delta, i,j \in \mathbb{N}} M_1(y_{ij}) + \lim_{y_{ij} > \delta, i,j \in \mathbb{N}} M_1(y_{ij}). \tag{2.13}$$

Now, since M_1 is an Orlicz function so we have $M_1(\lambda x) \leqslant \lambda M_1(x), 0 < \lambda < 1$.
Therefore we have,

$$\lim_{y_{ij} \leqslant \delta,\ i,j \in \mathbb{N}} M_1(y_{ij}) \leqslant M_1(2) \lim_{y_{ij} \leqslant \delta,\ i,j \in \mathbb{N}} (y_{ij}). \tag{2.14}$$

For $y_{ij} > \delta$, we have $y_{ij} < \frac{y_{ij}}{\delta} < 1 + \frac{y_{ij}}{\delta}$. Now, since M_1 is non-decreasing and convex, it follows that,

$$M_1(y_{ij}) < M_1(1 + \frac{y_{ij}}{\delta}) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1(\frac{2y_{ij}}{\delta}). \tag{2.15}$$

Since M_1 satisfies the $\triangle_2$- condition we have,

$$\begin{aligned} M_1(y_{ij}) &< \frac{1}{2} K \frac{y_{ij}}{\delta} M_1(2) + \frac{1}{2} K M_1(\frac{2y_{ij}}{\delta}) \\ &< \frac{1}{2} K \frac{y_{ij}}{\delta} M_1(2) + \frac{1}{2} K \frac{y_{ij}}{\delta} M_1(2) \\ &= K \frac{y_{ij}}{\delta} M_1(2). \end{aligned} \tag{2.16}$$

This implies that,

$$M_1(y_{ij}) < K \frac{y_{ij}}{\delta} M_1(2). \tag{2.17}$$

Hence,we have

$$\lim_{y_{ij} > \delta,\ i,j \in \mathbb{N}} M_1(y_{ij}) \leqslant max\{1, K\delta^{-1} M_1(2) \lim_{y_{ij} > \delta, i,j \in \mathbb{N}} (y_{ij})\}. \tag{2.18}$$

Therefore from (2.12),and (2.13) we have

$$I - \lim_{ij} M_1(y_{ij}) = 0.$$

$$\Rightarrow I - \lim_{ij} M_1 M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

This implies that $x = (x_{ij}) \in \ {}_2({}_0BV^I_\sigma(M_1M_2))$.Hence $X(M_2) \subseteq X(M_1M_2)$ for $X = \ {}_2({}_0BV^I_\sigma)$. The other cases can be proved in similar way.

(b) Let $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M_1)) \cap {}_2({}_0BV_\sigma^I(M_2))$. Let $\epsilon > 0$ be given.Then $\exists\rho > 0$ such that,

$$I - \lim M_1(\frac{|\phi_{mnij}(x)|}{\rho}) = 0, \tag{2.19}$$

and

$$I - \lim M_2(\frac{|\phi_{mnij}(x)|}{\rho}) = 0. \tag{2.20}$$

Therefore

$$I - \lim_{ij}(M_1 + M_2)(\frac{|\phi_{mnij}(x)|}{\rho}) = I - \lim_{ij} M_1(\frac{|\phi_{mnij}(x)|}{\rho}) + I - \lim_{ij} M_2(\frac{|\phi_{mnij}(x)|}{\rho}),$$

from eqs (2.19) and (2.20)

$$\Rightarrow I - \lim_{ij}(M_1 + M_2)(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

we get

$$x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M_1 + M_2)).$$

Hence we get ${}_2({}_0BV_\sigma^I(M_1)) \cap {}_2({}_0BV_\sigma^I(M_2)) \subseteq {}_2({}_0BV_\sigma^I(M_1 + M_2))$.
For $X = {}_2BV_\sigma^I, {}_2({}_0M_{BV_\sigma}^I), {}_2(M_{BV_\sigma}^I)$ the inclusion are similar. □

Corollary 2.1. *$X \subseteq X(M)$ for $X = {}_2(BV_\sigma^I), {}_2BV_\sigma^I, {}_2({}_0M_{BV_\sigma}^I)$ and ${}_2M_{BV_\sigma}^I$.*

Proof. For this let $M(x) = x$, for all $x = (x_{ij}) \in X$. Let us suppose that $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I)$. Then for any given $\epsilon > 0$ we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |\phi_{mnij}(x)| \geqslant \epsilon\} \in I.$$

Now let

$$A_1 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |\phi_{mnij}(x)| < \epsilon\} \in I,$$

be such that $A_1^c \in I$. Now consider , for $\rho > 0$,

$$M(\frac{|\phi_{mnij}(x)|}{\rho}) = \frac{|\phi_{mnij}(x)|}{\rho} < \frac{\epsilon}{\rho} < \epsilon.$$

$\Rightarrow I - \lim M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0$, which implies that $x = (x_{ij}) \in {}_2({}_0BV_\sigma^I(M))$. Hence we have

$${}_2({}_0BV_\sigma^I) \subseteq {}_2({}_0BV_\sigma^I(M)).$$

$$\Rightarrow X \subseteq X(M)$$

and the other cases will be proved similarly. □

Theorem 2.3. *For any Orlicz function M, the spaces* $_2({}_0BV_\sigma^I(M))$ *and* $_2({}_0M^I_{BV_\sigma})$ *are solid and monotone.*

Proof. Here we consider $_2({}_0BV_\sigma^I)$ and for $_2({}_0BV_\sigma^I(M))$ the proof shall be similar.
Let $x = x_{ij} \in {}_2({}_0BV_\sigma^I(M))$ be an arbitrary element ,$\Rightarrow \exists \rho > 0$ such that

$$I - \lim_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho}) = 0.$$

Let α_{ij} be a sequence of scalars with $|\alpha_{ij}| \leqslant 1$ for $i, j \in \mathbb{N}$.
Now, M is an Orlicz function. Therefore

$$M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) = M(\frac{|\alpha_{ij}||\phi_{mnij}(x)|}{\rho})$$

$$\leqslant |\alpha_{ij}|M(\frac{|\phi_{mnij}(x)|}{\rho})$$

$\Rightarrow M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) \leqslant M(\frac{|\phi_{mnij}(x)|}{\rho})$ for all $i, j \in \mathbb{N}$.

$$\Rightarrow I - \lim_{ij} M(\frac{|\alpha_{ij}\phi_{mnij}(x)|}{\rho}) = 0.$$

Thus we have $(\alpha_{ij}x_{ij}) \in {}_2({}_0BV_\sigma^I(M))$. Hence $_2({}_0BV_\sigma^I(M))$ is solid. Therefore $_2({}_0BV_\sigma^I(M))$ is monotone. Since every solid sequence space is monotone. □

Theorem 2.4. *For any Orlicz function M,the space* $_2BV_\sigma^I(M)$ *and* $_2(M_{BV_\sigma^I}(M))$ *are neither solid nor monotone in general.*

Proof. Here we give counter example for establishment of this result. Let $X = {}_2BV_\sigma^I$ and $_2(M_{BV_\sigma^I})$.
Let us consider $I = I_f$ and $M(x) = x$, for all $x = x_{ij} \in [0, \infty)$. Consider,the K-step space $X_K(M)$ of $X(M)$ defined as follows:
Let $x = (x_{ij}) \in X(M)$ and $y = (y_{ij}) \in X_K(M)$ be such that $(y_{ij}) = (x_{ij})$, if i,j is even and $(y_{ij}) = 0$, otherwise.
Consider the sequence (x_{ij}) defined by $(x_{ij}) = 1$ for all $i, j \in \mathbb{N}$. Then $x = (x_{ij}) \in {}_2BV_\sigma^I(M)$ and $_2M_{BV_\sigma^I}(M)$, but K-step space preimage does not belong to $BV_\sigma^I(M)$ and $_2M^I_{BV_\sigma}(M)$. Thus $_2BV_\sigma^I(M)$ and $_2M^I_{BV_\sigma}(M)$ are not monotone and hence they are not solid. □

Theorem 2.5. *For an Orlicz function M, the spaces* $_2BV_\sigma^I(M)$ *and* $_2BV_\sigma^I(M)$ *are sequence algebra.*

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in {}_2({}_0(BV_\sigma^I(M)))$ be any two arbitrary elements. $\Rightarrow \rho_1, \rho_2 > 0$ such that,

$$I - \lim_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho_1}) = 0,$$

and

$$I - lim_{ij} M(\frac{|\phi_{mnij}(y)|}{\rho_2}) = 0.$$

Let $\rho = \rho_1\rho_2 > 0$. Then

$$M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho}) = M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho_1\rho_2})$$

$$\Rightarrow I - \lim_{ij} M(\frac{|\phi_{mnij}(x)\ \phi_{mnij}(y)|}{\rho}) = 0.$$

Therefore we have $(x_{ij}y_{ij}) \in\ _2({}_0BV_\sigma^I(M))$. Hence $_2({}_0BV_\sigma^I(M))$ is sequence algebra. □

Theorem 2.6. *For any Orlicz function M, the spaces* $_2({}_0BV_\sigma^I(M))$ *and* $_2BV_\sigma^I(M)$ *are not convergence free.*

Proof. To show this let $I = I_f$ and $M(x) = x$, for all $x = [0, \infty)$. Now consider the double sequence $(x_{ij}), (y_{ij})$ which defined as follows:

$$x_{ij} = \frac{1}{i+j} and\ \ y_{ij} = i + j, \forall i, j \in \mathbb{N}.$$

Then we have (x_{ij}) belong to both $_2({}_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$, but (y_{ij}) does not belong to $_2({}_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$. Hence, the spaces $_2({}_0BV_\sigma^I(M))$ and $_2BV_\sigma^I(M)$ are not convergence free. □

Theorem 2.7. *Let M be an Orlicz function. Then*

$$_2({}_0BV_\sigma^I(M)) \subseteq\ _2BV_\sigma^I(M) \subseteq\ _2({}_\infty BV_\sigma^I(M)).$$

Proof. For this let us consider $x = (x_{ij}) \in\ _2({}_0BV_\sigma^I(M))$. It is obvious that it must belong to $_2BV_\sigma^I(M)$. Now consider

$$M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) \leqslant M(\frac{|\phi_{mnij}(x)|}{\rho}) + M\Big(\frac{|L|}{\rho}\Big).$$

Now taking the limit on both sides we get

$$I - lim_{ij} M(\frac{|\phi_{mnij}(x) - L|}{\rho}) = 0.$$

Hence $x = (x_{ij}) \in\ _2BV_\sigma^I(M)$.
Now it remains to show that $_2(BV_\sigma^I(M)) \subseteq\ _2({}_\infty BV_\sigma^I(M))$. For this let us consider $x = (x_{ij}) \in\ _2BV_\sigma^I(M) \Rightarrow \exists\rho > 0$ s.t

$$I - \lim_{ij} M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) = 0.$$

Now consider

$$M\Big(\frac{|\phi_{mnij}(x)|}{\rho}\Big) \leqslant M\Big(\frac{|\phi_{mnij}(x) - L|}{\rho}\Big) + M\Big(\frac{|L|}{\rho}\Big).$$

Now taking the supremum on both sides we get

$$\sup_{ij} M(\frac{|\phi_{mnij}(x)|}{\rho}) < \infty.$$

Hence $x = (x_{ij}) \in\ _2({}_\infty BV_\sigma^I(M))$. □ □

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