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Practical Algorithmic Optimizations for Finding Maximal Matchings in Induced Subgraphs of Grids and Minimum Cost Perfect Matchings in Bipartite Graphs

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Abstract

In this paper we present practical algorithmic optimizations addressing two problems. The first one is concerned with computing a maximal matching in an induced subgraph of a grid graph. For this problem we present a faster sequential algorithm using bit operations and a way of implementing it in a parallel environment. The second problem is concerned with computing minimum cost perfect matchings in bipartite graphs. For this problem we extend the idea behind the Hopcroft-Karp maximum matching algorithm and then we consider a more general situation in which multiple minimum cost perfect matchings need to be computed in the same graph, under certain cost restrictions. We present experimental results for all the proposed optimizations.

Keywords: Minimum cost perfect matching, maximal matching, maximum flow, grid graph.
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1. Introduction

The problem of computing maximum or maximal matchings in bipartite graphs has been considered many times in the scientific literature. Many of the proposed algorithms use the fact that computing a maximum matching in a bipartite graph is equivalent to computing a maximum flow in a slightly modified graph. Thus, results from the theory of network flows can be used for computing maximum matchings. If only a maximal matching is needed, then simpler greedy-type algorithms can be employed. In this paper we present several practical algorithmic improvements for some of the algorithms used for computing maximal matchings in grid graphs and minimum cost perfect matchings in bipartite graphs.

The rest of this paper is structured as follows. In Section 2 we define the main terms and techniques used in this paper. In Section 3 we discuss related work. In Section 4 we present

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faster algorithms for computing maximal matchings in induced subgraphs of grid graphs based on algorithms which use bit operations. In Section 5 we extend an idea used for computing maximum matchings in bipartite graphs to the computation of minimum cost perfect matchings. The idea consists of using multiple edge-disjoint augmenting paths per iteration in order to reduce the number of iterations of the algorithm. In Section 6 we consider another perfect matching problem. In this problem we are interested in computing a minimum cost perfect matching in a complete bipartite graph under certain restrictions regarding the cost computation. The cost of the matching is considered to be equal to the sum of the costs of the edges from the matching *except* for the cost of the minimum cost edge from the matching (i.e. the minimum cost edge of the matching is considered to have cost 0 when computing the cost of the matching). In Section 7 we present experimental evaluations of all the algorithms discussed in this paper. Finally, in Section 8 we conclude.

2. Terms and Definitions

A bipartite graph is a graph whose vertices can be split into two sets L (left) and R (right). We consider the vertices to be numbered from 1 to $|L|$ in the left set and from 1 to $|R|$ in the right set (it is acceptable to have vertices with the same number in the graph, because they will be differentiated based on the set L or R to which they belong). Every edge (x, y) of the graph is between a node $x \in L$ and a node $y \in R$. A matching in a bipartite graph is a set of edges such that no two edges in the set have a common vertex. A maximum matching is a matching of maximum cardinality. A maximal matching is a matching to which no more edges can be added (i.e. all the edges outside of the matching have a common vertex with at least one edge from the matching). A perfect matching is a matching in which every node of the graph is an endpoint of an edge from the matching (such a matching may exist only when $|L| = |R|$).

In order to reduce the maximum matching problem to a maximum flow problem we need to construct a directed graph as follows. We will have a special node S called the *source* and another special node T called the *sink*. We will also keep all the nodes from the given bipartite graph. Each edge (x, y) of the original bipartite graph will be replaced by a directed arc from x to y having capacity 1. We will also add capacity 1 arcs from S to every node $x \in L$ and from every node $y \in R$ to T. In case the edges of the bipartite graph have costs these costs are maintained on the directed arcs from the nodes $x \in L$ to the nodes $y \in R$ (we will denote by $c(x, y)$ the cost of the edge between $x \in L$ and $y \in R$). All the arcs having S or T as an endpoint will have cost 0.

One of the best known maximum flow algorithms is the Edmonds-Karp algorithm (Edmonds & Karp, 1972). This algorithm can be summarized as follows: As long as possible find a shortest path from S to T in the residual graph and augment the flow along that path. When arc costs are involved the algorithm can be adjusted in order to find a minimum cost path from S to T in the residual graph. Note that the residual graph may contain negative costs. This version of the Edmonds-Karp algorithm is known as the *successive shortest path* algorithm (Todinov, 2013). A simple breadth-first search algorithm is used for finding a shortest path in the first case (i.e. when edge costs are not involved), while a minimum cost path algorithm needs to be used in the second case (i.e. when edge costs are involved), for instance, Bellman-Ford-Moore (Papaefthymiou & Rodrigue, 1997) or even Dijkstra's algorithm (Todinov, 2013) after modifying the graph's arc costs in order to remove negative costs. Thus, the algorithm consists of multiple iterations, in each of which the flow is increased along a single path. The most time consuming part in each iteration is the traversal of the graph in order to find an augmenting path. In a graph with V vertices and E arcs finding the shortest augmenting path takes $O(V + E)$ time when no costs are involved and $O(V \cdot E)$ time when costs are involved (or $O(V + E \cdot log(V))$ or $O(E + V \cdot log(V))$ time when Dijkstra's algorithm is used on the modified residual graph costs). Then, augmenting the flow along the found path is easy (it takes only $O(V)$ time). In the case of bipartite graphs it is sufficient to find a path from S to an unmatched vertex in R (because this vertex is directly connected to T through an existing arc in the residual graph).

3. Related Work

The best algorithm for computing a maximum matching in sparse bipartite graphs is the Hopcroft-Karp algorithm (Hopcroft & Karp, 1973), which has a time complexity of $O(E \cdot \sqrt{V})$ where V is the number of vertices and E is the number of edges of the graph. For dense bipartite graphs the algorithm proposed in (Alt *et al.*, 1991) has a slightly better time complexity of $O(V^{1.5}\sqrt{\frac{E}{log(V)}})$. Both of these algorithms have a better time complexity than the Edmonds-Karp algorithm for finding a maximum flow presented in the previous section. However, due to its simplicity, the Edmonds-Karp algorithm is used in many practical implementations. Moreover, experimental evaluations showed that for some types of bipartite graphs some modified versions of the Edmonds-Karp algorithm (which use breadth-first search from all the source's neighbors for finding augmenting paths) are faster than the Hopcroft-Karp algorithm, despite having a worse theoretical time complexity (Cherkassky *et al.*, 1998).

Edmonds-Karp is not the only algorithm for computing maximum flows in graphs. In fact, many such algorithms were proposed in the scientific literature. Some of the most popular ones are Dinic's algorithm (Dinic, 1970), Karzanov's algorithm (Karzanov, 1974) and the push-relabel maximum flow algorithm (Goldberg & Tarjan, 1986).

A minimum cost perfect bipartite matching can be computed in $O(V^3)$ time using the Hungarian algoritm (Munkres, 1957). The *successive shortest path* algorithm for minimum cost maximum flows can be implemented in $O(V \cdot (E + V \cdot log(V)))$ time in order to compute a minimum cost maximum matching by using Fibonacci heaps (Fredman & Tarjan, 1987). The algorithm consists of $O(V)$ iterations and each iteration runs in $O(E + V \cdot log(V))$ time. Dynamic versions of the minimum cost perfect bipartite matching problem, in which edge costs can be changed, have also been considered (Mills-Tettey *et al.*, 2007).

Maximum matchings can also be computed in general graphs, not just bipartite graphs (see, for instance, Gabow's algorithm (Gabow, 1976), having an $O(V^3)$ time complexity). Minimum cost perfect matchings have also been considered in some special classes of graphs, e.g. graphs induced by points in the plane (Varadarajan, 1998). Greedy algorithms for maximal matchings, including parallel versions, were presented in (Blelloch *et al.*, 2012). The problem of maintaining maximal matchings in dynamic graphs has been addressed in (Neiman & Solomon, 2013).

4. Faster Algorithm for Maximal Matchings in Induced Subgraphs of Grid Graphs using Bit Operations

We consider an $M \cdot N$ grid graph in which every node has a coordinate (x, y) $(0 \leq x \leq N-1$, $0 \leq y \leq M-1)$ and some nodes are missing. The graph is defined by the implicit adjacency structure of the existing nodes (i.e. two nodes at distance 1 in the grid are neighbors). We are interested in computing a maximal matching in this graph. Note that a maximal matching simply implies that no other edge of the graph can be added to the matching and not that the matching has maximum cardinality. Computing a maximum cardinality matching can be done easily, because the graph is bipartite (we can separate the nodes into two groups based on the parity of their sum of x and y coordinates) and there are many polynomial-time maximum matching algorithms in such graphs (see Section 3).

Computing a maximal matching can be achieved faster, in only $O(M \cdot N)$ time. Let's consider the following Greedy algorithm (Algorithm 1) which traverses the grid graph in increasing order of the y-coordinate and for each y in inceasing order of the x-coordinate.

Algorithm 1 Greedy $O(M \cdot N)$ Algorithm for Finding a Maximal Matching in an Induced Subgraph of a Grid Graph

```
C = 0 {At the end of the algorithm C will be the size of the maximal matching.}
for y = 0 to M − 1 do
  for x = 0 to N − 1 do
    if node (x, y) exists in the graph then
      if y > 0 and node (x, y − 1) exists in the graph and node (x, y − 1) is not matched then
        Match the nodes (x, y) and (x, y − 1)
        C = C + 1
      else if x > 0 and node (x − 1, y) exists in the graph and node (x − 1, y) is not matched
      then
        Match the nodes (x, y) and (x − 1, y)
        C = C + 1
      end if
    end if
  end for
end for
```

We can implement a faster version of the Algorithm 1 by using bit operations. Note that the presented algorithm will only compute the size of the maximal matching and not the matching itself. The speed increase is due to using bit operations and handling multiple nodes at the same time. We will split each row of the grid graph (corresponding to a y-coordinate) into blocks of B bits. Block i of each row contains bits referring to the coordinates $i \cdot B, \ldots, (i+1) \cdot B - 1$. We will denote by $block(y, i)$ the block i of the row corresponding to the coordinate y. We will have bit j of $block(y, i)$ set to 1 if the node $(i \cdot B + j, y)$ exists in the graph, and set to 0 otherwise $(0 \leq j \leq B-1)$. We will traverse the graph from $y = 0$ to $y = M - 1$ as in Algorithm 1. During the traversal we will maintain a row of blocks corresponding to the previous row in which 1 bits will correspond to existing unmatched nodes. When considering a new row y, the first step is to perform an **AND** between the current row and the previous row of unmatched nodes. All the 1 bits in the result of this operation will represent nodes from the current row which are matched to nodes from the previous row. After performing this match we will clear the matched 1 bits from the current row. The next step is to match nodes from the current row which are adjacent to each other faster than $O(N)$ time. In order to achieve this we will need to use a preprocessing step. For each sequence S of B bits we will compute $MCnt(S)$ that is the number of pairs of adjacent bits matched in S and $MRes(S)$ the B-bit sequence containing the remaining unmatched 1 bits of S. We will start with $MCnt(S) = 0$ and $MRes(S) = S$. Then we will traverse all the bits j of S from 1 to $B-1$. If $MRes(S)(j) = 1$ and $Mres(S)(j-1) = 1$ then we increase $MCnt(S)$ by 1 and we clear the bits j and $j-1$ in $MRes(S)$. Thus, we can compute $MCnt(S)$ and $MRes(S)$ in $O(B)$ time, obtaining a preprocessing time of $O(2^B \cdot B)$. Within the same time complexity we will also compute for each B-bit sequence S the number of 1 bits in S, $BCnt(S)$.

With these values we can perfom the matching on the current row y. We will consider each block i from 0 to $(N-1)/B$ and we will maintain the state of the current row as a sequence of blocks $crow$. First we copy $block(y, i)$ to $crow(i)$. Then, if $i > 0$ and bit $B-1$ of $crow(i-1)$ is 1 and bit 0 of $crow(i)$ is 1 we match these two bits and then we set them to zero. Afterwards we replace $crow(i)$ by $MRes(crow(i))$. The detailed algorithm is presented in Algorithm 2.

The time complexity of Algorithm 2 is $O(2^B \cdot B + M \cdot N/B)$. The $O(2^B \cdot B)$ term is the time complexity of the preprocessing stage and the $O(M \cdot N/B)$ is the time complexity of the actual algorithm. The presented algorithm can even be implemented in a parallel manner. First of all the preprocessing stage is obviously parallelizable: each of the 2^B values of the tables $MRes$, $MCnt$ and $BCnt$ ca be computed independently. In order to parallelize the actual algorithm we will need to refactor it first. We will first perform all the horizontal matchings on each of the M rows. We can first perform the matching within each block of each row independently in parallel and store the result in a variable specific to each $(row, block)$ pair (this means that we would have such a variable for each block of each row). Then we can handle the matching between bit 0 of odd-numbered blocks and bit $B-1$ of the preceding even-numbered block in parallel, followed by another stage in which we handle the matching between bit 0 of even-numbered blocks and bit $B-1$ of the preceding odd-numbered block in parallel. Then we can handle matchings between nodes on different rows. In order to parallelize this stage we will first consider all the rows corresponding to odd y coordinates being matched to the adjacent row with a smaller and even y coordinate. Obviously, each block of all of these $M/2$ (we consider integer division) rows can be handled independently in parallel. Then we will consider all the rows corresponding to even y coordinates being matched to the adjacent row with a smaller and odd y coordinate. Each block of these $M - M/2$ rows can also be handled independently, in parallel. The parallel algorithm presented here can use up to 2^B processors in the preprocessing stage and up to $M \cdot N/B$ processors in the maximal matching computation stage. Note that the result of the parallel version may differ from the result of the sequential algorithm (Algorithm 2) because a different maximal matching will be computed (due to the different order of performing the vertical and horizontal matches).

Algorithm 2 Greedy $O(2^B \cdot B + M \cdot N/B)$ Algorithm for Finding a Maximal Matching in an Induced Subgraph of a Grid Graph

```
Compute the tables MRes, MCnt and BCnt.
C = 0
prow(i) = 0 (0 ≤ i ≤ (N − 1)/B)
for y = 0 to M − 1 do
   crow(i) ← block(y, i) (0 ≤ i ≤ (N − 1)/B)
   for i = 0 to (N − 1)/B do
      vmatch(i) = crow(i) AND prow(i)
      C = C + BCnt(vmatch(i)))
      crow(i) = crow(i) XOR vmatch(i)
   end for
   C = C + MCnt(crow(0))
   crow(0) = MRes(crow(0))
   for i = 1 to (N − 1)/B do
      if bit B − 1 of crow(i − 1) is 1 and bit 0 of crow(i) is 1 then
         C = C + 1
         Clear bit B − 1 of crow(i − 1) and bit 0 of crow(i).
      end if
      C = C + MCnt(crow(i))
      crow(i) = MRes(crow(i))
   end for
   prow(i) = crow(i) (0 ≤ i ≤ (N − 1)/B)
end for
```

5. Using a Maximal Set of Edge-Disjoint Paths for Reducing the Number of Iterations of Minimum Cost Perfect Matching Algorithms

The best algorithm for computing a maximum matching in a sparse bipartite graph is the Hopcroft-Karp algorithm (Hopcroft & Karp, 1973) which has a time complexity of $O(E \cdot \sqrt{V})$. The main idea behind that algorithm is to enhance a standard augmenting path algorithm as follows. After each BFS traversal of the graph in order to find an augmenting path, the matching will not be increased only along one path, but rather along a maximal set of edge-disjoint shortest paths (note that in this case edge-disjoint paths are also vertex-disjoint paths, because they are paths in a shortest path tree; the only common vertex is the source S).

The same idea can be used when computing a minimum cost perfect matching. At each iteration of the *successive shortest path* algorithm (Todinov, 2013) we need to find a minimum cost path in the residual graph. Note that the residual graph may have arcs with negative costs, but does not have negative cycles. Thus, we either need to use a shortest path algorithm which supports negative costs (e.g. Bellman-Ford-Moore (Papaefthymiou & Rodrigue, 1997)) or we need to modify the costs in order to obtain non-negative costs only and, thus, use Dijkstra's algorithm (Todinov, 2013).

No matter what shortest path algorithm we use, at the end of the algorithm we have the minimum cost of a path starting at the source and ending at each node $x \in R$. We can sort all these nodes in ascending order of the cost of the minimum cost path to reach them (ignoring the unreachable nodes, if any). Then, rather than only increasing the matching along the minimum cost augmenting path, we can consider these nodes in sorted order. For each node x we trace back its shortest path to the source. If the matching was already augmented at the current iteration along at least one edge of the path, then we ignore node x and we move on to the next one. If the current path does not intersect with any of the paths along which the matching was augmented at the current iteration then we can augment the matching along this path and mark its edges in order to know that no other shortest path containing (some of) these edges can be used for augmenting the matching at the current iteration.

A direct implementation of this modified matching augmentation algorithm takes $O(V^2)$ time per iteration, because there may be $O(V)$ verified paths and each verification may take $O(V)$ time. On the other hand, we cannot guarantee that the matching will be augmented along more than one path. A scenario in which all the paths have the first edge in common (from the source to a vertex $x \in L$) and the minimum cost path has only this edge in common with the other paths is quite possible. Since $O(V^2)$ may be a higher time complexity than that of computing the minimum cost paths, we may end up increasing the running time of the algorithm instead of decreasing it. Thus, we need to reduce the time complexity of the matching augmentation part. This can be achieved as follows. Let's remember that the minimum cost paths are paths in a shortest path tree (where the length of a path is its cost). We will consider the paths in the same order as before and we will consider all the edges to be initially unmarked. If the last edge of the current path is not marked then we will be able to augment the matching along the current path. After augmenting the matching along the current path, let $x \in L$ be the first vertex on the path (after the source). We will traverse the whole subtree of the shortest path tree rooted at x and we will mark all the edges of this subtree. Augmenting the matching along all the possible paths takes at most $O(V)$ time overall (because the paths are edge-disjoint). Marking the edges of the shortest path tree also takes at most $O(V)$ time overall, because there are $O(V)$ edges in the shortest path tree and each edge is marked at most once. Thus, the matching augmentation algorithm takes only $O(V)$ time plus the time needed for sorting the paths in increasing order of their costs (e.g. $O(V \cdot log(V))$ time).

The first version of the algorithm proposed in this section is described in pseudocode in Algorithm 3. The input to the algorithm consists of two maps: *dist*, containing the cost of the shortest path from S to every vertex $x \in R$ (we consider $dist(x) = +\infty$ if the vertex x is not reachable from S), and *parent*, containing the *parent* in the shortest path tree for each vertex of the graph. Left-set vertices x are denoted as (x, L) in the algorithm and right-set vertices y are denoted as (y, R). In order to maintain the pseudocode simpler, we will mark the graph vertices instead of the edges (because, as mentioned earlier, in this case the edge-disjoint paths are also vertex-disjoint). The second version of the algorithm is described in pseudocode in Algorithm 4. The input to the Algorithm 4 also consists of the same two maps *dist* and *parent*, together with an extra map, *children*, which contains the children in the shortest path tree of each vertex of the graph.

This algorithm basically augments the matching along a maximal set of edge-disjoint paths (in fact, because they are paths of a shortest path tree, the paths are also vertex-disjoint except for the source vertex). It is important. however, to consider these paths in increasing order of their costs,

Algorithm 3 Increasing the Matching Along a Maximal Set of Edge-Disjoint Paths - The $O(V^2)$ Algorithm

Input: dist, parent.

Set all the vertices of the graph as unmarked.
Sort the vertices $x \in R$ in increasing order of $dist(x)$.
for $x \in R$ in increasing order of $dist(x)$ such that $dist(x) < +\infty$ **do**
  $y = (x, R)$, $OK = true$
  while $y \neq S$ **and** $OK = true$ **do**
    if vertex y is marked **then**
      $OK = false$
    else
      $y = parent(y)$
    end if
  end while
  if $OK = true$ **then**
    Increase the matching along the shortest path from S to (x, R) (the reverse of the path can be found by following the parent pointers starting from (x, R)).
    $y = (x, R)$
    while $y \neq S$ **do**
      Mark vertex y as marked.
      $y = parent(y)$
    end while
  end if
end for

Algorithm 4 Increasing the Matching Along a Maximal Set of Edge-Disjoint Paths - The $O(V)$ Algorithm

Input: dist, parent, children.

Set all the vertices of the graph as unmarked.
Sort the vertices $x \in R$ in increasing order of $dist(x)$.
for $x \in R$ in increasing order of $dist(x)$ such that $dist(x) < +\infty$ **do**
  if (x, R) is not marked **then**
    Increase the matching along the shortest path from S to (x, R) (the reverse of the path can be found by following the parent pointers starting from (x, R)).
    Let (y, L) be the first node on the path from S to (x, R) after S. Recursively mark all the vertices located in vertex (y, L)'s subtree of the shortest path tree. The *children* map will be used for retrieving for each vertex v the set $children(v)$ that is the set of the shortest path tree children of the vertex v.
  end if
end for

in order to make sure that the residual graph at the next iteration does not contain negative cycles. The reason for which this optimization works is as follows. In a perfect mtaching every vertex has to be matched. When augmenting the matching along a shortest path to a node x, even if x is not the right-side node with the minimum cost path, the point is that any future minimum cost path to node x (in any future residual graph) will not have a lower cost than the current shortest path to x. So there is no reason for us not to augment the matching along that path, as long as this does not block paths with lower costs along which the matching could have been augmented.

Note that this optimization is not correct in a maximum matching algorithm. It is not correct to augment the matching along a path which does not have the globally minimum cost, because we are not sure if the right side vertex x needs to be in the optimal matching or not. And since vertices added to the matching are never removed by this algorithm, it is possible to make a mistake in this case.

This optimization does not change the theoretical time complexity of the minimum cost perfect matching algorithm, because we cannot provide any extra guarantees regarding the number of augmenting paths per iteration (and, thus, we cannot provide guarantees regarding the reduction in the number of iterations).

6. Minimum Cost Perfect Matching With the Minimum Cost Edge Ignored

In this section we consider the following problem. Given a complete bipartite graph with n nodes on the left side and n nodes on the right side and costs on its edges, we want to find a minimum cost perfect matching in which the cost is defined as the sum of the costs of all the edges in the matching except for the cost of the edge with the smallest cost.

A simple method for solving this problem is to iterate over all the edges (i, j) and fix them as the smallest edge in the matching. Then we would compute a (usual) minimum cost maximum matching in the bipartite graph from which left node i, right node j and all the edges (i', j') with $c(i', j') < c(i, j)$ (or $c(i', j') = c(i, j)$ and the edge (i', j') was considered before the edge (i, j)) are removed. If the maximum matching M has size $n-1$ then we found a potential solution, as follows. The potential solution consists of the $n - 1$ edges of the found matching plus the edge (i, j). The cost of the matching (according to the definition used in this section) is equal to the sum of the costs of the $n - 1$ edges of M. Note that the fixed edge (i, j) is the edge with the minimum cost in the perfect matching (the one whose cost is not considered towards the cost of the matching). Once the edge (i, j) was fixed we needed to minimize the total cost of the other $n - 1$ edges of the perfect matching. Moreover, the other edges of the perfect matching needed to have costs which were larger than or equal to $c(i, j)$. The minimum cost maximum matching M contains the $n - 1$ edges we were looking for, in case its cardinality is $n - 1$. If its cardinality is less than $n - 1$ then we can conclude that there is no perfect matching containing the edge (i, j) as the minimum cost edge.

This solution requires the computation of $O(n^2)$ independent minimum cost maximum matchings. The key to obtain a better solution is to notice that the $O(n^2)$ matchings that we need to compute are not totally independent. We will sort all the n^2 edges first in ascending cost order and then we will consider them in this order. For the first edge we will compute the minimum cost maximum matching from scratch. Let's assume that we reached the edge (i, j). This time we will not compute the new matching from scratch. Instead, let's consider the matching M obtained for the previous edge in the sorted order. We will remove from M any edge with an endpoint at the left node i or the right node j (if any). Then we will remove from M all the edges with a cost smaller than $c(i, j)$ (or equal to $c(i, j)$ but for which the corresponding edge was considered before the current edge (i, j) in the sorted order). All the other edges of M will be maintained. We will start the (usual) minimum cost maximum matching for the edge (i, j) with all the remaining edges from M as part of the matching. Note that the algorithm may replace some of these edges by other edges. This can happen if the reverse of an edge (x, y) ($x \in L$ and $y \in R$) from M is, at some point, part of the shortest path from S to T in the (new) residual graph. When considering the maximum matching problem as a maximum flow problem, the fact that an edge (x, y) is removed from the matching means that the flow is pushed back along that edge (in order to be redirected somewhere else).

By applying the optimizations from the previous paragraph we expect that the number of iterations required for computing each new minimum cost maximum matching will be significantly reduced.

This problem can also be viewed as a dynamic minimum cost perfect matching problem, in which the edge costs can be modified (for instance, instead of removing edges from the graph we can consider that their cost increased to $+\infty$).

7. Experimental Results

We implemented the three optimizations presented in this paper and compared them against their unoptimized versions. All the tests were run on a machine running Windows 7 with an Intel Atom N450 1.66 GHz CPU and 1 GB RAM. All the algorithms were implemented in the C++ programming language and the code was compiled using the G++ compiler version 3.3.1.

First we tested our new algorithm for computing a maximal matching in an induced subgraph of a grid graph. We chose $M = N = 2048$ and we randomly generated the induced subgraph - each point (x, y) ($0 \leq x \leq N - 1$, $0 \leq y \leq M - 1$) had an equal probability of being part of the subgraph or not. Thus, each of the tested subgraphs had approximately 50% of the nodes of the full grid graph. We generated 100 subgraphs and ran Algorithm 1 (the unoptimized version) and Algorithm 2 (the optimized version) on each of them. We computed the total running time for all the graphs. Algorithm 1 took 5.3 seconds. For Algorithm 2 we considered two values for B: $B = 16$ and $B = 8$. For $B = 16$ the running time was 2.74 sec and for $B = 8$ it was 1.13 sec. Note that in this case we computed the tables *MCnt*, *Mres* and *BCnt* each time (i.e. for each of the 100 tests). However, when running the algorithm on multiple tests with the same value of B, these tables only need to be computed once, in the beginning. Thus, we changed the algorithm in order to compute these tables only once in the very beginning and not for each of the 100 tests. The new running times were 0.92 sec for $B = 16$ and 1.13 sec for $B = 8$. Note that the running time is unchanged for $B = 8$ because the sizes of the tables are small and the time needed to compute them is negligible compared to the time needed to compute the matching. However, for $B = 16$, when the sizes of the tables increase significantly, it is much better to compute the tables in the beginning and reuse them for each test.

Then we repeated the tests for induced subgraphs of grid graphs containing 75% and 100% of the nodes of a full grid graph. For graphs with 75% of the nodes of a full grid graph the running times of our optimized algorithm were: 1.21 sec for $B = 8$ and 0.93 sec for $B = 16$ (note that we only considered the case when the tables are computed just once). The running time of the unoptimized version was 4.69 sec. When considering the full grid graph we obtained the following running times: 0.90 sec for $B = 8$ and 0.55 sec for $B = 16$ for the optimized version and 3.40 sec for the unoptimized version.

We did not test other values of B because the implementation would become less feasible. For $B > 16$ the sizes of the precomputed tables would become too large. For $B = 8$ and $B = 16$ we were able to make use of existing C/C++ data types (*unsigned char* and *unsigned short int*) in order to store a block. For $B \neq 8$ and $B \neq 16$ (and $B \leq 16$) we cannot exactly fit a block into an existing C/C++ data type.

Second we tested the improvement brought by the use of multiple edge-disjoint paths for augmenting the matching in a minimum cost bipartite perfect matching algorithm. The expected improvement consisted in a reduction of the number of iterations. The standard algorithm would use n iterations where $|L| = |R| = n$. We chose $n = 256$ and we generated 100 complete bipartite graphs. The cost of each edge was chosen to be a random integer between 1 and 10000 (inclusive). The unoptimized algorithm we used was the standard *successive shortest path* algorithm with the Bellman-Ford-Moore algorithm for computing minimum cost paths at each iteration. The optimized algorithm simply included the $O(n^2)$ matching augmentation along multiple edge-disjoint minimum paths described in Section 5. We measured both the total running time and the total number of iterations. The total running time (for all the 100 graphs) and the total number of iterations of the standard algorithm were: 23.17 sec and 25600 iterations. With our optimization the total running time was 2.3 sec and the total number of iterations was 1022. We notice that, even with the most basic implementation of our optimization, the running time was reduced 10 times and the number of iterations was reduced 25 times. Although the running time improvements may not translate directly when other minimum cost path computation algorithms are used (e.g. Dijkstra's algorithm) or when the $O(V + V \cdot log(V))$ matching augmentation optimization is used (instead of the $O(V^2)$ version), the improvement in the number of iterations does not depend on these algorithms and, thus, it is applicable to any implementation of the *successive shortest path* minimum cost bipartite perfect matching algorithm.

Then we considered the same testing scenario, except that the edge costs were chosen as random integers between 1 and 2 (inclusive). The total running time of our optimized algorithm was 2.98 sec and the total number of iterations was 5100. The number of iterations of the standard algorithm remained the same (as expected), but its running time dropped to 13.83 sec.

We also considered the complete bipartite graph with the following costs $c(x, y) = min(x, y)$ $(1 \leq x, y \leq n)$ and $n = 256$. In this case our optimization did not reduce the number of iterations at all (due to the special structure of the bipartite graph it was never able to augment the matching along more than one path per iteration). However, the running time with our optimization enabled was almost identical to the unoptimized version. We conclude that our optimization has a great potential for reducing the number of iterations of the *successive shortest path* minimum cost perfect matching algorithm and even in the pathological cases when it cannot reduce the number of iterations, it doesn't cause any significant overhead.

Although there is a large difference between the number of iterations obtained by our optimization in the cases of random complete bipartite graphs and in the case of the specific bipartite graph from the previous paragraph, we did not consider other types of bipartite graphs for testing. Understanding the correlation between the performance of our optimization and the specific properties of the costs of the edges of the bipartite graph is an interesting topic, but we defer its study to a later date, because we feel that this topic is more appropriate for a separate, more experimentally focused, paper.

For the problem presented in Section 6 we tested our optimization of not recomputing the perfect matching from scratch each time. The first minimum cost perfect matching algorithm that we used was the one which contained our matching augmentation optimization presented in Section 5 and tested earlier. We generated 10 bipartite graphs with $n = 128$ and edge costs randomly selected between 1 and 10000 (inclusive). We computed the total execution time and the total number of iterations of the *successive shortest path* algorithm. When the matching was computed from scratch each time ($O(n^2)$ times) the total running time was 805 sec and the total number of iterations was 1305200. When we applied our optimization from Section 6, the total running time was 221 sec and the total number of iterations was 244886. When using the standard *successive shortest path* algorithm in order to compute a minimum cost perfect matching (and not using our optimization of not recomputing the matching from scratch each time), the total running time was 5798 sec and the total number of iterations was 20610157. When we applied our optimization from Section 6 and also used the standard minimum cost perfect matching algorithm the total running time was 232 sec and the total number of iterations was 251789. We can see that our optimization of not recomputing each minimum cost perfect matching from scratch is very effective. When combined with the optimization presented in Section 5, of augmenting the matching along multiple paths at each iteration, we obtained the best results. However, even when just the standard *successive shortest path* minimum cost perfect matching algorithm is used in conjunction with our optimization from Section 6 the improvements over the naive unoptimized version are significant. Nevertheless, more tests may need to be performed in the future in order to understand sufficiently well how good our proposed optimization really is.

8. Conclusions

In this paper we presented three practical algorithmic optimizations addressing problems like computing maximal matchings in induced subgraphs of grid graphs or computing minimum cost perfect matchings in bipartite graphs (under certain restrictions). The proposed optimizations were evaluated experimentally and compared against the unoptimized algorithms. The execution time was significantly reduced in each case, thus proving the validity and effectiveness of our optimizations.

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Reich Type Contractions on Cone Rectangular Metric Spaces Endowed with a Graph

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Abstract

In this paper we prove some fixed point theorems for Reich type contractions on cone rectangular metric spaces endowed with a graph without assuming the normality of cone. The results of this paper extends and generalize several known results from metric, rectangular metric, cone metric and cone rectangular metric spaces in cone rectangular metric spaces endowed with a graph. Some examples are given which illustrate the results.

Keywords: Graph, cone rectangular metric space, Reich type contraction, fixed point.
2010 MSC: 47H10, 54H25.

1. Introduction

In 1906, the French mathematician M. Fréchet Fréchet (1906) introduced the concept of metric spaces. After the work of Fréchet several authors generalized the concept of metric space by applying the conditions on metric function. In this sequel, Branciari Branciari (2000) introduced a class of generalized (rectangular) metric spaces by replacing triangular inequality of metric spaces by similar one which involves four or more points instead of three and improved Banach contraction principle Banach (1922) in such spaces. The result of Branciari is generalized and extended by several authors (see, for example, Flora *et al.* (2009); Bari & Vetro (2012); Chen (2012); Işik & Turkoglu (2013); Lakzian & Samet (2012); Arshad *et al.* (2013); Malhotra *et al.* (2013*a*,*b*) and the references therein).

Let (X, d) be a metric space and $T\colon X \to X$ be a mapping. Then T is called a Banach contraction if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \le \lambda d(x, y) \quad \text{for all} \quad x, y \in X. \tag{1.1}$$

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Banach contraction principle ensures the existence of a unique fixed point of a Banach contraction on a complete metric space.

Kannan Kannan (1968) introduced the following contractive condition: there exists $\lambda \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \quad \text{for all} \quad x, y \in X. \tag{1.2}$$

Reich Reich (1971) introduced the following contractive condition: there exist nonnegative constants λ, μ, δ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \leq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty) \quad \text{for all} \quad x, y \in X. \tag{1.3}$$

Examples show that (see Kannan (1968); Reich (1971)) the conditions of Banach and Kannan are independent of each other while the condition of Reich is a proper generalization of conditions of Banach and Kannan.

On the other hand, the study of abstract spaces and the vector-valued spaces can be seen in Kurepa (1934, 1987); Rzepecki (1980); Lin (1987); Zabreǐko (1997). L.G. Huang and X. Zhang Huang & Zhang (2007) reintroduced such spaces under the name of cone metric spaces and generalized the concept of a metric space, replacing the set of real numbers, by an ordered Banach space. After the work of Huang and Zhang Huang & Zhang (2007), Azam et al. Azam *et al.* (2009) introduced the notion of cone rectangular metric spaces and proved fixed point result for Banach type contraction in cone rectangular space. Malhotra et al. Malhotra *et al.* (2013*b*) generalized the result of Azam et al. Azam *et al.* (2009) in ordered cone rectangular metric spaces and proved some fixed point results for ordered Reich type contractions.

Recently, Jachymski Jachymski (2007) improved the Banach contraction principle for mappings on a metric space endowed with a graph. Jachymski Jachymski (2007) showed that the results of Ran and Reurings Ran & Reurings (2004) and Edelstein Edelstein (1961) can be derived by the results of Jachymski (2007). The results of Jachymski Jachymski (2007) was generalized by several authors (see, for example, Bojor (2012); Chifu & Petrusel (2012); Samreen & Kamran (2013); Asl *et al.* (2013); Abbas & Nazir (2013) and the references therein).

The fixed point results in cone rectangular metric spaces (also in rectangular metric spaces) endowed with a graph are not considered yet. In this paper, we prove some fixed point theorems for Reich type contractions on the cone rectangular metric spaces endowed with a graph. Our results extend the result of Jachymski Jachymski (2007) and the result of Malhotra et al. Malhotra *et al.* (2013*b*) into the cone rectangular metric spaces endowed with a graph. Some examples are provided which illustrate the results.

2. Preliminaries

First we recall some definitions about the cone rectangular metric spaces and graphs.

Definition 2.1. Huang & Zhang (2007) Let E be a real Banach space and P be a subset of E. The set P is called a cone if:
(i) P is closed, nonempty and $P \neq \{\theta\}$, here θ is the zero vector of E;
(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.
Given a cone $P \subset E$, we define a partial ordering " $\leq$ " with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \leq y$ but $x \neq y$. While $x \ll y$ if and only if $y - x \in P^0$, where P^0 denotes the interior of P.
Let P be a cone in a real Banach space E, then P is called normal, if there exist a constant $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 2.2. Huang & Zhang (2007) Let X be a nonempty set, E be a real Banach space. Suppose that the mapping $d : X \times X \to E$ satisfies:

(i) $\theta \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. In the following we always suppose that E is a real Banach space, P is a solid cone in E, i.e., $P^0 \neq \phi$ and " $\leq$ " is partial ordering with respect to P.

For examples and basic properties of normal and non-normal cones and cone metric spaces we refer Huang & Zhang (2007) and Rzepecki (1980).
The following remark will be useful in sequel.

Remark. Jungck *et al.* (2009) Let P be a cone in a real Banach space E, and $a, b, c \in P$, then:

(a) If $a \leq b$ and $b \ll c$ then $a \ll c$.
(b) If $a \ll b$ and $b \ll c$ then $a \ll c$.
(c) If $\theta \leq u \ll c$ for each $c \in P^0$ then $u = \theta$.
(d) If $c \in P^0$ and $a_n \to \theta$ then there exist $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ we have $a_n \ll c$.
(e) If $\theta \leq a_n \leq b_n$ for each n and $a_n \to a,\ b_n \to b$ then $a \leq b$.
(f) If $a \leq \lambda a$ where $0 \leq \lambda < 1$ then $a = \theta$.

Definition 2.3. Azam *et al.* (2009) Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$, satisfies:

(i) $\theta \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [rectangular property].

Then d is called a cone rectangular metric on X, and (X, d) is called a cone rectangular metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in (X, d)$. If for every $c \in E$, with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_n x_n = x$ or $x_n \to x$, as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $m \in \mathbb{N}$ we have $d(x_n, x_{n+m}) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone rectangular metric space. If the underlying cone is normal then (X, d) is called normal cone rectangular metric space.

The concept of cone metric space is more general than that of a metric space, because each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 2.1. Let $X = \mathbb{N}, E = \mathbb{R}^2$, $\alpha, \beta > 0$ and $P = \{(x, y) : x, y \geq 0\}$.
Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y, \\ 3(\alpha,\beta) & \text{if } x \text{ and } y \text{ are in } \{1,2\}, x \neq y, \\ (\alpha,\beta) & \text{otherwise.} \end{cases}$$

Now (X, d) is a cone rectangular metric space but (X, d) is not a cone metric space because it lacks the triangular property:

$$3(\alpha,\beta) = d(1,2) > d(1,3) + d(3,2) = (\alpha,\beta) + (\alpha,\beta) = 2(\alpha,\beta),$$

as $3(\alpha,\beta) - 2(\alpha,\beta) = (\alpha,\beta) \in P$.

Note that in above example (X, d) is a normal cone rectangular metric space. Following is an example of non-normal cone rectangular metric space.

Example 2.2. Let $X = \mathbb{N}, E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Then this cone is not normal (see Rezapour & Hamlbarani (2008)).
Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} \theta & \text{if } x = y, \\ 3e^t & \text{if } x, y \in \{1,2\}, x \neq y, \\ e^t & \text{otherwise.} \end{cases}$$

Then (X, d) is non-normal cone rectangular metric space but (X, d) is not a cone metric space because it lacks the triangular property.

Now we recall some basic notions from graph theory which we need subsequently (see also Jachymski (2007)).

Let X be a nonempty set and Δ denote the diagonal of the cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the rectangular distance between its vertices.

By G^{-1} we denote the conversion of a graph G, that is, the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}). \tag{2.1}$$

If x and y are vertices in a graph G, then a path in G from x to y of length l is a sequence $(x_i)_{i=0}^{l}$ of $l+1$ vertices such that $x_0 = x, x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \dots, l$. A graph G is called connected if there is a path between any two vertices of G. G is weakly connected if $\widetilde{G}$ is connected.

Throughout this paper we assume that X is nonempty set, G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.

Now we define the G-Reich contractions in a cone rectangular metric space.

Definition 2.4. Let (X, d) be a cone rectangular metric space endowed with a graph G. A mapping $T: X \to X$ is said to be a G-Reich contraction if:

(GR1) T is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

(GR2) there exist nonnegative constants λ, μ, δ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty) \tag{2.2}$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

An obvious consequence of symmetry of $d(\cdot, \cdot)$ and (2.1) is the following remark.

Remark. If T is a G-Reich contraction then it is both a G^{-1}-Reich contraction and a $\widetilde{G}$-Reich contraction.

Example 2.3. Any constant function $T: X \to X$ defined by $Tx = c$, where $c \in X$ is fixed, is a G-Reich contraction since $E(G)$ contains all the loops.

Example 2.4. Any Reich contraction on a X is a G_0-Reich contraction, where $E(G_0) = X \times X$.

Example 2.5. Let (X, d) be a cone rectangular metric space, $\sqsubseteq$ a partial order on X and $T: X \to X$ be an ordered Reich contraction (see Malhotra *et al.* (2013*b*)), that is, there exist nonnegative constants λ, μ, δ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty)$$

for all $x, y \in X$ with $x \sqsubseteq y$,. Then T is a G_1-Reich contraction, where $E(G_1) = \{(x, y) \in X \times X : x \sqsubseteq y\}$.

Definition 2.5. Let (X, d) be a cone rectangular metric space and $T: X \to X$ be a mapping. Then for $x_0 \in X$, a Picard sequence with initial value x_0 is defined by $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. The mapping T is called a Picard operator on X if T has a unique fixed point in X and for all $x_0 \in X$ the Picard sequence $\{x_n\}$ with initial value x_0 converges to the fixed point of T. The mapping T is called weakly Picard operator, if for any $x_0 \in X$, the limit of Picard sequence $\{x_n\}$ with initial value x_0, that is, $\lim_{n\to\infty} x_n$ exits (it may depend on x_0) and it is a fixed point of T.

Now we can state our main results.

3. Main results

Let (X, d) be a cone rectangular metric space, and G be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. The set of all fixed points of a self mapping T of X is denoted by Fix$T$, that is, Fix$T = \{x \in X : Tx = x\}$ and the set of all periodic points of T is denoted by $P(T)$, that is, $P(T) = \{x \in X : T^n x = x, \text{ for some } n \in \mathbb{N}\}$. Also we use the notation $X_T = \{x \in X : (x, Tx), (x, T^2 x) \in E(G)\}$. (X, d) is said to have the property (P) if:

whenever a sequence $\{x_n\}$ in X converges to x with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there is a subsequence $\{x_{k_n}\}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in \mathbb{N}$. (P)

Proposition 3.1. *Let (X, d) be a cone rectangular metric space endowed with a graph G and $T : X \to X$ be a G-Reich contraction. Then, if $x, y \in FixT$ are such $(x, y) \in E(G)$ then $x = y$.*

Proof. Let $x, y \in$ Fix$T$ and $(x, y) \in E(G)$, then by (GR2) we have

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty) \\ &= \lambda d(x, y) + \mu d(x, x) + \delta d(y, y) = \lambda d(x, y). \end{aligned}$$

As $\lambda < 1$, by (f) of Remark 2, we have $d(x, y) = \theta$, that is, $x = y$. □

Theorem 3.1. *Let (X, d) be a cone rectangular metric space endowed with a graph G. Let $T : X \to X$ be a G-Reich contraction. Then for every $x_0 \in X_T$ the Picard sequence $\{x_n\}$, is a Cauchy sequence.*

Proof. Let $x_0 \in X_T$ and define the iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. Since $x_0 \in X_T$ we have $(x_0, Tx_0) \in E(G)$ and T is a G-Reich contraction, by (GR1) we have $(Tx_0, T^2 x_0) = (x_1, x_2) \in E(G)$. By induction we obtain $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$.

Now since $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$ by (GR2) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq \lambda d(x_{n-1}, x_n) + \mu d(x_{n-1}, Tx_{n-1}) + \delta d(x_n, Tx_n) \\ &= \lambda d(x_{n-1}, x_n) + \mu d(x_{n-1}, x_n) + \delta d(x_n, x_{n+1}), \end{aligned}$$

that is,

$$d(x_n, x_{n+1}) \preceq \frac{\lambda + \mu}{1 - \delta} d(x_{n-1}, x_n) = \alpha d(x_{n-1}, x_n),$$

where $\alpha = \dfrac{\lambda + \mu}{1 - \delta} < 1$ (as $\lambda + \mu + \delta < 1$). Setting $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$, we obtain by induction that

$$d_n \preceq \alpha^n d_0 \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Note that, if $x_0 \in P(T)$ then there exists $k \in \mathbb{N}$ such that $T^k x_0 = x_k = x_0$ and by (3.1) we have

$$d_0 = d(x_0, x_1) = d(x_0, Tx_0) = d(x_k, Tx_k) = d(x_k, x_{k+1}) \preceq \alpha^k d(x_0, x_1) = \alpha^k d_0.$$

Since $\lambda \in [0,1)$ the above inequality yields a contradiction. Thus, we can assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.
As $x_0 \in X_T$ we have $(x_0, T^2x_0) = (x_0, x_2) \in E(G)$ and by (GR1) we obtain $(Tx_0, Tx_2) = (x_1, x_3) \in E(G)$. By induction we obtain $(x_n, x_{n+2}) \in E(G)$ for all $n \geq 0$. Therefore it follows from (GR2) that

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq d(Tx_{n-1}, Tx_{n+1}) \\ &\preceq \lambda d(x_{n-1}, x_{n+1}) + \mu d(x_{n-1}, Tx_{n-1}) + \delta d(x_{n+1}, Tx_{n+1}) \\ &\preceq \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+1})] + \mu d(x_{n-1}, x_n) \\ &\quad +\delta d(x_{n+1}, x_{n+2}), \end{aligned}$$

that is,

$$d(x_n, x_{n+2}) \preceq \frac{\lambda+\mu}{1-\lambda} d_{n-1} + \frac{\lambda+\delta}{1-\lambda} d_{n+1}$$

which together with (3.1) yields

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq \frac{\lambda+\mu+[\lambda+\delta]\alpha^2}{1-\lambda}\alpha^{n-1} d_0 \\ &\preceq \frac{2\lambda+\mu+\delta}{1-\lambda}\alpha^{n-1} d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2}) \preceq \beta\alpha^{n-1} d_0, \tag{3.2}$$

where $\beta = \frac{2\lambda+\mu+\delta}{1-\lambda} \succeq 0$. We shall show that the sequence $\{x_n\}$ is a Cauchy sequence.
We consider the value of $d(x_n, x_{n+p})$ in two cases.
If p is odd, say $2m+1$, then using rectangular inequality and (3.1) we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\preceq d(x_{n+2m}, x_{n+2m+1}) + d(x_{n+2m-1}, x_{n+2m}) + d(x_n, x_{n+2m-1}) \\ &= d_{n+2m} + d_{n+2m-1} + d(x_n, x_{n+2m-1}) \\ &\preceq d_{n+2m} + d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_n \\ &\preceq \alpha^{n+2m} d_0 + \alpha^{n+2m-1} d_0 + \alpha^{n+2m-2} d_0 + \cdots + \alpha^n d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2m+1}) \preceq \frac{\alpha^n}{1-\alpha} d_0. \tag{3.3}$$

If p is even, say $2m$, then using rectangular inequality, (3.1) and (3.2) we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) &\preceq d(x_{n+2m-1}, x_{n+2m}) + d(x_{n+2m-1}, x_{n+2m-2}) + d(x_n, x_{n+2m-2}) \\ &= d_{n+2m-1} + d_{n+2m-2} + d(x_n, x_{n+2m-2}) \\ &\preceq d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_{n+2} + d(x_n, x_{n+2}) \\ &\preceq \alpha^{n+2m-1} d_0 + \alpha^{n+2m-2} d_0 + \alpha^{n+2m-3} d_0 + \cdots + \alpha^{n+2} d_0 + \beta\alpha^{n-1} d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2m}) \preceq \frac{\alpha^n}{1-\alpha} d_0 + \beta\alpha^{n-1} d_0. \tag{3.4}$$

Since $\beta \geq 0$ and $\alpha < 1$, we have $\frac{\alpha^n}{1-\alpha}d_0, \beta\alpha^{n-1}d_0 \to \theta$ as $n \to \infty$ so it follows from (3.3), (3.4) and (a), (d) of Remark 2 that: for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+p}) \ll c \quad \text{for all} \quad p \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. □

Theorem 3.2. *Let (X, d) be a complete cone rectangular metric space endowed with a graph G and has the property (P). Let $T: X \to X$ be a G-Reich contraction such that $X_T \neq \emptyset$, then T is a weakly Picard operator.*

Proof. If $X_T \neq \emptyset$ then let $x_0 \in X_T$. By Theorem 3.1, the Picard sequence $\{x_n\}$, where $x_n = T^{n-1}x_0$ for all $n \in \mathbb{N}$, is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that

$$x_n \to u \quad \text{as} \quad n \to \infty. \tag{3.5}$$

We shall show that u is a fixed point of T. By Theorem 3.1 we have $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$, $d_n \preceq d(x_n, x_{n+1}) \preceq \alpha^n d_0$, where $\alpha = \frac{\lambda+\mu}{1-\delta} < 1$ and by the property (P) there exists a subsequence $\{x_{k_n}\}$ such that $(x_{k_n}, u) \in E(G)$ for all $n \in \mathbb{N}$. Also, we can assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. So, using (2.2) we have

$$\begin{aligned} d(u, Tu) &\preceq d(u, x_{k_n}) + d(x_{k_n}, x_{k_n+1}) + d(x_{k_n+1}, Tu) \\ &= d(u, x_{k_n}) + d_{k_n} + d(Tx_{k_n}, Tu) \\ &\preceq d(u, x_{k_n}) + \alpha^{k_n}d_0 + \lambda d(x_{k_n}, u) + \mu d(x_{k_n}, Tx_{k_n}) + \delta d(u, Tu) \\ &\preceq (1+\lambda)d(u, x_{k_n}) + (1+\mu)\alpha^{k_n}d_0 + \delta d(u, Tu), \end{aligned}$$

that is,

$$d(u, Tu) \preceq \frac{1+\lambda}{1-\delta}d(x_{k_n}, u) + \frac{1+\mu}{1-\delta}\alpha^{k_n}d_0 \tag{3.6}$$

Since $\alpha^{k_n}d_0 \to \theta$, $x_n \to u$ as $n \to \infty$ we can choose $n_0 \in \mathbb{N}$ such that, for every $c \in E$ with $\theta \ll c$ we have $d(x_{k_n}, u) \ll \dfrac{1-\delta}{2(1+\lambda)}c$ and $\alpha^{k_n}d_0 \ll \dfrac{1-\delta}{2(1+\mu)}c$ for all $n > n_0$. Therefore, it follows from (3.6) that: for every $c \in E$ with $\theta \ll c$ we have

$$d(u, Tu) \ll c \quad \text{for all} \quad n > n_0.$$

So, by (c) of Remark 2, we have $d(u, Tu) = \theta$, that is, $Tu = u$ theerefore $u \in \text{Fix}T$. Thus T is a weakly Picard operator. □

In the above theorem the mapping T is not necessarily a Picard operator. Indeed, such mapping T may has infinitely many fixed points. Following example verifies this fact.

Example 3.1. Let $X = \mathbb{N} = \bigcup_{k \in \mathbb{N}_0} N_k$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $N_k = \left\{2^k(2n-1) : n \in \mathbb{N}\right\}$ for all $k \in \mathbb{N}_0$. Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Let $d: X \times X \to E$ be defined by

$$d(x, y) = \begin{cases} \theta & \text{if } x = y, \\ 3e^t & \text{if } x, y \in \{1, 2\}, x \neq y, \\ e^t & \text{otherwise.} \end{cases}$$

Then (X, d) is a cone rectangular metric space endowed with graph G, where

$$E(G) = \Delta \bigcup_{k\in\mathbb{N}_0\setminus\{1\}} (N_k \times N_k) \bigcup \{(1, x) \colon x \in N_1\}.$$

Note that (X, d) is not a cone metric space. Define a mapping $T \colon X \to X$ by

$$Tx = \begin{cases} 2^k, & \text{if } x \in N_k, k \in \mathbb{N}_0 \setminus \{1\}; \\ 6, & \text{if } x = 2; \\ 1, & \text{if } x \in N_1 \setminus \{2\}. \end{cases}$$

Then it is easy to see that T is a G-Reich contraction with $\lambda \in [1/3, 1), \mu = \delta = 0$. All the conditions of Theorem 3.2 are satisfied and T has infinitely many fixed points, precisely $\mathrm{Fix}T = \left\{2^k \colon k \in \mathbb{N}_0 \setminus \{1\}\right\}$, therefore T is not a Picard operator but weakly Picard operator. Note that, if a Reich contraction on a cone rectangular metric space has a fixed point then it is unique therefore T is not a Reich contraction in (X, d) since $\mathrm{Fix}T$ is not singleton.

Remark. Unlike from Reich contraction, the above example shows that there may be more than one fixed points of a G-Reich contraction in a cone rectangular metric space and therefore a G-Reich contraction in a cone rectangular space need not be a Picard operator.

In following theorem we give a necessary and sufficient condition for T to be a Picard operator.

Theorem 3.3. *Let (X, d) be a complete cone rectangular metric space endowed with a graph G and has the property (P). Let $T \colon X \to X$ be a G-Reich contraction such that $X_T \neq \emptyset$, then T is a weakly Picard operator. Furthermore, the subgraph G_{Fix} defined by $V(G_{Fix}) = FixT$ is weakly connected if and only if T is a Picard operator.*

Proof. The existence of fixed point follows from Theorem 3.2. Let $u, v \in \mathrm{Fix}T$, then since G_{Fix} is weakly connected there exists a path $(x_i)_{i=0}^{l}$ in G_{Fix} from u to v, that is, $x_0 = u, x_l = v$ and $(x_{i-1}, x_i) \in E(G_{\mathrm{Fix}})$ for $i = 1, 2, \ldots, l$. Therefore by Proposition 3.1 and Remark 2 we obtain $u = v$. Thus, fixed point is unique and T is a Picard operator. □

Remark. In Jachymski (2007), for T to be a Picard operator Jachymski assumed that G must be weakly connected. From the above theorem it is clear that for T to be a Picard operator it is sufficient to take that $\mathrm{Fix}T$ is weakly connected. Next example will illustrate this fact.

Example 3.2. Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$, $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Let $d \colon X \times X \to E$ be defined by

$$d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{3}{10}e^t, \qquad d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{1}{5}e^t,$$

$$d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{3}\right) = \frac{3}{5}e^t, \qquad d(x, x) = \theta = 0 \quad \text{for all} \quad x \in X,$$

$$d\left(1, \frac{1}{n}\right) = \frac{n-1}{n}e^t \text{ for } n = 2, 3, 4, 5, \quad d(x, y) = d(y, x) \quad \text{for all} \quad x, y \in X,$$

Then (X, d) is a cone rectangular metric space endowed with graph G, where

$$E(G) = \Delta \cup \left\{ \left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{3}\right) \right\}.$$

Note that (X, d) is not a cone metric space. Define $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x = \frac{1}{2}, \frac{1}{5}; \\ \frac{1}{5}, & \text{if } x = \frac{1}{3}; \\ 1, & \text{if } x = \frac{1}{4}; \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Then T is a G-Reich contraction with $\lambda \in \left[\frac{2}{3}, 1\right), \mu = \delta = 0$. All the conditions of Theorem 3.3 are satisfied and T is a Picard operator and $\mathrm{Fix}T = \left\{\frac{1}{2}\right\}$. Note that the graph G is not weakly connected. Indeed, there is no path from 1 to $\frac{1}{n}$ or form $\frac{1}{n}$ to 1 for all $n = 2, 3, 4, 5$. Also, one can see that T is neither a Reich contraction in cone rectangular metric space (X, d) nor a G-Reich contraction with respect to the usual metric.

With suitable values of constants λ, μ and δ we obtain the following corollaries.

Corollary 3.1. *Let (X, d) be a complete cone rectangular metric space endowed with a graph G and has the property (P). Let $T : X \to X$ be a G-contraction, that is,*

(G1) *T is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
(G2) *there exists $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \preceq \lambda d(x, y) \text{ for all } x, y \in X \text{ with } (x, y) \in E(G).$$

Then, if $X_T \neq \emptyset$ then T is a weakly Picard operator. Furthermore, the subgraph G_{Fix} defined by $V(G_{Fix}) = FixT$ is weakly connected if and only if T is a Picard operator.

Corollary 3.2. *Let (X, d) be a complete cone rectangular metric space endowed with a graph G and has the property (P). Let $T : X \to X$ be a G-Kannan contraction, that is,*

(GK1) *T is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
(GK2) *there exists $\lambda \in [0, 1/2)$ such that*

$$d(Tx, Ty) \preceq \lambda[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X \text{ with } (x, y) \in E(G).$$

Then, if $X_T \neq \emptyset$ then T is a weakly Picard operator. Furthermore, the subgraph G_{Fix} defined by $V(G_{Fix}) = FixT$ is weakly connected if and only if T is a Picard operator.

Following corollary is a fixed point result for an ordered Reich contraction (see Malhotra *et al.* (2013*b*)) and a generalization of result of Ran and Reurings Ran & Reurings (2004) in cone rectangular metric spaces.

Corollary 3.3. *Let (X, d) be a complete cone rectangular metric space endowed with a partial order $\sqsubseteq$ and $T: X \to X$ be a mapping. Suppose the following conditions hold:*

(A) *T is an ordered Reich contraction;*
(B) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;*
(C) *T is nondecreasing with respect to $\sqsubseteq$;*
(D) *if $\{x_n\}$ is a nondecreasing sequence in X and converging to some z, then $x_n \sqsubseteq z$.*

Then T is a weakly Picard operator. Furthermore, $FixT$ is well ordered (that is, all the elements of $FixT$ are comparable) if and only if T is a Picard operator.

Proof. Let G be a graph defined by $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \sqsubseteq y\}$. Then by conditions (A) and (C), T is a G-Reich contraction and by condition (B) we have $X_T \neq \emptyset$. Also by condition (D) we see that property (P) is satisfied. Now proof follows from Theorem 3.3. □

Conclusion. In the present paper we have proved the existence and uniqueness of fixed point theorems for a G-Reich contraction in cone rectangular metric spaces endowed with a graph. We note that the results of this paper generalize the ordered version of theorem of Reich (see Reich (1971) and Malhotra *et al.* (2013*b*)). Note that, in usual metric spaces the fixed point theorem for G-contractions generalizes and unifies the ordered version as well as the cyclic version of corresponding fixed point theorems (see Kirk *et al.* (2003) and Kamran *et al.* (2013)). We conclude with an open problem that: is it possible to prove the cyclic version of the result of Reich in cone rectangular metric spaces or rectangular metric spaces?

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Hadamard Product of Simple Sets of Polynomials in $\mathbb{C}^n$

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Abstract

In this paper we give some convergence properties of Hadamard product set of polynomials defined by several simple monic sets of several complex variables in complete Reinhardt domains and in hyperelliptical regions too.

Keywords: Basic sets of polynomials, Hadamard product, complete Reinhardt domains, hyperelliptical regions.
2010 MSC: 47H10, 54H25.

1. Introduction

In 1933, Whittaker (Whittaker, 1933), (Whittaker, 1949) introduced the subject of basic sets of polynomials of a single complex variable. This subject is developed by several authors using one and several complex variables. It is of fundamental importance in the theory of basic sets of polynomials of several complex variables to define some kinds of basic sets of polynomials in $\mathbb{C}^n$. This is the main aim of this paper. We will define and study Hadamard products of basic sets of polynomials in complete Reinhardt domains and in hyperelliptical regions.

We start with basic concepts, notations and terminology on this paper.

Let $\mathbb{C}$ represent the field of complex variables. In the space $\mathbb{C}^2$ of the two complex variables z and w, the successive monomial $1, z, w, z^2, zw, w^2, ...$ are arranged so that the enumeration number of the monomial $z^j w^k$ in the above sequence is

$$\frac{1}{2}(j+k)(j+k)+k; \quad j,k \geq 0.$$

The enumeration number of the last monomial of a polynomial $P(z,w)$ in two complex variables is called the degree of the polynomial. A sequence $\{P_i(z;w)\}_0^\infty$ of polynomials in two complex variables in which the order of each polynomial is equal to its degree is called a simple set

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(see (Kishka, 1993), (Kumuyi & Nassif, 1986) and (Sayyed & Metwally, 1998)). Such a set is conveniently denoted by $\{P_i(z;w)\}$, where the last monomial in $P_{m,n}(z,w)$ is $z^m w^n$.

If further, the coefficient of this last monomial is 1, the simple set is termed monic. Thus, in the simple monic set $\{P_{m,n}(z;w)\}$ the polynomial $P_{m,n}(z,w)$ is represented as follows.

$$P_{m,n}(z,w) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} P_{k-j,j}^{m,n} z^{k-j} w^j \quad (P_{m,n}^{m,n} = 1; P_{m+n-j,j}^{m,n} = 0, j > n).$$

Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables. The following definition is introduced in (Mursi & Makar, 1955*a*,*b*).

Definition 1.1. A set of polynomials $\{P_{\mathbf{m}}[\mathbf{z}]\} = \{P_0, P_1, P_2, ..., P_n, ...\}$ is said to be basic when every polynomial in the complex variables z_s; $s \in I = \{1, 2, 3, ..., n\}$ can be uniquely expressed as a finite linear combination of the elements of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus according to (Mursi & Makar, 1955*b*), the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if and only if there exists a unique row-finite matrix $\overline{P}$ such that $\overline{P}P = P\overline{P} = \mathbf{I}$, where $P = [P_{\mathbf{m},\mathbf{h}}]$ is the matrix of coefficients, $\overline{P}$ is the matrix of operators of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and $\mathbf{I}$ is the infinite unit matrix.

Similar definition for a simple monic set can be extended to the case of several complex variables by replacing m, n by $(\mathbf{m}) = (m_1, m_2, m_3, ..., m_n)$, j, k by $(\mathbf{h}) = (h_1, h_2, h_3, ..., h_n)$ and z, w by $\mathbf{z}$, where each of $(\mathbf{m})$ and $(\mathbf{h})$ be multi-indices of non-negative integers.

The fact that the simple monic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of several complex variables is necessarily basic follows from the observation that the matrix $[P_{\mathbf{m},\mathbf{h}}]$ of coefficients of the polynomials of the set is a lower triangular matrix with non-zero diagonal elements. (These elements are each equal to 1 for monic sets).

Definition 1.2. The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be algebraic of degree ℓ when its matrix of coefficients P satisfies the usual identity

$$\alpha_0 P^{\ell} + \alpha_1 P^{\ell-1} + ... + \alpha_{\ell} I = 0.$$

Thus, we have a relation of the form

$$\overline{P}_{\mathbf{m},\mathbf{h}} = \delta_{\mathbf{m},\mathbf{h}}\gamma_0 + \sum_{s_1=1}^{\ell-1} \gamma_{s_1} P_{\mathbf{m},\mathbf{h}}^{(s_1)},$$

where $P_{\mathbf{m},\mathbf{h}}^{(s_1)}$ are the elements of the power matrix P^{s_1} and $\gamma_{s_1}, \quad s_1 = 0, 1, 2, ..., \ell - 1$ are constant numbers. In the space of several complex variables $\mathbb{C}^n$. Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables, a closed complete Reinhardt domain of radii $\alpha_s r (> 0)$; $s \in I = \{1, 2, 3, ..., n\}$ is here denoted by $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ and is given by

$\bar{\Gamma}_{[\alpha\mathbf{r}]} = \bar{\Gamma}_{\left[\alpha_1 r, \alpha_2 r, ..., \alpha_n r\right]} = \{\mathbf{z} \in \mathbb{C}^n : |z_s| \le \alpha_s r \quad ; s \in I\}$, where α_s are positive numbers. The open complete Reinhardt domain is here denoted by $\Gamma_{[\alpha\mathbf{r}]}$ and is given by

$\Gamma_{[\alpha\mathbf{r}]} = \Gamma_{\left[\alpha_1 r, \alpha_2 r, ..., \alpha_n r\right]} = \{\mathbf{z} \in \mathbb{C}^n : |z_s| < \alpha_s r \quad ; s \in I\}$.

Consider unspecified domain containing the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha\mathbf{r}]}$. This domain will be of radii $\alpha_s r_1$; $r_1 > r$, then making a contraction to this domain, we will get the domain $\bar{D}([\alpha\mathbf{r}^+]) = \bar{D}([\alpha_1 r^+, \alpha_2 r^+, ..., \alpha_n r^+])$, where r^+ stands for the right-limit of r_1 at r.

Now let $\mathbf{m} = (m_1, m_2, ..., m_n)$ be multi-indices of non-negative integers. The entire function $f(\mathbf{z})$ of several complex variables has the following representation:

$$f(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}.$$

Suppose now that the function $f(\mathbf{z})$, is given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$$

is regular in $\bar{\Gamma}_{[\alpha\mathbf{r}]}$ and

$$M[f; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha\mathbf{r}]}} \left| f(\mathbf{z}) \right|.$$

For the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\left\{\overline{P}_{\mathbf{m}}[\mathbf{z}]\right\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

$$\overline{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \overline{P}_{\mathbf{h}}[\mathbf{z}].$$

Let $N_{\mathbf{m}} = N_{m_1, m_2, ..., m_n}$ be the number of non-zero coefficients $\overline{P}_{\mathbf{m},\mathbf{h}}$ in the last equality.

A basic set satisfying the condition

$$\lim_{\langle\mathbf{m}\rangle\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{\langle\mathbf{m}\rangle}} = 1, \tag{1.1}$$

is called, as in (Mursi & Makar, 1955*a*,*b*) and (Kishka & El-Sayed Ahmed, 2003) a Cannon set.

Let $\{P_{\mathbf{m}}[\mathbf{z}]\}$ be a basic set of polynomials of the several complex variables z_s; $s \in I$, then the Cannon sum for this set in the complete Reinhardt domains is given as follows:

$$\Omega(P_{\mathbf{m}}, [\alpha\mathbf{r}]) = \prod_{s=1}^{n} (\alpha_s r)^{<\mathbf{m}>-\mathbf{m}_s} \sum_{\mathbf{h}} \left| \bar{P}_{\mathbf{m},\mathbf{h}} \right| M(P_{\mathbf{m}}, [\alpha\mathbf{r}]),$$

where

$$M(P_{\mathbf{m}}, [\alpha\mathbf{r}]) = \max_{\bar{\Gamma}_{[\alpha\mathbf{r}]}} \left| P_{\mathbf{m}}[\mathbf{z}] \right|.$$

The Cannon function is defined by:

$$\Omega(P, [\alpha\mathbf{r}]) = \lim_{\langle\mathbf{m}\rangle\to\infty} \left\{ \Omega(P_{\mathbf{m}}, [\alpha\mathbf{r}]) \right\}^{\frac{1}{\langle\mathbf{m}\rangle}}.$$

When this associated series converges uniformly to $f(\mathbf{z})$ in some domain it is said to represent $f(\mathbf{z})$ in that domain; in other words, as in the classical terminology of Whittaker for a single complex variable (see (Whittaker, 1949)), the basic set $P_{\mathbf{m}}[\mathbf{z}]$ will be effective in that domain. For more information about basic sets of polynomials we refer to ((Abul-Ez, 2000)-(Whittaker, 1949)).

The convergence properties of basic sets of polynomials are classified according to the classes of functions represented by their associated basic series and also to the domain in which are represented.

Concerning the effectiveness of the basic set of polynomials of several complex variables in complete Reinhardt domains, we have the following results from (Mursi & Makar, 1955*a*,*b*).

Theorem 1.1. *(Mursi & Makar, 1955a,b) The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the closed complete Reinhardt $\overline{\Gamma}_{[\alpha_s \mathbf{r}]}$ is that*

$$\Omega(P; r_s) = \prod_{s=1}^{n} \alpha_s r. \tag{1.2}$$

In the space of several complex variables $\mathbb{C}^n$, an open elliptical region $\sum_{s=1}^{n} \frac{|z_s|^2}{{r_s}^2} < 1$ is here denoted by $\mathbf{E}_{r_s}$ and its closure $\sum_{s=1}^{n} \frac{|z_s|^2}{{r_s}^2} \leq 1$; is denoted by $\overline{\mathbf{E}}_{r_s}$, where r_s; $s \in I$ are positive numbers. In terms of the introduced notations these regions satisfy the following inequalities:

$$\mathbf{E}_{r_s} = \{\mathbf{w} : |\mathbf{w}| < 1\}$$

$$\overline{\mathbf{E}}_{r_s} = \{\mathbf{w} : |\mathbf{w}| \leq 1\},$$

where $\mathbf{w} = (w_1, w_2, w_3, ..., w_n)$, $w_s = \frac{z_s}{{r_s}^2}$; $s \in I$. Suppose now that the function $f(\mathbf{z})$, is given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$$

is regular in $\overline{\mathbf{E}}_{\mathbf{r_s}}$ and

$$M[f; r_s] = \sup_{\overline{\mathbf{E}}_{\mathbf{rs}}} \left| f(\mathbf{z}) \right|.$$

Then it follows that $\{|z_s| \leq r_s t_s; \quad |t_s| = 1\} \subset \overline{\mathbf{E}}_{\mathbf{r_s}}$; hence

$$|a_{\mathbf{m}}| \leq \frac{M[f; \rho_s]}{\rho^{\mathbf{m}} t^{\mathbf{m}}} = \frac{M[f; \rho_s]}{\prod_{s=1}^{n} \rho_s^{m_s} t_s^{m_s}} \leq \inf_{|t|=1} \frac{M[f; \rho_s]}{\prod_{s=1}^{n} (\rho_s t_s)^{m_s}}$$
$$= \sigma_{\mathbf{m}} \frac{M[f; \rho_s]}{\prod_{s=1}^{n} \rho_s^{m_s}}$$

for all $0 < \rho_s < r_s$; $s \in I$, where

$$\sigma_{\mathbf{m}} = \inf_{|t|=1} \frac{1}{t^{\mathbf{m}}} = \frac{\{\langle \mathbf{m} \rangle\}^{\frac{\langle \mathbf{m} \rangle}{2}}}{\prod_{s=1}^{n} m_s^{\frac{m_s}{2}}}$$

and $1 \le \sigma_{\mathbf{m}} \le (\sqrt{n})^{\langle \mathbf{m} \rangle}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0$; $s \in I$. Thus, it follows that

$$\lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} (r_s)^{\langle \mathbf{m} \rangle - m_s}} \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \le \frac{1}{\prod_{s=1}^{n} \rho_s} \quad ; \quad \rho_s < r_s; s \in I$$

and since ρ_s can be chosen arbitrary near to r_s; $s \in I$, we conclude that

$$\lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} (r_s)^{\langle \mathbf{m} \rangle - m_s}} \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \le \frac{1}{\prod_{s=1}^{n} r_s}.$$

Now, write

$$G(P_{\mathbf{m}}; r_s) = \max_{\mu,\nu} \sup_{\overline{\mathbf{E}}_{\mathbf{r_s}}} \left| \sum_{\mathbf{j}=\mu}^{\nu} \overline{P}_{\mathbf{m};\mathbf{j}} P_{\mathbf{j}}[\mathbf{z}] \right|,$$

where, r_s; $s \in I$ are positive numbers.

The Cannon sum of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ for $\overline{\mathbf{E}}_{r_s}$ will be

$$\Omega(P_{\mathbf{m}}; r_s) = \sigma_{\mathbf{m}} \prod_{s=1}^{n} \{r_s\}^{\langle \mathbf{m} \rangle - m_s} G(P_{\mathbf{m}}; r_s)$$

and the Cannon function for the same set is

$$\Omega(P; r_s) = \lim_{\langle \mathbf{m} \rangle \to \infty} \{\Omega(P_{\mathbf{m}}; r_s)\}^{\frac{1}{\langle \mathbf{m} \rangle}}.$$

Concerning the effectiveness of the basic set of polynomials of several complex variables in hyperellipse, we have the following results from (El-Sayed Ahmed & Kishka, 2003).

Theorem 1.2. *(El-Sayed Ahmed & Kishka, 2003) The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the closed hyperellipse $\overline{\mathbf{E}}_{r_s}$ is that*

$$\Omega(P; r_s) = \prod_{s=1}^{n} r_s.$$

Convergence properties (effectiveness) for Hadamard product set simple monic sets of polynomials of a single complex variable is introduced by Melek and El-Said in (Melek & El-Said, 1985). In (Nassif & Rizk, 1988) Nassif and Rizk introduced an extension of this product in the case of two complex variables using spherical regions. In (El-Sayed Ahmed, 2006), the same author has studied this problem in $\mathbb{C}^n$ using hepespherical regions. It should be mentioned here the study of this problem in Clifford analysis (see (Abul-Ez, 2000)). For more details on basic sets of polynomials in Clifford setting, we refer to (Abul-Ez, 2000; Abul-Ez & De Almeida, 2013; Abul-Ez & Constales, 2003; Aloui *et al.*, 2010; Aloui & Hassan, 2010; Hassan, 2012; Saleem *et al.*, 2012) and others. In the present paper, we aim to investigate the extent of a generalization of this Hadamard product set in $\mathbb{C}^n$ using hyperspherical regions.

In (Nassif & Rizk, 1988), Nassif and Rizk introduced the following definition.

Definition 1.3. Let $\{P_{m,n}(z,w)\}$ and $\{q_{m,n}(z,w)\}$ be two simple monic sets of polynomials, where

$$P_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} P_{i,j}^{m,n} z^i w^j,$$

$$q_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} q_{i,j}^{m,n} z^i w^j.$$

Then the Hadamard product of the sets $\{P_{m,n}(z,w)\}$ and $\{q_{m,n}(z,w)\}$ is the simple monic set $\{U_{m,n}(z,w)\}$ given by

$$U_{m,n}(z,w) = \sum_{(i,j)=0}^{(m,n)} U_{i,j}^{m,n} z^i w^j,$$

where

$$U_{i,j}^{m,n} = \frac{\sigma_{m,n}}{\sigma_{i,j}} P_{i,j}^{m,n} q_{i,j}^{m,n}; \quad ((i,j) \le (m,n)),$$

and

$$\sigma_{m,n} = \inf_{|t|=1} \frac{1}{t^{m+n}} = \frac{\{m+n\}^{\frac{m+n}{2}}}{m^{\frac{m}{2}} n^{\frac{n}{2}}}.$$

In this paper, we give an inevitable modification in the definition of Hadamard product of basic sets of polynomials of two complex variables as to yield favorable results in the case of several complex variables in complete Reinhardt domains in $\mathbb{C}^n$, by using k basic sets of polynomials instead of two sets.

Now, we are in a position to extend the above product by using k basic sets of polynomials of several complex variables in complete Reinhardt domains, so we will denote these polynomials by $\{P_{1,\mathbf{m}}[\mathbf{z}]\}, \{P_{2,\mathbf{m}}[\mathbf{z}]\}, ..., \{P_{k,\mathbf{m}}[\mathbf{z}]\}$ and in general write $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1,2,3,...,k$.

Definition 1.4. Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1,2,3,...,k$ be simple monic sets of polynomials of several complex variables, where

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}. \tag{1.3}$$

Then the Hadamard product of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ is the simple monic set $\{H_{\mathbf{m}}[\mathbf{z}]\}$ given by

$$H_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} H_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \tag{1.4}$$

where

$$H_{\mathbf{m},\mathbf{h}} = \left(\prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}}\right). \tag{1.5}$$

If we substitute by $k = 2$ and consider polynomials of two complex variables instead of several complex variables, then we will obtain Definition 1.3. It should be remarked here that Definition 1.4 is different from that used in (Metwally, 2002).

2. Effectiveness in complete Reinhardt domains

In this section, we will study the effectiveness of the extended Hadamard product of simple monic sets of polynomials of several complex variables defined by (1.4) and (1.5) in closed complete Reinhardt domains and at the origin.

Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ be simple monic sets of polynomials of several complex variables z_s; $s \in I$, so that we can write

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \tag{2.1}$$

where

$$P^{m_1,m_2,\ldots,m_n}_{s_2,m_1,m_2,\ldots,m_n} = 1; \quad s_2 = 1, 2, \ldots, k\,.$$

The normalizing functions of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are defined by (see (Nassif & Rizk, 1988))

$$\mu(P_{s_2}; \alpha_s r) = \lim_{\langle\mathbf{m}\rangle\to\infty} \sup \Big\{ M[P_{s_2,\mathbf{m}}; \alpha_s r] \Big\}^{\frac{1}{\langle\mathbf{m}\rangle}}, \tag{2.2}$$

where $M[P_{s_2,\mathbf{m}}; \alpha_s r]$ are defined as follows:

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha r]}} \Big| P_{s_2,\mathbf{m}}[\mathbf{z}] \Big|.$$

Notice that the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are monic. By applying Cauchy's inequality in (2.2), we have

$$|P_{s_2,\mathbf{m},\mathbf{h}}| \leq \frac{1}{\prod_{s=1}^{n} (\alpha_s r)^{\langle\mathbf{m}\rangle}} \sup_{\bar{\Gamma}_{[\alpha r]}} \Big| P_{s_2,\mathbf{m}}[\mathbf{z}] \Big|,$$

which implies that

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] \geq \prod_{s=1}^{n} (\alpha_s r)^{\langle\mathbf{m}\rangle}.$$

It follows from (2.2) that

$$\mu(P_{s_2}; \alpha_s r) \geq \prod_{s=1}^{n} \alpha_s r. \tag{2.3}$$

Next, we show if ρ is positive number greater than r, then

$$\mu(P_{s_2}; \alpha_s \rho) \leq \frac{\prod_{s=1}^{n} \alpha_s \rho}{\prod_{s=1}^{n} \alpha_s r} \mu(P_{s_2}; \alpha_s r), \quad \alpha_s \rho > \alpha_s r. \tag{2.4}$$

In fact, this relation follows by applying (2.2) to the inequality

$$M[P_{s_2,\mathbf{m}}; \alpha_s r] \leq K \left(\frac{\prod_{s=1}^{n} \alpha_s \rho}{\prod_{s=1}^{n} \alpha_s r} \right)^{\langle\mathbf{m}\rangle} M[P_{s_2,\mathbf{m}}; \alpha_s r],$$

which in its turn, is derivable from (2.2), Cauchy's inequality and the supremum of $\mathbf{z}^{\mathbf{m}}$, where $K = O(\langle \mathbf{m} \rangle + 1)$.

Now, let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, and that $\{H^*_{\mathbf{m}}[\mathbf{z}]\}$ is the set defined as follows

$$H^*_{\mathbf{m}}[\mathbf{z}] = \prod_{s_2=1}^{k} P_{s_2,\mathbf{m}}[\mathbf{z}]. \tag{2.5}$$

The following fundamental result is proved.

Theorem 2.1. *If, for any* $\alpha_s r > 0$

$$\mu(P_{s_2}; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r, \tag{2.6}$$

then

$$\mu(H^*; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r. \tag{2.7}$$

Proof. We first observe that, if ρ be any finite number greater than r , then by (2.1), (2.2) and (2.9), we obtain that

$$\mu(P_{s_2}; \alpha_s \rho) = \prod_{s=1}^{n} \alpha_s \rho. \tag{2.8}$$

Now, given $r^* > r$, we choose finite number r' such that

$$\alpha_s r < \alpha_s r' < \alpha_s r^*. \tag{2.9}$$

Then by (2.1) and (2.6), we obtain that

$$M(P_{s_2,\mathbf{h}}; \alpha_s r) < \eta \prod_{s=1}^{n} \alpha_s (r')^{\langle \mathbf{h} \rangle} \quad \text{where} \quad \eta > 1, \tag{2.10}$$

where $\langle \mathbf{h} \rangle = h_1 + h_2 + h_3 + \ldots h_n$. Also from (2.4), we can write

$$H^*_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} \prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}} P_{s_2,\mathbf{h}}[\mathbf{z}].$$

Hence (2.9) and (2.10) lead to

$$M[H^*_{\mathbf{m}}; \alpha_s r] \leq \eta K \Big(1 - \Big(\frac{\prod_{s=1}^{n} \alpha_s r'}{\prod_{s=1}^{n} \alpha_s r^*} \Big)^{n} \Big)^{-n} M[P_{s_2,\mathbf{m}}; \alpha_s r^*],$$

Making $\langle \mathbf{m} \rangle \to \infty$ and applying (2.7), we get

$$\mu(H^*;\alpha_s r) = \lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \sigma_{\mathbf{m}} M[H^*_{\mathbf{m}};\alpha_s r] \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \mu(P_{s_2};\alpha_s r^*) = \prod_{s=1}^{n} \alpha_s r^*,$$

which leads to the equality (2.6), by the choice of r^* near to r, and our theorem is therefore proved. □

Remark. From Theorem 2.1, if we consider the simple monic sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ accord to condition (2.6), then it is not hard to prove by induction for the j-power sets $\{P^{(j)}_{s_2,\mathbf{m}}[\mathbf{z}]\}$ that

$$\mu(P^{(j)}_{s_2};\alpha_s r) = \prod_{s=1}^{n} \alpha_s r. \tag{2.11}$$

Now, we give the following result.

Theorem 2.2. *Let* $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ *be simple monic algebraic sets of polynomials of several complex variables, which accord to condition (10). Then the set will be effective in the closed complete Reinhardt domain* $\bar{\Gamma}_{[\alpha r]}$.

Proof. Suppose that the monomial $\mathbf{z}^{\mathbf{m}}$ admit the representation

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}].$$

Since the set $\{P_{1,\mathbf{m}}[\mathbf{z}]\}$ is algebraic, we find there exists a relation of the form

$$\overline{P}_{1,\mathbf{m},\mathbf{h}} = \sum_{j=1}^{k} a_j P^{(j)}_{1,\mathbf{m},\mathbf{h}}; \quad ((\mathbf{h}) \leq (\mathbf{m})), \tag{2.12}$$

where k is a finite positive integer which together with the coefficients $(a_j)^k_{j=1}$, is independent of the indices $(\mathbf{m})$, $(\mathbf{h})$. The coefficients $P^{(j)}_{1,\mathbf{m},\mathbf{h}}$ are defined by

$$P^{(j)}_{1,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=1}^{(\mathbf{m})} P^{(j)}_{1,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}; \quad 1 \leq j \leq k.$$

It follows that

$$\left| P^{(j)}_{1,\mathbf{m},\mathbf{h}} \right| (\alpha_s r)^{\langle \mathbf{m} \rangle} \leq \sigma_{\mathbf{h}} M[P^{(j)}_{1,\mathbf{m}};\alpha_s r]. \tag{2.13}$$

According to (2.11) for given $r^* > r$ and from the definition corresponding to $\mu(P^{(j)}_1;\alpha_s r)$, we deduce that

$$M[P^{(j)}_{1,\mathbf{h}};\alpha_s r] < K(\alpha_s r^*)^{\langle \mathbf{h} \rangle}. \tag{2.14}$$

Applying (2.13) and (2.14) in (2.12), we obtain that

$$\left| \overline{P}^{(j)}_{1,\mathbf{m},\mathbf{h}} \right| < \zeta \beta K \frac{\prod_{s=1}^{n} (\alpha_s r^*)^{\langle \mathbf{m} \rangle}}{\prod_{s=1}^{n} (\alpha_s r)^{\langle \mathbf{m} \rangle}}, \tag{2.15}$$

where

$$\beta = \max\{|a_j|; 0 \leq j \leq k\} \quad \text{and} \quad \zeta \quad \text{is a constant.} \tag{2.16}$$

In view of the representation

$$\mathbf{z^m} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}],$$

the Cannon sum of the set $\{P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]\}$ will be

$$\Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} |\overline{P}^{(j)}_{\mathbf{m},\mathbf{h}}| M[P^{(j)}_{1,\mathbf{h}}; \alpha_s r], \tag{2.17}$$

where,

$$M[P^{(j)}_{1,\mathbf{h}}; \alpha_s r] = \sup_{\bar{\Gamma}_{[\alpha \mathbf{r}]}} |P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]|. \tag{2.18}$$

Therefore (2.14), (2.15) and (2.17) (for $r^* > r$) give

$$\Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) < \zeta K \beta \prod_{s=1}^{n} (\alpha_s r^*)^{\langle \mathbf{m} \rangle}. \tag{2.19}$$

Hence the Cannon function of the set $\{P^{(j)}_{1,\mathbf{m}}[\mathbf{z}]\}$ turns out to be

$$\Omega(P^{(j)}_{1}; \alpha_s r) = \lim_{\langle \mathbf{m} \rangle \to \infty} \left\{ \Omega(P^{(j)}_{1,\mathbf{m}}; \alpha_s r) \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} = \prod_{s=1}^{n} \alpha_s r^*,$$

which, by the choice of r^*, implies that

$$\Omega(P^{(j)}_{1}; \alpha_s r) = \prod_{s=1}^{n} \alpha_s r.$$

As very similar, we can obtain that the sets $\{P^{(j)}_{\nu,\mathbf{m}}[\mathbf{z}]\}$; $\nu = 2, 3, 4, ..., k$ will be effective in the losed complete Reinhardt domain $\bar{\Gamma}_{[\alpha r]}$. Our theorem is therefore proved. □

3. Effectiveness in hyperelliptical regions

Now, we are in a position to extend the above product by using k basic sets of polynomials of several complex variables, so we will denote these polynomials by $\{P_{1,\mathbf{m}}[\mathbf{z}]\}, \{P_{2,\mathbf{m}}[\mathbf{z}]\}, ..., \{P_{k,\mathbf{m}}[\mathbf{z}]\}$ and in general write $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$.

Definition 3.1. Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, where

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z^h}. \tag{3.1}$$

Then the Hadamard product of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ is the simple monic set $\{H_{\mathbf{m}}[\mathbf{z}]\}$ given by

$$H_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} H_{\mathbf{m},\mathbf{h}}\mathbf{z}^{\mathbf{h}}, \tag{3.2}$$

where

$$H_{\mathbf{m},\mathbf{h}} = \left(\frac{\sigma_{\mathbf{m}}}{\sigma_{\mathbf{h}}}\right)^{k-1}\left(\prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}}\right). \tag{3.3}$$

If we substitute by $k = 2$ and consider polynomials of two complex variables instead of several complex variables, then we will obtain Definition 1.3. It should be remarked here that Definition 1.4 is different from that used in (Metwally, 2002).

Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ be simple monic sets of polynomials of several complex variables z_s; $s \in I$, so that we can write

$$P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}}\mathbf{z}^{\mathbf{h}}, \tag{3.4}$$

where

$$P^{m_1,m_2,...,m_n}_{s_2,m_1,m_2,...,m_n} = 1; \quad s_2 = 1, 2, ..., k\,.$$

The normalizing functions of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are defined by (see (Nassif & Rizk, 1988))

$$\mu(P_{s_2}; r_s) = \lim_{\langle\mathbf{m}\rangle\to\infty} \sup\left\{\sigma_{\mathbf{m}} M[P_{s_2,\mathbf{m}}; r_s]\right\}^{\frac{1}{\langle\mathbf{m}\rangle}}, \tag{3.5}$$

where $M[P_{s_2,\mathbf{m}}; r_s]$ are defined as follows:

$$M[P_{s_2,\mathbf{m}}; r_s] = \sup_{\overline{\mathbf{E}}_{\mathbf{rs}}}\left|P_{s_2,\mathbf{m}}[\mathbf{z}]\right|.$$

Notice that the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ are monic. By applying Cauchy's inequality, we deduce

$$|P_{s_2,\mathbf{m},\mathbf{h}}| \leq \frac{\sigma_{\mathbf{m}}}{[\prod_{s=1}^{n} r_s]^{\langle\mathbf{m}\rangle}} \sup_{\overline{\mathbf{E}}_{\mathbf{rs}}}\left|P_{s_2,\mathbf{m}}[\mathbf{z}]\right|,$$

which implies that

$$M[P_{s_2,\mathbf{m}}; r_s] \geq \frac{[\prod_{s=1}^{n} r_s]^{\langle\mathbf{m}\rangle}}{\sigma_{\mathbf{m}}}.$$

It follows from (3.4) that

$$\mu(P_{s_2}; r_s) \geq r_s. \tag{3.6}$$

Next, we show if ρ_s are positive numbers greater than r_s, then

$$\mu(P_{s_2}; \rho_s) \leq \frac{\prod_{s=1}^{n} \rho_s}{\prod_{s=1}^{n} r_s}\mu(P_{s_2}; r_s), \quad \rho_s > r_s. \tag{3.7}$$

In fact, this relation follows by applying (3.4) to the inequality

$$M[P_{s_2,\mathbf{m}};\rho_s] \leq K\Big(\frac{\prod_{s=1}^{n}\rho_s}{\prod_{s=1}^{n} r_s}\Big)^{\langle\mathbf{m}\rangle} M[P_{s_2,\mathbf{m}};r_s],$$

which in its turn, is derivable from (3.4), Cauchy's inequality and the supremum of $\mathbf{z}^{\mathbf{m}}$, where $K = O(\langle\mathbf{m}\rangle + 1)$.

Now, let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, ..., k$ be simple monic sets of polynomials of several complex variables, and that $\{H^*_{\mathbf{m}}[\mathbf{z}]\}$ is the set defined as follows

$$H^*_{\mathbf{m}}[\mathbf{z}] = \prod_{s_2=1}^{k} P_{s_2,\mathbf{m}}[\mathbf{z}]. \tag{3.8}$$

The following fundamental result is proved.

Theorem 3.1. *If, for any* $r_s > 0$

$$\mu(P_{s_2}; r_s) = \prod_{s=1}^{n} r_s, \tag{3.9}$$

then

$$\mu(H^*; r_s) = \prod_{s=1}^{n} r_s. \tag{3.10}$$

Proof. We first observe that, if ρ be any finite number greater than r , then by (3.4), (3.5) and (3.7), we obtain that

$$\mu(P_{s_2};\rho_s) = \prod_{s=1}^{n} \rho_s. \tag{3.11}$$

Now, given $r^*_s > r_s$, we choose finite number r'_s such that

$$r_s < r'_s < r^*. \tag{3.12}$$

Then by (3.4) and (3.8), we obtain that

$$M(P_{s_2,\mathbf{h}}; r_s) < \frac{\eta}{\sigma_{\mathbf{h}}}[\prod_{s=1}^{n} r'_s]^{\langle\mathbf{h}\rangle} \quad \text{where} \quad \eta > 1, \tag{3.13}$$

where $\langle\mathbf{h}\rangle = h_1 + h_2 + h_3 + \ldots h_n$. Also from (3.7), we can write

$$H^*_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} \prod_{s_2=1}^{k} P_{s_2,\mathbf{m},\mathbf{h}} P_{s_2,\mathbf{h}}[\mathbf{z}].$$

Hence (3.9) and (3.10) lead to

$$\mu(H^*; r_s) = \lim_{\langle \mathbf{m} \rangle \to \infty} \sup \left\{ \sigma_{\mathbf{m}} M[H^*_{\mathbf{m}}; r_s] \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \mu(P_{s_2}; r^*_s) = \prod_{s=1}^{n} r^*_s,$$

which leads to the equality (3.8), by the choice of r^*_s near to r_s, and our theorem is therefore proved. □

Remark. From Theorem 3.1 if we consider the simple monic sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ accord to condition (3.8), then it is not hard to prove by induction for the j-power sets $\{P^{(j)}_{s_2,\mathbf{m}}[\mathbf{z}]\}$ that

$$\mu(P^{(j)}_{s_2}; r_s) = \prod_{s=1}^{n} r_s. \tag{3.14}$$

Remark. It should be remarked that the results of this paper improve some results in (El-Sayed Ahmed, 2006, 2013).

4. Conclusion

We have obtained some essential and important results for the effectiveness of the Hadamard product set of polynomials in complete Reinhardt domains and in heperelliptical regions. From the established theorems, representations and convergence of power set of the the Hadamard product set are introduced in complete Reinhardt domains and in heperelliptical regions too. Various problems relating to the properties of the Hadamard set of simple basic sets of polynomials are treated with particular emphasis on distinction between the single and several complex variables cases. An important result is established for the relationship between the Cannon functions of simple sets of polynomials in several complex variables and those of the directly Hadamard sets.

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Fractional Order Differential Equations Involving Caputo Derivative

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Abstract

In this paper, the Banach contraction principle and Schaefer theorem are applied to establish new results for the existence and uniqueness of solutions for some Caputo fractional differential equations. Some examples are also discussed to illustrate the main results.

Keywords: Caputo derivative, Banach fixed point theorem, Fractional differential equations.
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1. Introduction

The theory of fractional differential equations has excited in recent years a considerable interest both in mathematics and in applications, (see (Bengrine & Dahmani, 2012; Delbosco & Rodino, 1996; Diethelm & Walz, 1998; El-Sayed, 1998)). In particular, existence and uniqueness of solutions for fractional differential equations have attracted the attention of many mathematicians (Diethelm & Ford, 2002; Houas & Dahmani, 2013; Zhang, 2011; Ntouyas, 2012; Su, 2009; Yang, 2012; Zhang, 2011).
This paper deals with the existence and uniqueness of solutions to the following problem

$$\begin{aligned} D^{\alpha}x(t)+f\left(t,y(t),D^{\delta}y(t)\right)&=0, t\in J, \\ D^{\beta}y(t)+g\left(t,x(t),D^{\sigma}x(t)\right)&=0, t\in J, \\ x(0)=y(0)=0, x(1)-\lambda_1 x(\eta)=0, y(1)-\lambda_1 y(\eta)&=0, \\ x''(0)=y''(0)=0, x''(1)-\lambda_2 x''(\xi)=0, y''(1)-\lambda_2 y''(\xi)&=0, \end{aligned} \tag{1.1}$$

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where $D^{\alpha}, D^{\beta}, D^{\delta}$ and D^{σ}, are the Caputo fractional derivatives, $3 < \alpha, \beta \leq 4, \delta \leq \alpha - 1, \sigma \leq \beta - 1, 0 < \xi, \eta < 1$, $J = [0, 1]$, λ_1, λ_2 are real constants satisfying $\lambda_1 \eta \neq 1, \lambda_2 \xi \neq 1$ and f, g are two functions which will be specified later.
This paper is organized as follows: In section 2, we present some preliminaries and lemmas. In section 3, we present our main results for the existence and uniqueness of solutions of (1.1). In section 4, some examples are treated to illustrate our results.

2. Preliminaries

To present our main results, we need the the following two definitions:

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0, \tag{2.1}$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2.2. The fractional derivative of $f \in C^n([0, \infty[)$ in the Caputo's sense is defined as:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, n-1 < \alpha, n \in N^*. \tag{2.2}$$

More details about fractional calculus can be found in (Mainardi, 1997; Podlubny *et al.*, 2002).
We need also to introduce the spaces:
$X = \{x : x \in C([0,1]), D^{\sigma} x \in C([0,1])\}$ and $Y = \{y : y \in C([0,1]), D^{\delta} y \in C([0,1])\}$. For these spaces, we associate respectively the norms $\| x \|_X = \| x \| + \| D^{\sigma} x \|; \| x \| = \sup_{t \in J} |x(t)|, \| D^{\sigma} x \| = \sup_{t \in J} \left| D^{\sigma} x(t) \right|$ and $\| y \|_Y = \| y \| + \| D^{\delta} y \|; \| y \| = \sup_{t \in J} |y(t)|, \| D^{\delta} y \| = \sup_{t \in J} \left| D^{\delta} y(t) \right|$. It is clear that, $(X, \| \,.\, \|_X)$ and $(Y, \| \,.\, \|_Y)$, are two Banach spaces.
Also, $(X \times Y, \|(x,y)\|_{X \times Y})$ is a Banach space. Its norm is given by $\|(x,y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$.

The following lemmas are crucial for our main results (Kilbas & Marzan, 2005; Lakshmikantham & Vatsala, 2008):

Lemma 2.1. *For $\alpha > 0$, the general solution of the equation $D^{\alpha} x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.3}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1, n = [\alpha] + 1$.

Lemma 2.2.

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.4}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We prove also the following lemma which is needed to present the integral solution for the problem (1.1):

Lemma 2.3. *Let* $h \in C([0,1])$, $t \in J$, $3 < \alpha \leq 4$. *Then the solution of the equation*

$$D^{\alpha} x(t) + h(t) = 0, \tag{2.5}$$

where,

$$\begin{aligned} x(0) &= 0, x(1) - \lambda_1 x(\eta) = 0, \\ x''(0) &= 0, x''(1) - \lambda_2 x''(\xi) = 0 \end{aligned} \tag{2.6}$$

is given by the following expression

$$\begin{aligned} x(t) = &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h\left(s, y(s), D^{\delta} y(s)\right) ds \\ &+\frac{\lambda_1 t}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} h\left(s, y(s), D^{\delta} y(s)\right) ds \\ &-\frac{1}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h\left(s, y(s), D^{\delta} y(s)\right) ds \\ &+\frac{\left(\lambda_2 - \lambda_2 \lambda_1 \eta^3\right) t + (\lambda_2 \lambda_1 \eta - \lambda_2) t^3}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^{\xi} (\xi - s)^{\alpha-3} h\left(s, y(s), D^{\delta} y(s)\right) ds \\ &-\frac{\left(1 - \lambda_1 \eta^3\right) t + (\lambda_1 \eta - 1) t^3}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^1 (1-s)^{\alpha-3} h\left(s, y(s), D^{\delta} y(s)\right) ds. \end{aligned} \tag{2.7}$$

Proof: Let $c_i \in \mathbb{R}, i = 0,1,2,3$. Then by lemmas 2.1, 2.2, the general solution of (2.5) can be written as:

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3. \tag{2.8}$$

Using (2.6), we immediately get $c_0 = c_2 = 0$. On the other hand, we have

$$\begin{aligned} c_1 = &-\frac{\lambda_1}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} h(s) ds + \frac{1}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s) ds \\ &-\frac{\lambda_2\left(1 - \lambda_1 \eta^3\right)}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^{\xi} (\xi - s)^{\alpha-3} h(s) ds \\ &+\frac{(1 - \lambda_1 \eta)}{6(\lambda_1 \eta - 1)(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^1 (1-s)^{\alpha-3} h(s) ds. \end{aligned} \tag{2.9}$$

To obtain the value of c_3, we remarque that

$$c_3 = -\frac{\lambda_2}{6(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^{\xi} (\xi - s)^{\alpha-3} h(s) ds + \frac{1}{6(\lambda_2 \xi - 1)\Gamma(\alpha - 2)} \int_0^1 (1-s)^{\alpha-3} g(s) ds. \tag{2.10}$$

Finally, substituting the values of c_1 and c_3 in (2.8), we obtain (2.7).

3. Main results

We begin by introducing the quantities:

$$
\begin{aligned}
N_1 &= \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)},\\
N_2 &= \frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} + \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\\
&\quad+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)},\\
N_3 &= \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)} + \frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)},\\
N_4 &= \frac{1}{\Gamma(\beta-\delta+1)} + \frac{|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)} + \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\\
&\quad+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}.
\end{aligned}
$$

We impose also the hypotheses:

$(H1)$: The functions $f\ ,g\ :[0,1]\times\mathbb{R}^2\rightarrow\mathbb{R}$ are continuous.

$(H2)$: There exist non negative functions $a_i, b_i \in C([0,1])\,, i=1,2$ such that for all $t\in[0,1]$ and $(x_1,y_1)\,,(x_2,y_2)\in\mathbb{R}^2$, the inequalities

$$
\begin{aligned}
|f(t,x_1,y_1)-f(t,x_2,y_2)| &\le a_1(t)|x_1-x_2|+b_1(t)|y_1-y_2|,\\
|g(t,x_1,y_1)-g(t,x_2,y_2)| &\le a_2(t)|x_1-x_2|+b_2(t)|y_1-y_2|,
\end{aligned}
\tag{3.1}
$$

are valid, and

$$
\omega_1=\sup_{t\in J} a_1(t)\,,\omega_2=\sup_{t\in J} b_1(t)\,,\varpi_1=\sup_{t\in J} a_2(t)\,,\varpi_2=\sup_{t\in J} b_2(t)\,.
$$

$(H3)$: There exist positive constants L_1 and L_2 such that

$$
|f(t,x,y)|\le L_1, |g(t,x,y)|\le L_2 \text{ for each } t\in J \text{ and all } x,y\in\mathbb{R}.
$$

Our first main result is given by the following theorem:

Theorem 3.1. *Assume that (H2) holds and suppose that*

$$
(N_1+N_2)(\omega_1+\omega_2)+(N_3+N_4)(\varpi_1+\varpi_2)<1. \tag{3.2}
$$

Then the problem (1.1) has a unique solution on J.

Proof: We apply Banach fixed point theorem. So, we consider the operator $\phi: X\times Y\rightarrow X\times Y$ defined by:

$$
\phi(x,y)(t):=(\phi_1 y(t)\,,\phi_2 x(t))\,, \tag{3.3}
$$

where

$$\begin{aligned}
\phi_1 y(t) \ : \ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds \\
&+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^{\eta} (\eta-s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds \\
&-\frac{1}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^{1} (1-s)^{\alpha-1} f\left(s, y(s), D^{\delta} y(s)\right) ds \\
&+\frac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{\xi} (\xi-s)^{\alpha-3} f\left(s, y(s), D^{\delta} y(s)\right) ds \\
&-\frac{\left(1-\lambda_1\eta^3\right)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{1} (1-s)^{\alpha-3} f\left(s, y(s), D^{\delta} y(s)\right) ds,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\phi_2 x(t) \ : \ &= -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g\left(s, x(s), D^{\sigma} x(s)\right) ds \\
&+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^{\eta} (\eta-s)^{\alpha-1} g\left(s, x(s), D^{\sigma} x(s)\right) ds \\
&-\frac{1}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^{1} (1-s)^{\alpha-1} g\left(s, x(s), D^{\sigma} x(s)\right) ds \\
&+\frac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{\xi} (\xi-s)^{\alpha-3} g\left(s, x(s), D^{\sigma} x(s)\right) ds \\
&-\frac{\left(1-\lambda_1\eta^3\right)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{1} (1-s)^{\alpha-3} g\left(s, x(s), D^{\sigma} x(s)\right) ds.
\end{aligned} \tag{3.5}$$

And we shall prove that ϕ is a contraction mapping.

Let $(x, y), (x_1, y_1) \in X \times Y$. Then, for each $t \in J$, we have:

$$\begin{aligned}
|\phi_1 y(t) - \phi_1 y_1(t)| \le & \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\
&+\tfrac{|\lambda_1| t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta} (\eta-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\
&+\tfrac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{1} (1-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\
&+\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi} (\xi-s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\
&+\tfrac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{1} (1-s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds.
\end{aligned} \tag{3.6}$$

Thanks to $(H2)$, we obtain

$$|\phi_1 y(t) - \phi_1 y_1(t)| \le \frac{(|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1)\left(\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right]\left(\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y-D^\delta y_1\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}. \tag{3.7}$$

Consequently,

$$|\phi_1 y(t) - \phi_1 y_1(t)| \le N_1(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right), \tag{3.8}$$

Hence,

$$\|\phi_1(y) - \phi_1(y_1)\| \le N_1(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right). \tag{3.9}$$

We have also,

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \le{}& \frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right| ds\\ &+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right| ds\\ &+\frac{t^{1-\sigma}}{|(\lambda_1\eta-1)|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right| ds\\ &+\left(\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} + \frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\right)\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right| ds\\ &+\left(\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} + \frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\right)\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right| ds.\end{aligned} \tag{3.10}$$

By $(H2)$, yields

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \le{}& \frac{(\omega_1+\omega_2)\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{\Gamma(\alpha-\sigma+1)}\\ &+\frac{(\omega_1+\omega_2)[|\lambda_1|\eta^\alpha+1]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\\ &+\frac{(\omega_1+\omega_2)\left[\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|\right]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\\ &+\frac{(\omega_1+\omega_2)\left[|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|\right]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}.\end{aligned} \tag{3.11}$$

This implies that,

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \le{}& \left[\frac{(\omega_1+\omega_2)}{\Gamma(\alpha-\sigma+1)} + \frac{\%(\omega_1+\omega_2)[|\lambda_1|\eta^\alpha+1]}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right]\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right)\\ &+\left[\frac{(\omega_1+\omega_2)\left[\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} + \frac{(\omega_1+\omega_2)\left[|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|\right]}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right).\end{aligned} \tag{3.12}$$

Therefore,

$$\left|D^{\sigma}\phi_1 y(t) - D^{\sigma}\phi_1 y_1(t)\right| \leq N_2(\omega_1 + \omega_2)\left(\|y - y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.13}$$

Consequently,

$$\left\|D^{\sigma}\phi_1(y) - D^{\sigma}\phi_1(y_1)\right\| \leq N_2(\omega_1 + \omega_2)\left(\|y - y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.14}$$

By (3.9) and (3.14), we can write

$$\|\phi_1(y) - \phi_1(y_1)\|_X \leq (N_1 + N_2)(\omega_1 + \omega_2)\left(\|y - y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.15}$$

With the same arguments as before, we have

$$\|\phi_2(x) - \phi_2(x_1)\|_Y \leq (N_3 + N_4)(\varpi_1 + \varpi_2)\left(\|x - x_1\| + \left\|D^{\sigma}x - D^{\sigma}x_1\right\|\right). \tag{3.16}$$

Using (3.15) and (3.16), we can state that

$$\|\phi(x,y) - \phi(x_1,y_1)\|_{X\times Y} \leq \begin{bmatrix} (N_1 + N_2)(\omega_1 + \omega_2) \\ +(N_3 + N_4)(\varpi_1 + \varpi_2) \end{bmatrix}(\|(x - x_1, y - y_1)\|_{X\times Y}). \tag{3.17}$$

Thanks to (3.2), we conclude that ϕ is contraction. As a consequence of Banach fixed point theorem, we deduce that ϕ has a unique fixed point which is a solution of (1.1).

The second main result is based on Schaefer theorem. We have:

Theorem 3.2. *Suppose that (H1) and (H3) are satisfied. Then, the problem* (1.1) *has at least one solution on J.*

Proof: A: Thanks to (H1), we can state that the operator ϕ is continuous on $X \times Y$.

B: We will prove that ϕ maps bounded sets into bounded sets in $X \times Y$.

So, taking $\rho > 0$, and $(x,y) \in B_{\rho} := \{(x,y) \in X \times Y; \|(x,y)\|_{X\times Y} \leq \rho\}$, then for each $t \in J$, we have:

$$\begin{aligned}
|\phi_1 y(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds \\
& + \frac{|\lambda_1| t}{|\lambda_1\eta - 1|\Gamma(\alpha)}\int_0^{\eta} (\eta - s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds \\
& + \frac{t}{|\lambda_1\eta - 1|\Gamma(\alpha)}\int_0^{\eta} (\eta - s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds \\
& + \frac{\left|\lambda_2 - \lambda_2\lambda_1\eta^3\right| t + |\lambda_2\lambda_1\eta - \lambda_2| t^3}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-2)}\int_0^{\xi} (\xi - s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds \\
& + \frac{\left|1 - \lambda_1\eta^3\right| t + |\lambda_1\eta - 1| t^3}{6|\lambda_1\eta - 1||\lambda_2\xi - 1|\Gamma(\alpha-2)}\% \int_0^{1} (1 - s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds.
\end{aligned} \tag{3.18}$$

The condition $(H3)$ implies that

$$\begin{aligned} |\phi_1 y(t)| &\le \frac{L_1(|\lambda_1\eta-1||\lambda_1|\eta^{\alpha}+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{L_1\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left(\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right)\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)} \\ &\le L_1\left[\begin{array}{c} \frac{|\lambda_1\eta-1||\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ +\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\end{array}\right]. \end{aligned} \tag{3.19}$$

Then,

$$\|\phi_1(y)\| \le L_1 N_1. \tag{3.20}$$

For D^{σ}, we have the following inequalities

$$\begin{aligned} \left|D^{\sigma}\phi_1 y(t)\right| \le& \frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ &+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ &+\frac{t^{1-\sigma}}{|(\lambda_1\eta-1)|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ &+\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)}\end{array}\right)\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\ &+\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{\%6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)}\end{array}\right)\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds. \end{aligned} \tag{3.21}$$

By $(H3)$ again, yields the following formula

$$\begin{aligned} \left|D^{\sigma}\phi_1 y(t)\right| &\le L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right] \\ &\quad +L_1\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\end{array}\right] \\ &\le L_1\left[\begin{array}{c}\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ +\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{\%|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\end{array}\right]. \end{aligned} \tag{3.22}$$

Hence, we can write

$$\left\|D^{\sigma}\phi_1(y)\right\| \le L_1 N_2. \tag{3.23}$$

Using (3.20) and (3.23), we obtain

$$\|\phi_1(y)\|_X \le L_1(N_1+N_2). \tag{3.24}$$

As before, we obtain

$$\|\phi_2(x)\|_Y \leq L_2(N_3+N_4). \tag{3.25}$$

By (3.24) and (3.25), we get

$$\|\phi(x,y)\|_{X\times Y} \leq L_1(N_1+N_2)+L_2(N_3+N_4). \tag{3.26}$$

Therefore,

$$\|\phi(x,y)\|_{X\times Y} < \infty. \tag{3.27}$$

C: Now, we prove the equi-continuity of ϕ.
Let us take $(x,y)\in B_\rho$, and $t_1,t_2\in J$, with $t_1<t_2$. We have:

$$\begin{aligned}
|\phi_1 y(t_2)-\phi_1 y(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right)\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\frac{|\lambda_1|(t_2-t_1)}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\frac{(t_1-t_2)}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|(t_2-t_1)+|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&+\frac{\left|1-\lambda_1\eta^3\right|(t_1-t_2)+|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds \\
&\leq \frac{L_1\left(t_1^\alpha-t_2^\alpha+2\left(t_2^\alpha-t_1^\alpha\right)\right)}{\Gamma(\alpha+1)}+\frac{L_1|\lambda_1|\eta^\alpha(t_2-t_1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{L_1(t_1-t_2)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\
&+\frac{L_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}(t_2-t_1)+L_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)} \\
&+\frac{L_1\left|1-\lambda_1\eta^3\right|(t_1-t_2)+L_1|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}.
\end{aligned} \tag{3.28}$$

Therefore,

$$\begin{aligned}
|\phi_1 y(t_2)-\phi_1 y(t_1)| &\leq L_1\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2-t_1) \\
&+L_1\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_1-t_2) \\
&+L_1\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right]\left(t_2^3-t_1^3\right)+\frac{L_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right) \\
&+\frac{L_1}{\Gamma(\alpha+1)}\left(t_1^\alpha-t_2^\alpha\right)+\frac{2L_1}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha.
\end{aligned} \tag{3.29}$$

We have also

$$\begin{aligned}
\left|D^{\sigma}\phi_1 y(t_2) - D^{\sigma}\phi_1 y(t_1)\right| \le \tfrac{1}{\Gamma(\alpha-\sigma)}\int_0^{t_1}\left((t_1-s)^{\alpha-\sigma-1}-(t_2-s)^{\alpha-\sigma-1}\right)\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
+\tfrac{1}{\Gamma(\alpha-\sigma)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
+\tfrac{|\lambda_1|\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
+\tfrac{\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{1}(1-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
+\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds \\
+\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)}\end{array}\right]\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}y(s)\right)\right|ds.
\end{aligned} \tag{3.30}$$

The condition $(H3)$ implies that

$$\begin{aligned}
\left|D^{\sigma}\phi_1 y(t_2) - D^{\sigma}\phi_1 y(t_1)\right| \le \tfrac{L_1}{\Gamma(\alpha-\sigma+1)}\left(t_1^{\alpha-\sigma}-t_2^{\alpha-\sigma}+2(t_2-t_1)^{\alpha-\sigma}\right) \\
+L_1\left[\begin{array}{c}\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ +\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\end{array}\right]\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)+L_1\left[\begin{array}{c}\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ +\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\end{array}\right]\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right) \\
+\tfrac{L_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)+\tfrac{L_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).
\end{aligned} \tag{3.31}$$

The inequalities (3.29) and (3.31) imply that:

$$\|\phi_1 y(t_2)-\phi_1 y(t_1)\|_X \le L_1\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2-t_1)$$
$$+L_1\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_1-t_2)+L_1\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right]\left(t_2^3-t_1^3\right)$$
$$+\frac{L_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right)+\frac{L_1}{\Gamma(\alpha+1)}\left(t_1^\alpha-t_2^\alpha+2(t_2-t_1)^\alpha\right)$$
$$+\frac{L_1}{\Gamma(\alpha-\sigma+1)}\left(t_1^{\alpha-\sigma}-t_2^{\alpha-\sigma}+2(t_2-t_1)^{\alpha-\sigma}\right)+L_1\left[\begin{array}{c}\frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\\+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\end{array}\right]\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)$$
$$+L_1\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\right]\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)$$
$$+\frac{L_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)+\frac{L_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right). \tag{3.32}$$

With the same arguments as before, we can write

$$\|\phi_2 x(t_2)-\phi_2 x(t_1)\|_Y \le L_2\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\beta}{|\lambda_1\eta-1|\Gamma(\beta+1)}+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right](t_2-t_1)(3.33)$$
$$+L_2\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\beta+1)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right](t_1-t_2)+L_2\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right]\left(t_2^3-t_1^3\right)$$
$$+\frac{L_2|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_1^3-t_2^3\right)+\frac{L_2}{\Gamma(\beta+1)}\left(t_1^\beta-t_2^\beta+2(t_2-t_1)^\beta\right)$$
$$+\frac{L_2}{\Gamma(\alpha-\beta+1)}\left(t_1^{\beta-\delta}-t_2^{\beta-\delta}+2(t_2-t_1)^{\beta-\delta}\right)+L_2\left[\begin{array}{c}\frac{|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)}\\+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\end{array}\right]\left(t_2^{1-\delta}-t_1^{1-\delta}\right)$$
$$+L_2\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\right]\left(t_1^{1-\delta}-t_2^{1-\delta}\right)$$
$$+\frac{L_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_2^{3-\delta}-t_1^{3-\delta}\right)+\frac{L_2|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_1^{3-\delta}-t_2^{3-\delta}\right).$$

Thanks to (3.32) and (3.33), we can state that $\|\phi(x,y)(t_2)-\phi(x,y)(t_1)\|_{X\times Y}\to 0$ as $t_2\to t_1$. Combining **A, B, C** and using Arzela-Ascoli theorem, we conclude that ϕ is completely continuous operator.

D: We shall show that

$$\Omega := \{(x,y)\in X\times Y, (x,y)=\mu\phi(x,y), 0<\mu<1\}, \tag{3.34}$$

is a bounded set.

Let $(x,y)\in\Omega$, then $(x,y)=\mu\phi(x,y)$, for some $0<\mu<1$. Thus, for each $t\in J$, we have:

$$y(t)=\mu\phi_1 y(t), x(t)=\mu\phi_2 x(t).$$

Therefore,

$$\begin{aligned}\frac{1}{\mu}|y(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds \\ &+\frac{|\lambda_1|t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds \\ &+\frac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds \\ &+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds \\ &+\frac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds.\end{aligned} \tag{3.35}$$

Thanks to $(H3)$, we can write

$$\begin{aligned}\frac{1}{\mu}|y(t)| \leq\ & \frac{L_1(|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ &+\frac{L_1\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}.\end{aligned} \tag{3.36}$$

Thus,

$$|y(t)| \leq \mu L_1\left[\begin{array}{c}\frac{(|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ +\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\end{array}\right]. \tag{3.37}$$

Hence,

$$|y(t)| \leq \mu N_1 L_1, t \in J. \tag{3.38}$$

On the other hand,

$$\begin{aligned}\frac{1}{\mu}\left|D^{\sigma}y(t)\right| \leq\ & \frac{1}{\Gamma(\alpha-\delta)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds \\ &+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds \\ &+\frac{t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds \\ &+\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds \\ &+\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^{\delta}(s)\right)\right|ds.\end{aligned} \tag{3.39}$$

Thanks to $(H3)$, we have

$$\frac{1}{\mu}\left|D^{\sigma}y(t)\right| \leq L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right] + L_1\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]. \tag{3.40}$$

Therefore,

$$\left|D^{\sigma}y(t)\right| \leq \mu L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right] + \mu L_1\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right]. \tag{3.41}$$

Thus,

$$\left|D^{\beta}y(t)\right| \leq \mu L_1 N_2, t \in J. \tag{3.42}$$

From (3.38) and (3.42) , we get

$$\|y\|_X \leq \mu L_1 (N_1 + N_2). \tag{3.43}$$

Analogously, we can obtain

$$\|x\|_Y \leq \mu L_2 (N_3 + N_4). \tag{3.44}$$

It follows from (3.43) and (3.4) that

$$\|(x, y)\|_{X\times Y} \leq \mu L_1 (N_1 + N_2) + \mu L_2 (N_3 + N_4). \tag{3.45}$$

Hence,

$$\|\phi (x, y)\|_{X\times Y} < \infty. \tag{3.46}$$

This shows that the set Ω is bounded. Thanks to **A,B,C** and **D**, we conclude that ϕ has at least one fixed point. Theorem 3.2 is thus proved.

4. Examples

Example 4.1. Let us consider the coupled equations:

$$\begin{cases} D^{\frac{7}{2}}x(t)+\dfrac{|y(t)|}{7\left(\pi t^2+3\right)^2(2+|y(t)|)}+\dfrac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|\left|D^{\frac{7}{3}}y(t)\right|}{7\pi(1+e^t)^2\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)}=0, t\in[0,1], \\ D^{\frac{11}{3}}y(t)+\dfrac{3\pi|x(t)|}{\left(5e^{t^2}+3\sqrt{\pi}\right)(1+|x(t)|)}+\dfrac{\pi e^{-2\pi t}\left|D^{\frac{5}{2}}x(t)\right|}{5\left(t+3\sqrt{\pi}\right)^2\left(1+\left|D^{\frac{5}{2}}x(t)\right|\right)}=0, t\in[0,1], \\ x(0)=0, x(1)-\frac{3}{4}x\left(\frac{1}{3}\right)=0, y(0)=0, y(1)-\frac{3}{4}y\left(\frac{1}{3}\right)=0, \\ x''(0)=0, x''(1)-\frac{4}{5}x''\left(\frac{2}{3}\right)=0, y''(0)=0, y''(1)-\frac{4}{5}y''\left(\frac{2}{3}\right)=0. \end{cases} \tag{4.1}$$

It is clear that

$$
\begin{aligned}
f(t,x,y) &= \frac{|x|}{7\left(\pi t^2+3\right)^2(2+|x|)} + \frac{\sqrt{\pi}e^{-\pi t}\,|\cos(\pi t)|\,|y|}{7\pi(1+e^t)^2(2+|y|)}, t\in[0,1], x,y\in\mathbb{R},\\
g(t,x,y) &= \frac{3\pi|x|}{\left(5e^{t^2}+3\sqrt{\pi}\right)(1+|x|)} + \frac{\pi e^{-2\pi t}|y|}{5\left(t+3\sqrt{\pi}\right)^2(1+|y|)}, t\in[0,1], x,y\in\mathbb{R}.
\end{aligned}
$$

For $x, y, x_1, y_1 \in \mathbb{R}, t \in [0,1]$, we have

$$
|f(t,x,y)-f(t,x_1,y_1)| \le \frac{1}{7\left(\pi t^2+3\right)^2}|x-x_1| + \frac{\sqrt{\pi}e^{-\pi t}}{7\pi(1+e^t)^2}|y-y_1|,
$$

and

$$
|g(t,x,y)-g(t,x_1,y_1)| \le \frac{3\pi}{\left(5e^{t^2}+3\sqrt{\pi}\right)}|x-x_1| + \frac{\pi e^{-2\pi t}}{5\left(t+3\sqrt{\pi}\right)^2}|y-y_1|.
$$

Hence,

$$
a_1(t) = \frac{1}{7\left(\pi t^2+3\right)^2}, b_1(t) = \frac{\sqrt{\pi}e^{-\pi t}}{7\pi(1+e^t)^2},
$$

and

$$
a_2(t) = \frac{3\pi}{5e^{t^2}+3\sqrt{\pi}}, b_2(t) = \frac{\pi e^{-2\pi t}}{5\left(t+3\sqrt{\pi}\right)^2}.
$$

These imply that

$$
\begin{aligned}
\omega_1 &= \sup_{t\in[0,1]} a_1(t) = \frac{1}{63}, \omega_2 = \sup_{t\in[0,1]} b_1(t) = \frac{\sqrt{\pi}}{28\pi},\\
\varpi_1 &= \sup_{t\in[0,1]} a_2(t) = \frac{3\pi}{5+3\sqrt{\pi}}, \varpi_2 = \sup_{t\in[0,1]} b_2(t) = \frac{1}{45},
\end{aligned}
$$

$$
N_1 = 1,08935, N_2 = 3,444, N_3 = 0,77571, N_4 = 2,51754,
$$

and,

$$
(N_1+N_2)(\omega_1+\omega_2) + (N_3+N_4)(\varpi_1+\varpi_2) = 0,16329 + 0,36466 = 0,52795 < 1.
$$

So by Theorem 3.1, the problem (4.1) has a unique solution (x,y) on $[0,1]$.

Example 4.2. The following example illustrates Theorem 3.2. We take:

$$\begin{cases} D^{\frac{15}{4}}x(t) + \frac{1}{(t^2+1)\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)} + \frac{2e^{-t}|\cos(ty(t))|}{7(1+e^t)^2\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)} = 0, t \in [0,1], \\ D^{\frac{10}{3}}y(t) + \frac{1}{\left(e^{t^2}+1\right)(1+|x(t)|)} + \frac{e^{-t}}{(t+1)^2\left(1+\left|D^{\frac{5}{2}}x(t)\right|\right)} = 0, t \in [0,1], \\ x(0) = 0, x(1) - \frac{3}{4}x\left(\frac{1}{3}\right) = 0, y(0) = 0, y(1) - \frac{3}{4}y\left(\frac{1}{3}\right) = 0, \\ x''(0) = 0, x''(1) - \frac{4}{5}x''\left(\frac{2}{3}\right) = 0, y''(0) = 0, y''(1) - \frac{4}{5}y''\left(\frac{2}{3}\right) = 0. \end{cases} \quad (4.2)$$

We have

$$f(t,x,y) = \frac{1}{\left(t^2+1\right)(2+|y|)} + \frac{2e^{-t}|\cos(tx)|}{7(1+e^t)^2(2+|y|)}$$

and

$$g(t,x,y) = \frac{1}{\left(e^{t^2}+1\right)(1+|x|)} + \frac{e^{-t}}{(t+1)^2(1+|y|)}.$$

So by Theorem 3.2, the problem (4.2) has at least one solution on $[0,1]$.

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On the Divisibility of Trinomials by Maximum Weight Polynomials over $\mathbb{F}_2$

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Abstract

Divisibility of trinomials by given polynomials over finite fields has been studied and used to construct orthogonal arrays in recent literature. Dewar et al. (Dewar *et al.*, 2007) studied the division of trinomials by a given pentanomial over $\mathbb{F}_2$ to obtain the orthogonal arrays of strength at least 3, and finalized their paper with some open questions. One of these questions is concerned with generalizations to the polynomials with more than five terms. In this paper, we consider the divisibility of trinomials by a given maximum weight polynomial over $\mathbb{F}_2$ and apply the result to the construction of the orthogonal arrays of strength at least 3.

Keywords: Divisibility of trinomials, Maximum weight polynomials, Orthogonal arrays.
2010 MSC: 11T55, 05B15, 94A55.

1. Introduction

Sparse irreducible polynomials such as trinomials over $\mathbb{F}_2$ are widely used to perform arithmetic in extension fields of $\mathbb{F}_2$ due to fast modular reduction. In particular, primitive trinomials and maximum-length shift register sequences generated by them play an important role in various applications such as stream ciphers (see (Golomb, 1982), (Jambunathan, 2000)). But even irreducible trinomials do not exist for every degree. When a primitive (respectively irreducible) trinomial of a given degree does not exist, an almost primitive (respectively irreducible) trinomial, which is a reducible trinomial with primitive (respectively irreducible) factor, may be used as an alternative (Brent & Zimmermann, 2004). This encouraged the researchers to study divisibility of trinomials by primitive or irreducible polynomials (Cherif, 2008), (Golomb & Lee, 2007), (Kim & Koepf, 2009). The divisibility of trinomials by primitive polynomials is also related to orthogonal arrays.

Let f be a polynomial of degree m over $\mathbb{F}_2$ and let $a = (a_0, a_1, \cdots)$ be a shift-register sequence with characteristic polynomial f. Denote by C_n^f the set of all subintervals of this sequence with

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length n, where $m < n \leq 2m$, together with the zero vector of length n. Munemasa (Munemasa, 1998) observed that very few trinomials of degree at most $2m$ are divisible by a given primitive trinomial of degree m and proved that if f is a primitive trinomial satisfying certain properties, then C_n^f is an orthogonal array of strength 2 having the property of being very close to an orthogonal array of strength 3. Munemasa's work was extended in (Dewar *et al.*, 2007). The authors considered the divisibility of a trinomial of degree at most $2m$ by a given pentanomial f of degree m and obtained the orthogonal arrays of strength 3. They suggested some open questions in the end of their paper. One of them is to extend the results to finite fields other than $\mathbb{F}_2$. In this regard, Panario et al. (Panario *et al.*, 2012) characterized the divisibility of binomials and trinomials over $\mathbb{F}_3$. Another question in (Dewar *et al.*, 2007) is related to extend the results to the polynomials with more than five terms. In this paper we analyze the division of trinomials by a maximum weight polynomial over $\mathbb{F}_2$.

In the theory of shift register sequences it is well known that the lower the weight, i.e. the number of nonzero coefficients of the characteristic polynomial of shift register sequence, is, the faster is the generation of the sequence. But Ahmadi and Menezes (Ahmadi & Menezes, 2007) point out the advantage of maximum weight polynomials over $\mathbb{F}_2$ in the implementation of fast arithmetic in extension fields.

We show that no trinomial of degree at most $2m$ is divisible by a given maximum weight polynomial f of degree m, provided that $m > 7$. Using this result we can also obtain the orthogonal arrays of strength at least 3. The rest of the paper is organized as follows. In Section 2, some basic definitions and results are given and in Section 3, some properties of maximum weight polynomials and shift register sequences generated by them are mentioned. We focus on the divisibility of trinomials by maximum weight polynomials in Section 4, and conclude in Section 5.

2. Preliminaries

A *period* of a nonzero polynomial $f(x) \in \mathbb{F}_q[x]$ with $f(0) \neq 0$ is the least positive integer e for which $f(x)$ divides $x^e - 1$. A polynomial $f(x) \in \mathbb{F}_q[x]$ is called *reducible* if it has nontrivial factors; otherwise *irreducible*. A polynomial $f(x)$ of degree m is called *primitive* if it is irreducible and has period $2^m - 1$. The *reciprocal polynomial* of $f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \in \mathbb{F}_q[x]$ with $a_m \neq 0$ is defined by

$$f^*(x) = x^m f(1/x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_{m-1}x + a_m.$$

We refer to (Lidl & Niederreiter, 1994) for more information on the polynomials over finite fields. Throughout this paper we only consider a binary field $\mathbb{F}_2$ and all the polynomials are assumed to be in $\mathbb{F}_2[x]$, unless otherwise specified.

A *shift-register sequence* with characteristic polynomial $f(x) = x^m + \sum_{i=0}^{m-1} c_i x^i$ is the sequence $a = (a_0, a_1, \cdots)$ defined by the recurrence relation

$$a_{n+m} = \sum_{i=0}^{m-1} c_i a_{i+n}$$

for $n \geq 0$.

A subset C of $\mathbb{F}_2^n$ is called an *orthogonal array* of strength t if for any $t-$ subset $T = \{i_1, i_2, \cdots, i_t\}$ of $\{1, 2, \cdots, n\}$ and any $t-$tuple $(b_1, b_2, \cdots, b_t) \in \mathbb{F}_2^t$, there exist exactly $|C|/2^t$ elements $c = (c_1, c_2, \cdots, c_n)$ of C such that $c_{i_j} = b_j$ for all $1 \leq j \leq t$ (Munemasa, 1998). From the definition, if C is an orthogonal array of strength t, then it is also an orthogonal array of strength s for all $1 \leq s \leq t$.

The next theorem, due to Delsarte, relates orthogonal arrays to linear codes.

Theorem 2.1. (Delsarte, 1973) *Let C be a linear code over $\mathbb{F}_q$. Then C is an orthogonal array of maximum strength t if and only if $C^\perp$, its dual code, has minimum weight $t + 1$.*

Munemasa (Munemasa, 1998) described the dual code of the code generated by a shift-register sequence in terms of multiples of its primitive characteristic polynomial and Panario et al. (Panario *et al.*, 2012) generalized this result as follows by removing the primitiveness condition for the characteristic polynomial.

Theorem 2.2. (Panario *et al.*, 2012) *Let $a = (a_0, a_1, \cdots)$ be a shift register sequence with minimal polynomial f, and suppose that f has degree m with m distinct roots. Let ρ be the period of f and $2 \leq n \leq \rho$. Let C_n^f be the set of all subintervals of the shift register sequence a with length n, together with the zero vector of length n. Then the dual code of C_n^f is given by*

$$(C_n^f)^\perp = \{(b_1, \cdots, b_n) : \sum_{i=0}^{n-1} b_{i+1} x^i \text{ is divisible by } f.\}$$

A *maximum weight polynomial* is a degree-m polynomial of weight m (where m is odd) over $\mathbb{F}_2$(Ahmadi & Menezes, 2007), namely,

$$f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + x + 1 = \frac{x^{m+1} + 1}{x + 1} + x^l$$

If you take

$$g(x) = (x + 1) f(x) = x^{m+1} + x^{l+1} + x^l + 1,$$

then the weight of $g(x)$ is 4, and its middle terms are consecutive, so reduction using $g(x)$ instead of $f(x)$ is possible and can be effective in the arithmetic of an extension field $\mathbb{F}_{2^m}$ as if the reduction polynomial were a trinomial or a pentanomial. This fact motivated us to consider the divisibility of trinomials by maximum weight polynomials.

3. Character of shift register sequence generated by a maximum weight polynomial

In this section we state a simple property of maximum weight polynomials and characterize the shift register sequences generated by them.

Proposition 3.1. *Let $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$. If $f(x)$ is irreducible, then* $\gcd(m, l) = 1$.

Proof. Suppose $\gcd(m,l) = d > 1, m = m_1 d$ and $l = l_1 d$. Then we have

$$\begin{aligned} g(x) &:= (x+1)f(x) = x^{m+1} + x^{l+1} + x^l + 1 \\ &= x^{l+1}(x^{m-l}+1) + (x^l+1) = x^{l+1}(x^{m_1 d - l_1 d}+1) + (x^{l_1 d}+1) \\ &= x^{l+1}(x^{d(m_1-l_1)}+1) + (x^{l_1 d}+1). \end{aligned}$$

So $(x^d+1)/(x+1)$ is a factor of $f(x)$, which means $f(x)$ is reducible. □

Proposition 3.2. *Let $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$ be a primitive polynomial and*

$$a_{n+m} = \sum_{i=0}^{m-1} a_{n+i} + a_{n+l} (n \geq 0)$$

be a shift-register sequence with characteristic polynomial f. Then for all positive integer n,

$$a_{n+m} = a_{n-1} + a_{n-1+l} + a_{n+l}.$$

Proof. Since $f(x)$ is the characteristic polynomial of $(a_0, a_1, \cdots)$, we get $a_l = a_0 + a_1 + \cdots + a_m$ where $a_0, a_1, \cdots, a_{m-1}$ are initial values not all of which are zero. We use induction on n.

If $n = 1$,

$$\begin{aligned} a_{m+1} &= a_1 + \cdots + a_l + a_{l+2} + \cdots + a_m \\ &= a_0 + (a_0 + \cdots + a_l + a_{l+1} + a_{l+2} + \cdots + a_m) + a_{l+1} \\ &= a_0 + a_l + a_{l+1}. \end{aligned}$$

Now assume that the equation $a_{n+m} = a_{n-1} + a_{n-1+l} + a_{n+l}$ holds true for all positive integers less or equal to n. Then,

$$\begin{aligned} a_{m+n+1} &= a_{n+1} + \cdots + a_{n+l} + a_{n+l+2} + \cdots + a_{n+m} \\ &= (a_0 + \cdots + a_m) + (a_0 + \cdots + a_n) + a_{n+l+1} \\ &\quad + (a_{m+1} + \cdots + a_{m+n}) \\ &= a_l + (a_0 + \cdots + a_n) + a_{n+l+1} + (a_0 + a_l + a_{l+1}) \\ &\quad + (a_1 + a_{l+1} + a_{l+2}) + \cdots + (a_{n-1} + a_{l+n-1} + a_{l+n}) \\ &= a_n + a_{l+n} + a_{n+l+1} \end{aligned}$$

This completes the proof. □

4. Divisibility of trinomials by maximum weight polynomials

In this section we consider the divisibility of trinomials by maximum weight polynomials, provided that the degree of the trinomial does not exceed double the degree of the maximum weight polynomial. Let $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$ and suppose that $f(x)$ divides a trinomial $g(x)$ with

$$g(x) = f(x)h(x) = (x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1) \cdot \sum_{k=0}^{t} x^{i_k},$$

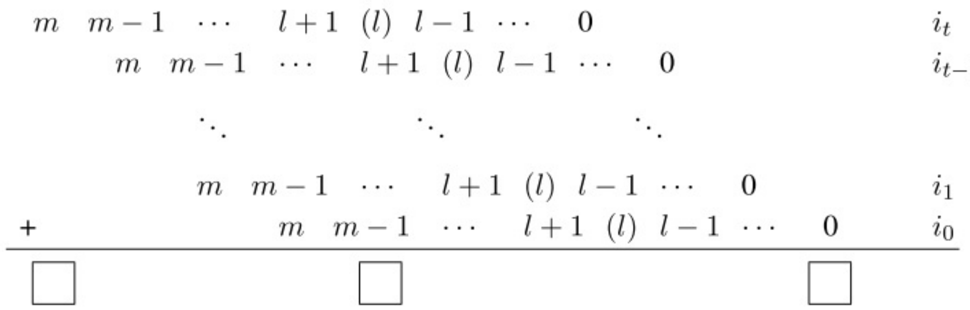

Figure 1. An illustration of equation $g(x) = f(x) \sum_{k=0}^{t} x^{i_k}$

where x^{i_k}s are the non-zero terms of $h(x)$ and $0 = i_0 < i_1 < \cdots < i_t$. The above equation can be illustrated as in Figure 1.
Here (l) stands for the missing terms. We adopt the same terminology as in (Dewar *et al.*, 2007), (Panario *et al.*, 2012). In particular, if the sum of coefficients in the same column of Figure 1 is 0, then we write that the corresponding terms x^i *cancel* and if the sum is 1 then we say that one of the corresponding terms is *left-over*. The proof of our main results will be done with Figure 1. Since the most top-left term $m + i_t$ and the most bottom-right term $0 + i_0$ are trivial left-over terms, we have only one left-over term undetermined. Below a *left-over term* means the left-over term which is neither $m + i_t$ nor $0 + i_0$. And we always assume that $m + i_0$ is in the same column as $s + i_t, 0 \le s \le m - 1$ and denote the number of terms in $h(x)$ as N.

Lemma 4.1. *Let* $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$ *and* $g(x)$ *be a trinomial of degree at most* $2m$ *divisible by* $f(x)$ *with* $g(x) = f(x)h(x)$. *Then* N *equals to 3 or 5.*

Proof. Since $g(x)$ is a trinomial and $f(x)$ has an odd number of terms, $h(x)$ also has an odd number of terms, that is, t is even. Suppose that N is greater or equal to 7. If $s \ge l$ then for every even number k, $m + i_{t-k}$ is a left-over term. Since $t \ge 6$, we have more than 2 left-over terms which contradicts the assumption.

Consider the case of $s < l$. First assume that there exists a unique left-over term to the left of $m + i_0$. It is sufficient to show $l \ge 3$ because if so, $0 + i_2$ is an extra left-term which leads to a contradiction. Observe a position $l + i_t$. If $l + i_t \ge m + i_{t-2}$ then clearly $l \ge i_{t-2} - i_0 \ge 4$, so we have done. Assume that $l + i_t < m + i_{t-2}$. Then $l + i_t \ge m + i_{t-4}$ because if not, then $m + i_{t-2}$ and $m + i_{t-4}$ are left-over terms. Thus we have $l \ge i_{t-4} - i_0$. If $l + i_t > m + i_{t-4}$ then $l > 2$ and if $l + i_t = m + i_{t-4}$ then $i_{t-4} - i_0 > 2$ because if $i_{t-4} - i_0 = 2$ then $m + i_{t-5} = l + i_{t-1}$ and so an extra left-over term appears.

Next assume that there is no left-over term to the left of $m+i_0$. Then it is clear that $m+i_{t-2} = l+i_t$ and $l \ge i_{t-2} - i_0 \ge 5$ hence $0 + i_2$ and $0 + i_4$ are left-over terms; contradiction. □

Lemma 4.2. *Under the same condition as in Lemma 1, if* $s < l$ *then* $m + i_0$ *cannot be a left-over term.*

Proof. Assume that $m + i_0$ is a left-over term. Then all the remaining terms in other columns must cancel and by Lemma 1 $N = 3$ or $N = 5$. If $N = 3$, then $l + i_1 > m + i_0$ from $s < l$ and thus an extra left-over term occurs in the column of $l + i_1$. Now assume that N is 5. We see easily $l + i_t = m + i_{t-2}$ and $i_t - i_{t-1} = 1$. If there is an extra left-over term to the left of $m + i_0$, then we have done. If there is no any extra left-over term to the left of $m + i_0$, then $i_2 - i_1 = 2$ because if $i_2 - i_1 = 1$ then $m + i_1 = l + i_{t-1}$ and so $m + i_1$ is an extra left-over term and if $i_2 - i_1 > 2$ then $l - 2 + i_t = l - 1 + i_{t-1} = m - 2 + i_2$ and so $l - 2 + i_t$ is an extra left-over term. Then from the condition $i_t \leq m$, it follows $l \geq 3$ and thus $0 + i_2$ is an extra left-over term; contradiction. □

Theorem 4.1. *Let* $f(x) = x^m + x^{m-1} + \cdots + x^{l+1} + x^{l-1} + \cdots + 1 \in \mathbb{F}_2[x]$. *If* $g(x)$ *is a trinomial of degree at most* $2m$ *divisible by* $f(x)$ *with* $g(x) = f(x)h(x)$, *then*

1) $f(x)$ *is one of the polynomial exceptions given in Table 1.*

2) $f(x)$ *is the reciprocal of one of the polynomials listed in the previous item.*

Table 1. Table of polynomial exceptions

No	$g(x)$	$f(x)$	$h(x)$
1	$x^5 + x^4 + 1$	$x^3 + x + 1$	$x^2 + x + 1$
2	$x^6 + x^4 + 1$	$x^3 + x^2 + 1$	$x^3 + x^2 + 1$
3	$x^9 + x^7 + 1$	$x^5 + x^3 + x^2 + x + 1$	$x^4 + x + 1$
4	$x^7 + x^5 + 1$	$x^5 + x^4 + x^3 + x + 1$	$x^2 + x + 1$
5	$x^8 + x^5 + 1$	$x^5 + x^4 + x^3 + x^2 + 1$	$x^3 + x^2 + 1$
6	$x^{14} + x^{13} + 1$	$x^7 + x^6 + x^5 + x^4 + x^3 + x + 1$	$x^7 + x^5 + x^2 + x + 1$
7	$x^{13} + x^{10} + 1$	$x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 1$	$x^6 + x^5 + x^3 + x^2 + 1$

Proof. We divide into three cases: $s > l$ or $s = l$ or $s < l$.

Case 1 : $s > l$.

Since $h(x)$ has an odd number of terms, $s \leq m - 2$ and $m + i_0$ is a left-over term, hence all the remaining terms in other columns must cancel. There is no missing term to the left of $s + i_t$, and therefore $m + i_{t-2}$ is a left-over term. This means $i_0 = i_{t-2}$, namely, $N = 3$. Since $m - 1 + i_0$ must cancel, $s = l + 1$ and $m - 2 + i_0$ cancels up automatically from $i_t - i_{t-1} = 1$. We see easily that $l = 1$ or $m - 3 + i_0$ is a missing term because $m - 3 + i_0$ must cancel up. If $l = 1$, then clearly $m = 5$ and we get the 5th polynomial in Table 1. If $m - 3 + i_0$ is a missing term, then $l = m - 3$. Since $l - 1 + i_0$ must cancel up, l must equal to 2 and so we get the 4th polynomial in Table 1.

Case 2 : $s = l$.

In this case, $m + i_0$ cannot be a left-over term because the number of non-zero terms in column of $m + i_0$ is even. If there is a unique left-over term to the left of $m + i_0$, then it must be $m - 1 + i_t$ or $m + i_2$.

Case 2.1 : $m - 1 + i_t$ is a unique left-over term to the left of $m + i_0$.

Clearly $i_{t-1} = i_t - 2$. If $N = 3$ then $m - 1 + i_0$ is an extra left-term and if $N = 5$ then $m + i_{t-2}$ is so. This contradicts to the assumption.

Case 2.2 : $m + i_2$ is a unique left-over term to the left of $m + i_0$.

This is the case of $N = 5$ and $i_t - i_{t-1} = i_2 - i_1 = 1$. $m - 1 + i_0$ cancels automatically because $m - 1 + i_0 = l + i_{t-1}$. Thus we have only two possible cases: $l = 1$ or $l \neq 1, l + i_2 = m - 2 + i_0$. Assume that $l = 1$ then $m - 3 + i_0$ must be in the column of $l + i_2$ and $m - 5 + i_0$ must cancel with $0 + i_1$ so we get the 7th polynomial in Table 1. And assume that $l \neq 1, l + i_2 = m - 2 + i_0$ then $i_{t-1} - i_2 = 1$ and observing $m - 4 + i_0$ implies that $m - 4 = l, l - 3 \neq 0$ or $m - 4 > l, l = 3$. In these two cases we have an extra left-over term $l-2+i_0$; contradiction.

Case 2.3 : There is no left-over term to the left of $m+i_0$.

It is obvious that $N=3$ and $i_t-i_1=1$. If $i_1-i_0>3$ then we have two left-over terms among $j+i_0(1\le j\le 3)$. Hence i_1-i_0 is less or equals to 3. Examining all cases for i_1-i_0 we get the reciprocals of the 1st, 3rd and 4th polynomials in Table 1.

Case 3 : $s<l$.

By lemma 2, $m+i_0$ is not a left-over term. So there exists $z(1\le z\le t-1)$ such that $m+i_0=l+i_z$.

Case 3.1 : $m+i_0=l+i_{t-1}$.

Clearly we have $i_{t-1}\ge i_t-3$. First assume that $i_{t-1}=i_t-3$. Then l equals to $m-1$ or $m-2$. If $l=m-1$, then $l-1+i_t=m-2+i_t$ is a left-over term so $l-3+i_t=l+i_{t-1}=m+i_0$ and $h(x)$ has three terms. Since the unique left-over term has already been determined, $0+i_t=l-1+i_{t-1}=l+i_0$ and we get the 3rd polynomial in Table 1. If $l=m-2$, then $m-1+i_t$ is a left-over term and $m+i_0$ must cancel with $0+i_t$ which means $i_1-i_0=2$ and $l=3$. But then $1+i_0$ appears as an extra left-over term; contradiction.

Next assume that $i_{t-1}=i_t-2$. When $l\ne m-1$, $m-1+i_t$ is a left-over term and $l\le m-3$ because if $l=m-2$ then $m+i_{t-1}$ is an extra left-over term. $l+i_t$ must cancel with $m+i_{t-2}$ and in fact N is 5. Thus $i_2-i_1=1$. By the condition $m+i_0=l+i_{t-1}$, we have $i_1-i_0=1$. Since $m-1+i_0$ must cancel up, $l-2=0$ or $m-3=l$. If $l-2=0$ then we get the 6th polynomial in Table 1 and the equation $m-3=l$ leads to a contradiction due to an extra left-over term in column of $l-3+i_0$. When $l=m-1$, clearly N is 3 from the conditin $l+i_{t-1}=m+i_0$. By research of possible values of l we get the reciprocals of the 2nd and 5th polynomials in Table 1.

Next assume that $i_{t-1}=i_t-1$. If $N=5$ then $m+i_{t-2}$ is a left-over term and $i_{t-2}-i_1=1$, hence an extra left-over term occurs in the comumn of $l+i_t$. Thus N is 3. Since $l-1+i_t=l+i_{t-1}=m+i_0$, $l+1+i_{t-1}$ is a left-over term. If $m-1\ne l$, then $l-1=0$ from consideration of $m-1+i_0$ and therefore we get the 2nd polynomial in Table 1. If $m-1=l$, then $l-1$ cannot be zero, so we get the 1st polynomial in Table 1.

Case 3.2 : $m+i_0=l+i_2$.

In this case N is 5 and clearly $2\le l\le m-2$. Observe a column of $l+i_t$.

Case 3.2.1 : $m+i_2<l+i_t$.

We have a left-over term in the column of $l+i_t$ and $i_t-i_{t-1}=1$. Then $m+i_2$ must cancel with $l-1+i_t$ and also i_2-i_1-1. By the condition $l+i_2=m+i_0$, $m-1+i_0$ must cancel with $l+i_1$. From $i_t\le m$ we have $l\ge 3$ and $i_1-i_0=1$ because if not, then $1+i_0$ is an extra left-over term. Hence l equals to $m-2$. Since $m-1+i_0$ must cancel up, $l-4\ne 0$. Observing the term $l-1+i_0$, we see that $l-5=0$ and then $l-2+i_0$ appears as an extra left-over term; contradiction.

Case 3.2.2 : $m+i_2=l+i_t$.

Assume that $m-1+i_t$ is a left-over term. Then clearly $l<m-2$ and $i_t-i_{t-1}=2$. If $i_2-i_0=2$, then $m+i_0$ must concel with $l+i_{t-1}$ which contradicts to the condition $m+i_0=l+t_{t-2}$. And if $i_2-i_0>2$, then an extra left-over term occurs in the column of $l+1+i_t$ or $l+2+i_t$ which again leads to a contradiction.

Now assume that $m-1+i_t$ is not a left-over term. Then $i_t-i_{t-1}=1$ and $m+i_1$ cancels with $l+i_{t-1}$ or $m+i_1<l+i_{t-1}$. If $m+i_1$ cancels with $l+i_{t-1}$ then $m+i_1$ is a left-over term and $i_2-i_1=1$. From $i_t\le m$, we have $0\le l-2$. Since if $i_1-i_0\ge 2$ then $1+i_0$ is an extra left-over term, $i_1-i_0=1$ and $l=m-2=4$. Then $l+2+i_0$ appears as an extra left-over term; contradiction. If $m+i_1<l+i_{t-1}$

then $m + i_1$ must cancel with $m - 2 + i_2$ or $m - 3 + i_2$. Briefly considering as above, we arrive at a contradiction in both cases.

Case 3.2.3 : $m + i_2 > l + i_t$.

You shall see that $l \leq m - 3, i_t - i_{t-1} = 1$ and $m + i_2$ is a left-over term. Since $m - 1 + i_2$ must cancel, $m - 1 + i_2 = l + i_t$ or $m - 1 + i_2 = m + i_1$. In the first case $i_2 - i_1 = 3$ because $l + i_{t-1} = m - 2 + i_2 = l - 1 + i_t$. Since $m + i_1 < l + i_{t-1}$, l is greater or equals to 3. If $i_1 - i_0 > 1$ then $1 + i_0$ is an extra left-over term and if $i_1 - i_0 = 1$ then $l = 3$ and $m - 2 + i_0$ is an extra left-over term, which leads to a contradiction. In the second case we have $l + i_t = m + i_0$; contradiction.

Case 3.3 : $m + i_0 = l + i_1$.

In this case we have $l \geq 3$ from $i_t \leq m$. First assume that $1 + i_0$ is a left-over term. Then clearly $i_1 - i_0 = 2, l + i_0 = 0 + i_2$ and $l + 1 + i_0 = l - 1 + i_1 = 1 + i_2 = 0 + i_{t-1}$. Since $l + 2 + i_0 = l + i_1 = 2 + i_2 = 1 + i_{t-1} = 0 + i_t$, we have $m = l + 2$. Then from $5 + i_2 = 4 + i_{t-1} = 3 + i_t$, we have $l = 5$ which corresponds the reciprocal of the 6th polynomial in Table 1.

Next assume that $1 + i_0$ is not a left-over term. Then $i_1 - i_0 = 1, l = m - 1$ and $0 + i_2$ is a left-over term because if not, then $0 + i_2 = l + i_0$ and thus $N = 3$ which is the case mentioned above. Considering the first and last terms in every rows, we have the following equations:

$$i_{t-1} - i_2 = 1, 0 + i_t = l + i_0, l + i_2 > m + i_1, i_2 - i_1 = 2,$$
$$0 + i_t = l + i_0, i_t - i_{t-1} = 2.$$

This implies the reciprocal of the 7th polynomial in Table 1. □

Note that every polynomial $f(x)$ listed in Table 1 has degree less than 8. From this fact we can immediately get the following corollary.

Corollary 4.1. *Let $f(x)$ be a maximum weight polynomial of odd degree m greater than 7 and $g(x)$ be a trinomial of degree at most $2m$. Then $g(x)$ is not divisible by $f(x)$.*

Combining these facts with Theorem 1 and Theorem 2, we get the following corollary on orthogonal arrays of strength 3.

Corollary 4.2. *Let $f(x)$ be a primitive maximum weight polynomial of odd degree m greater than 7. If $m \leq n \leq 2m$, then C_n^f is an orthogonal array of strength at least 3.*

5. Conclusion

In this paper, we analyzed the divisibility of trinomials by maximum-weight polynomials over $\mathbb{F}_2$ and used the result to obtain the orthogonal arrays of strength 3. More precisely, we showed that if $f(x)$ is a maximum-weight polynomial of degree m greater than 7, then $f(x)$ does not divide any trinomial of degree at most $2m$. Our work gives a partial answer to one of the questions posted in (Dewar *et al.*, 2007). As anticipated in (Dewar *et al.*, 2007), (Panario *et al.*, 2012), one seems to need some new techniques to give a complete answer to the question.

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Best Approximation in L^p-norm and Generalized (α, β)-growth of Analytic Functions

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Abstract

Let $0 < p \leq +\infty$ and $\Omega_R = \left\{z \in \mathbf{C}^n; \exp V_E(z) < R\right\}$, for some $R > 1$, where $V_E = \sup\left\{\frac{1}{d}\ln|P_d|, P_d \text{ polynomial of degree} \leq d, \|P_d\|_E \leq 1\right\}$ is the Siciak extremal function of a L-regular compact E.

The aim of this paper is the characterization of the generalized growth of analytic functions of several complex variables in the open set by means of the best polynomial approximation in L_p-norm on a compact E with respect to the set $\Omega_r = \left\{z \in \mathbf{C}^n; \exp V_E(z) \leq r\right\}, \quad 1 < r < R$.

Keywords: Extremal function, L-regular, generalized growth, best approximation of analytic function, L^p-norm.
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1. Introduction

Let E be a compact L-regular of $\mathbb{C}^n$. For an entire function f in $\mathbb{C}^n$ developed according an extremal polynomial basis $(A_k)_k$ (see Zeriahi (1987)), M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) have generalized growth in term of coefficients with respect the sequence $(A_k)_k$. The growth used by M. Harfoui was defined according to the functions α and β (see Harfaoui (2010), pp. 5, eq. (2.14)), with respect to the set:

$$\Omega_r = \left\{z \in \mathbb{C}^n,\ exp(V_E)(z) < r\right\},$$

where

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$$V_E = \sup\Big\{\frac{1}{d}\ \log|P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_E \leq 1\Big\}$$

is the Siciak's extremal function of E which is continuous in $\mathbb{C}^n$ (Because E is L-regular). The (α,β)-order and the (α,β)-type of f an entire function (or generalized order and generalized type) are defined respectively by:

$$\rho(\alpha,\beta) = \limsup_{r\to+\infty}\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(\log(r))} \quad \text{and} \quad \sigma(\alpha,\beta) = \limsup_{r\to+\infty}\frac{\alpha\Big(\|f\|_{\overline{\Omega}_r}\Big)}{\Big[\beta(r)\Big]^{\rho(\alpha,\beta)}},$$

where

$$\Big\|f\Big\|_{\overline{\Omega}_r} = \sup_{\overline{\Omega}_r}\Big|f(z)\Big|.$$

These results have been used to establish the generalized growth in terms of best approximation in L_p-norm for $p \geq 1$.

Let f be a function defined and bounded on E. For $k \in \mathbb{N}$ put

$$\pi_k^p(E,f) = \inf\Big\{\Big\|f - P\Big\|_{L^p(E,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\Big\},$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the family of all polynomials of degree $\leq k$ and μ the well-selected measure (The equilibrium measure $\mu = (dd^cV_E)^n$ associated to a L-regular compact E) (see Zeriahi (1983)) and $L^p(E,\mu)$, $p \geq 1$, is the class of all functions such that:

$$\Big\|f\Big\|_{L^p(E,\mu)} = \Big(\int_E |\ f\ |^p\ d\mu\Big)^{1/p} < \infty.$$

For an entire function $f \in \mathbb{C}^n$ M. Harfaoui established a precise relationship between the general growth with respect to the set (see (Harfaoui (2010)): $\Omega_r = \Big\{z \in \mathbb{C}^n :\ \exp(V_E)(z) < r\Big\}$, and the coefficients of the development of f with respect to the sequence $(A_k)_k$, called extremal polynomial (see Zeriahi (1987)). He used these results to give the relationship between the generalized growth of f and the sequence $(\pi_k^p(E,f))_k$. Note that M. Harfaoui did not study the case $0 < p < 1$ because the triangle inequality is not satisfied. A. Janik (see Janik (1991)) characterized the (α,β)-order of an analytic function g in Ω_R defned by

$$\Omega_R = \Big\{z \in \mathbb{C}^n,\ exp(V_E(z)) < R\Big\},\ \text{for some } R > 1,$$

by means of polynomial approximation and interpolation to g on on a L-regular compact E, with respect to the set

$$\Omega_r = \Big\{z \in \mathbb{C}^n,\ \exp(V_E(z)) < r,\quad 1 < r < R\Big\}.$$

In his work A. Janik used the best approximation defined, for a function defined and bounded on E, by:

$$\mathcal{E}_n^{(1)} = \mathcal{E}_n^{(1)}(f,E) = \|\ f - t_n\ \|,$$

$$\mathcal{E}_n^{(2)} = \mathcal{E}_n^{(2)}(f,E) = \|\ f - l_n\ \|,$$

$$\mathcal{E}^{(3)}_{n+1} = \mathcal{E}^{(3)}_{n+1}(f,E) =\| l_{n+1} - l_n \|,$$

where t_n denoted the nth Chebychev polynomial of the best approximation to f on E and l_n denoted the nth Lagrange interpolation polynomial for f with nodes at extremal points of E (see Siciak (1962)).

The (α,β)-order of an analytic function was defined as follows:

If E be a compact L-regular. If f is an analytic function in

$$\Omega_R = \Big\{z \in \mathbb{C}^n : \ \exp(V_E(z)) < R\Big\}$$

for some $R > 1$. We define the (α,β)-order of f (or generalized order) by

$$\rho(\alpha,\beta) = \limsup_{r\to R} \frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(R/(R-r))}$$

where $\big\|f\big\|_{\overline{\Omega}_r} = \sup_{\overline{\Omega}_r}\big|f(z)\big| = \sup\Big\{\ | f(z) |: \exp V_E(z) \le r, 1 < r < R\Big\}$.

In this work we study the generalized order and generalized type, which will be defined later, for an analytic function in the open set Ω_R, with respect to the set Ω_r in terms of coefficients of the analytic function in the development according to the sequence of extremal polynomials. So we obtain a generalization of the results of M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) and A. Janik (see Janik (1984), and Janik (1991)) replacing $\mathbb{C}^n$ by Ω_R and the entire function in $\mathbb{C}^n$ by analytic function in Ω_R .

After studying the generalized type of an analytic function in Ω_R, for some $R > 1$, we use this results to characterize the generalized type by means of best polynomial approximation on E in L_p-norm for $0 < p \le +\infty$.

Recall that the generalized growth used by M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) called (α,β)-growth was defined with respect to functions α and β defined as:

Let α and β be two positive, strictly increasing to infinity differentiable functions $]0,+\infty[$ to $]0,+\infty[$such that for every $c > 0$:

such that

$$\begin{cases} \lim\limits_{x\to+\infty} \dfrac{\alpha(cx)}{\alpha(x)} = 1, \\ \lim\limits_{x\to+\infty} \dfrac{\beta\Big(1 + x\omega(x)\Big)}{\beta(x)} = 1, \quad \lim\limits_{x\to+\infty} \omega(x) = 0, \\ \lim\limits_{x\to+\infty} \dfrac{d\Big(\beta^{-1}(c\alpha(x))\Big)}{\alpha(\log(x))} \le b. \\ \alpha\Big(x/\beta^{-1}(c\alpha(x))\Big) = (1 + o(x))\alpha(x), \qquad \text{for } x \to +\infty, \end{cases}$$

where $d(u)$ means the differential of u.

2. Definitions and notations

Before we give some definitions and results which will be frequently used in this paper.

Definition 2.1. (Siciak (1977)) Let E be a compact set in $\mathbb{C}^n$ and let $\|.\|_E$ denote the maximum norm on E. The function

$$V_E = \sup\left\{\frac{1}{d}\ \log|P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_E \leq 1, d \in \mathbb{N}\right\}$$

is called the Siciak's extremal function of the compact E.

Definition 2.2. Zeriahi (1983) A compact E in $\mathbb{C}^n$ is said to be L-regular if the extremal function, V_E, associated to E is continuous on $\mathbb{C}^n$.

Regularity is equivalent to the following Bernstein-Markov inequality (see Siciak (1962)): For any $\epsilon > 0$, there exists an open $U \supset E$ such that for any polynomial P, $\|P\|_U \leq e^{\epsilon . deg(P)}\|P\|_E$.

In this case we take $U = \left\{z \in \mathbb{C}^n; V_E(z) < \epsilon\right\}$.

Regularity also arises in polynomials approximation. For $f \in C(E)$, we let

$$\epsilon_d(E, f) = \inf\left\{\|f - P\|_E,\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the set of polynomials of degree at most d. Siciak (see Siciak (1977)) showed:

If E is L-regular, then $\limsup_{d\to+\infty}\Big(\varepsilon_d(E, f)\Big)^{1/d} = \frac{1}{r} < 1$ if and only if f has an analytic continuation to $\left\{z \in \mathbf{C}^n; V_E(z) < \log\left(\frac{1}{r}\right)\right\}$. It is known that if E is an compact L-regular of $\mathbb{C}^n$, there exists a measure μ, called extremal measure, having interesting properties (see Siciak (1962) and Siciak (1977)), in particular, we have:

(P_1) Bernstein-Markov inequality: $\forall\epsilon > 0$, there exists $C = C_\varepsilon$ is a constant such that

$$(BM) : \|P_d\|_E = C(1 + \varepsilon)^{s_k}\|P_d\|_{L^2(E,\mu)}, \tag{2.1}$$

for every polynomial of n complex variables of degree at most d.

(P_2) Bernstein-Waish (B.W) inequality:

For every set L-regular E and every real $r > 1$ we have:

$$\|f\|_E \leq M.r^{deg(f)}\Big(\int_E |\ f\ |^p\ .d\mu\Big)^{1/p} \tag{2.2}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.

Let $s : \mathbb{N} \to \mathbb{N}^n, k \to s(k) = (s_1(k), ..., s_n(k))$ be a bijection such that

$$|s(k + 1)| \geq |s(k)| \ \text{ where } \ |s(k)| = s_1(k) + ... + s_n(k).$$

A. Zeriahi (see Zeriahi (1987)) has constructed according to the Hilbert Schmidt method a sequence of monic orthogonal polynomials according to a extremal measure (see Siciak (1962)), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)} \tag{2.3}$$

such that $\left\|A_k\right\|_{L^p(E,\mu)} = \left[\inf\left\{\left\|z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)}\right\|_{L^2_{(E,\mu)}}, (a_1, a_2, ..., a_n) \in \mathbb{C}^n\right\}\right]^{1/s_k}$.

We need the following notations which will be used in the sequel: (N_1) $\nu_k = \nu_k(E) = \left\|A_k\right\|_{L^2(K,\mu)}$. (N_2) $a_k = a_k(E) = \left\|A_k\right\|_E = \max_{z\in K}\left|A_k(z)\right|$ and $\tau_k = \left(a_k\right)^{1/s_k}$, where $s_k = deg(A_k)$. With that notations and (B.W) inequality we have

$$\left\|A_k\right\|_{\overline{\Omega}_r} \leq a_k.r^{s_k} \tag{2.4}$$

where $s_k = deg(A_k)$. For more details (see Zeriahi (1983)).

Definition 2.3. Zeriahi (1983) Let E be a compact L-regular. If f is an analytic function in

$$\Omega_R = \left\{z \in \mathbb{C}^n :\ \exp(V_E(z)) < R\right\}$$

for some $R > 1$. We define the (α,β)-growth ((α,β)-order and (α,β)-type) of f (or generalized order) by $\rho(\alpha,\beta) = \limsup_{r\to R}\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(R/(R-r))}$, $\sigma(\alpha,\beta) = \limsup_{r\to R}\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\Big[\beta(R/(R-r))\Big]^{\rho(\alpha,\beta)}}$, where $\left\|f\right\|_{\overline{\Omega}_r} = \sup_{z\in\overline{\Omega}_r}\left|f(z)\right| = \sup\Big\{\mid f(z)\mid :\exp V_E(z) \leq r, 1 < r < R\Big\}$.

Note that in the classical case $\alpha(x) = \beta(x) = \log(x)$. We need the following lemma (see Zeriahi (1987)).

Lemma 2.1. *(Zeriahi (1987)) If E is a compact L-regular subset of $\mathbb{C}^n$, then for every $\theta > 1$, there exists an integer $N_\theta \geq 1$ and a constant $C_\theta > 0$ such that:*

$$\pi_k^p(E, f) \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^k}. \tag{2.5}$$

for every $k \geq 1$, every $r > 1$ and every $f \in 0\Big(\overline{\Omega}_{r\theta}\Big)$. If $f = \sum_{k=0}^{+\infty} f_k.A_k$ be an entire function, then for every $\theta > 1$, there exists $N_\theta \in \mathbb{N}^*$ and $C_\theta > 0$ such that

$$\Big|f_k\Big|\nu_k \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^{s_k}}, \tag{2.6}$$

for every $k \geq 0$ and $r > 1$. C_θ and N_θ do not depend on r or k, or f.

Note that the second assertion of the lemma is a consequence of the first assertion and it replaces Cauchy inequality for complex function defined on the complex plane $\mathbb{C}$.

3. Generalized order and coefficient characterizations with respect to extremal polynomial

The purpose of this section is to establish the relationship of the generalized growth of an analytic function in Ω_R with respect to the set $\Omega_r = \left\{z \in \mathbb{C} : \exp(V_E(z)) < r\right\}$ and coefficients of an entire function $f \in \mathbb{C}^n$ in the development with respect to the sequence of extremal polynomials.

Let $(A_k)_k$ be a basis of extremal polynomial associated to the set E defined the relation (2.3). We recall that $(A_k)_k$ is a basis of $O(\mathbb{C}^n)$ (the set of entire functions on $\mathbb{C}^n$). So if f is an entire function then $f = \sum_{k\geq 1} f_k.A_k$.

Put

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \mu(\alpha,\beta). \tag{3.1}$$

To prove the aim result of this section we need the following lemmas:

Lemma 3.1. *(Zeriahi (1987)) Let E be a compact L-regular subset of $\mathbb{C}^n$. Then*

$$\lim_{k\to+\infty} \left[\frac{|A_k(z)|}{\nu_k}\right]^{1/s_k} = \exp(V_E(z)), \tag{3.2}$$

for every $z \in \mathbb{C}^n \setminus \widehat{E}$ the connected component of $\mathbb{C}^n \setminus E$,

$$\lim_{k\to+\infty} \left[\frac{\|A_k\|_E}{\nu_k}\right]^{1/s_k} = 1. \tag{3.3}$$

Lemma 3.2. *For every $r > 1$ and $\mu > 0$, the maximum of the function*

$$x \to \omega(x,r) = x.\log(r/R) + \frac{x}{\beta^{-1}\left(\alpha(x)/\mu\right)}$$

is reached for $x = x_r$ solution of the equation

$$x = \alpha^{-1}\left\{\mu\beta\left[\frac{1 - d\log\left(\beta^{-1}(\alpha(x)/\mu)\right)/d(\log(x))}{\log\left(R/r\right)}\right]\right\}. \tag{3.4}$$

Proof. Put $G(x,\mu) = \beta^{-1}\left(\alpha(x)/\mu\right)$, then $\omega(x,r) = x.\log(r/R) + \dfrac{x}{G(x,\mu)}$. The maximum of the function $x \to \omega(x,r)$ is reached for $x = x_r$ solution of the equation of $\dfrac{d\omega(x,r)}{dx} = 0$. We have $\dfrac{\omega(x,r)}{dx} = 0 \Leftrightarrow \log(\dfrac{r}{R}) + \dfrac{G(x,\mu) - x.\dfrac{dG(x,\mu)}{dx}}{\left(G(x,\mu)\right)^2} = 0$, or $G(x,\mu) = \dfrac{1 - \dfrac{x}{G(x,\mu)}.\dfrac{dG(x,\mu)}{dx}}{\log(R/r)}$.

Since $\dfrac{dG(x,\mu)}{dx} = \dfrac{dG(x,\mu)}{d\log(x)}.\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.\dfrac{dG(x,\mu)}{d\log(x)}$, we get

$$G(x,\mu) = \frac{1 - \dfrac{1}{G(x,\mu)}.\dfrac{dG(x,\mu)}{d\log(x)}}{\log(R/r)} = \frac{1 - \dfrac{d\log G(x,\mu)}{d\log(x)}}{\log(R/r)}.$$

We deduce $x = x_r = \alpha^{-1}\left\{\mu\alpha\left[\dfrac{1 - d\Big(\beta^{-1}(\alpha(x)/\mu)\Big)/d(\log(x))}{\log\Big(R/r\Big)}\right]\right\}$. □

Lemma 3.3. *Let* $f = \sum_{k\geq 0} f_k.A_k$ *and E a L-regular compact. For every* $r \in]1, R[$, *we put*

$$\begin{cases} \overline{M}(f,r) = \sup_{k\in\mathbb{N}}\Big\{ \| f_k.A_k \|_E .r^k,\ r > 0\Big\} \\ \overline{\rho}(\alpha,\beta) = \limsup_{r\to R}\dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\beta(R/(R-r))} \end{cases}$$

then $\overline{\rho}(\alpha,\beta) \leq \mu(\alpha,\beta)$ *and* $\rho(\alpha,\beta) \leq \overline{\rho}(\alpha,\beta)$.

Proof. By the definition of μ (3.1) we have, for r sufficiently close to R and $\overline{\mu} = \mu + \epsilon$,

$$\log\Big(| f_k | .\tau_k^{s_k}.R^{s_k}\Big) \leq \frac{\alpha(s_k)}{\beta^{-1}\Big(\dfrac{1}{\overline{\mu}}.\alpha(s_k)\Big)}.$$

Then $\log\Big(| f_k | .\tau_k^{s_k}.r^{s_k}\Big) \leq s_k \log(r/R) + \dfrac{\alpha(s_k)}{\beta^{-1}\Big(\dfrac{1}{\overline{\mu}}.\alpha(s_k)\Big)}$. By the proprieties of α and β,the function $t \to \log(t)$ and the Lemma 3.3 we get $x_r = (1 + o(1))\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big)$ as $r \to R$. Indeed this result is a consequence of $\lim_{x\to+\infty}\left|\dfrac{d\Big(\beta^{-1}(c\alpha(x))\Big)}{\alpha(\log(x))}\right| \leq b$, $\log(1 + t) = (1 + o(t)).t$, $t \to 0$. Therefore $\log\Big(\| f_k.A_k \|_E .r^{s_k}\Big) \leq C_0.\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big)$, $k \in \mathbb{N}$. Passing to the maximum for the variable $k \in \mathbb{N}$ we obtain, for r sufficiently close to R $\log\Big(\overline{M}(f,r)\Big) \leq C_0.\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big)$, $k \in \mathbb{N}$. Then, by the proprieties of α, we obtain $\dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\beta(R/(R-r))} \leq \mu$. Passing to upper limit for $r \to R$ we have

$$(*)\quad \overline{\rho}(\alpha,\beta) \leq \mu.$$

Moreover we have for $z \in \Omega_r$ and $k \in \mathbb{N}$, $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} | f_k | .\|A_k\|_{\Omega_r} \leq \sum_{k\geq 0} | f_k | .\|A_k\|_E.r^{s_k}$. Write $r = \sqrt{r.R}.\sqrt{r/R}$, then $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} | f_k | .\|A_k\|_E.(\sqrt{r.R})^{s_k}.(\sqrt{r/R})^{s_k}$. Because $\sqrt{r/R} < 1$

then $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} \sup_{k\in_N}\Big(\mid f_k \mid .\|A_k\|_E.(\sqrt{r.R})^{s_k}\Big).(\sqrt{r/R})^{s_k}$ thus $\|f\|_{\overline{\Omega}_r} \leq \overline{M}(f,r')\sum_{k\geq 0}(\sqrt{r/R})^{s_k} \leq \overline{M}(f,r').\dfrac{1}{1-\sqrt{r/R}}$. where $r' = \sqrt{r.R}$. Therefore $\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \log\Big(\overline{M}(f,r')\Big) - \log\Big(1-\sqrt{r/R}\Big)$.
We have $\dfrac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(R/(R-r))} \leq \dfrac{\alpha\Big(\log(\overline{M}(f,\sqrt{r.R}) - \log(1-\sqrt{r/R})\Big)}{\beta(R/(R-\sqrt{r.R}))}.\dfrac{\beta(R/(R-\sqrt{r.R}))}{\beta(R/(R-r.R))}$. Passing to the upper limit we get

$$(**) \quad \rho(\alpha,\beta) \leq \overline{\rho}(\alpha,\beta).$$

By the relations (*) and (**) we obtain $\rho(\alpha,\beta) \leq \mu(\alpha,\beta)$. □

Theorem 3.1. *Let E be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$ such that*

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\Big(|f_k|.\tau_k^{s_k}.R^{s_k}\Big)}\right]} = \mu(\alpha,\beta) < \infty. \tag{3.5}$$

Then f is analytic in Ω_R, for some $R > 1$ and its (α,β)-order $\rho(\alpha,\beta) = \mu(\alpha,\beta)$.

Proof. It is known that for every polynomial P(see Siciak (1977))

$$| P(z) | \leq \| P \|_E \Big(\exp(V_E(z))\Big)^{deg(P)}, \quad \text{for every } z \in \mathbb{C}^n. \tag{3.6}$$

So for every $r \in]1,R[$, and for $P = f_k.A_k$ we get

$$| f_k.A_k(z) | \leq | f_k | . \| A_k \|_E \Big(\exp(V_E(z))\Big)^{s_k}, \text{for every} z \in \mathbb{C}^n. \tag{3.7}$$

Then for every $z \in \Omega_r$, we have $| f_k.A_k(z) | \leq | f_k | . \| A_k \|_E .r^{s_k}$. So, for every $r \in]1,R[$ the series $\sum f_k.A_k$ is convergent in Ω_r, whence $\sum_{k\geq 1} f_k.A_k$ is analytic in Ω_R.

Now we shall show that μ is the (α,β)-order of f. By the Lemma 3.3, to complete the proof of the theorem it suffices to show that $\rho(\alpha,\beta) \geq \mu(\alpha,\beta)$. By definition of ρ, we have, for every $\epsilon > 0$ there exists $r_\epsilon \in]1,R[$ such that for every $r \in]r_\epsilon, R[$ $\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \alpha^{-1}\Big[(\rho(\alpha,\beta)+\epsilon).\beta(R/(R-r))\Big]$. Applying (2.6) and (3.3) we have, for every $k \in \mathbb{N}$ and $r > 1$ sufficiently close to R

$$\log\Big(| f_k | \tau_k^{s_k}.R^{s_k}\Big) \leq -s_k\log\Big(r/R\Big) + \log\Big(C_0.\frac{(r-1)^{N_\theta}}{(R-r)^{-(2N+1)}}\Big) + \log\Big(\|f\|_{\overline{\Omega}_r}\Big), \tag{3.8}$$

then $\log\Big(| f_k | \tau_k^{s_k}.R^{s_k}\Big) \leq \varphi(r,s_k)$, where

$$\varphi(r,s_k) = -s_k\log\Big(r/R\Big) + \log\Big(C_0.\frac{(r-1)^{N_\theta}}{(R-r)^{-(2N+1)}} + \beta^{-1}\Big[(\rho(\alpha,\beta)+\epsilon).\beta(R/(R-r))\Big].$$

Put $\rho = \rho(\alpha,\beta)$ and $r_k = R.\left\{1 - \dfrac{1}{\beta^{-1}\left(\dfrac{1}{\rho+\epsilon}.\alpha\Big(\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}\Big)\right)}\right\}$. Replacing in the relation (3.8) r by r_k and applying the proprieties of the functions α and β:

$$\alpha\Big(x/\beta^{-1}(c\alpha(x))\Big) = (1+o(x))\alpha(x), \text{ for } c>0,\ x\to+\infty,$$

and the proprieties of the logarithm, we obtain $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq C_1.\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}$ where C_1 is a constant. Therefore $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq C_1.\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}$, thus

$$\beta\left(\frac{C_1.s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}\right) \geq \alpha(s_k)/(\rho+\epsilon).$$

Passing to the upper limit, after a simple calculus, we obtain $\mu(\alpha,\beta) \leq \rho(\alpha,\beta)$. □

4. Generalized type and coefficient characterizations with respect to extremal polynomial

The purpose of this section is to establish the relationship of the generalized type of an analytic function in Ω_R with respect to the set $\Omega_r = \Big\{z \in \mathbb{C} :\ \exp(V_E(z)) < r\Big\}$ and its coefficients in the development according to the sequence of extremal polynomials.

Let E be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$ be an analytic function of (α,β)-order $\rho = \rho(\alpha,\beta)$, and put:

$$\tau_E(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{4.1}$$

We need the following proposition:

Proposition 4.1. *Let $f = \sum_{k\geq 0} f_k.A_k$ and E a L-regular compact. For every $r \in]1,R[$, we put*

$$\begin{cases} \overline{M}(f,r) = \sup_{k\in\mathbb{N}}\Big\{\mid f_k \mid . \parallel A_k \parallel_E .r^{s_k}\Big\} \\ \overline{\sigma_1}(\alpha,\beta) = \limsup_{r\to R} \dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\Big(\beta(R/(R-r))\Big)^{\rho(\alpha,\beta)}} \end{cases}$$

then $\sigma(\alpha,\beta) \leq \overline{\sigma_1}(\alpha,\beta)$.

Proof. For $z \in \Omega_r$ and $k \in \mathbb{N}$, using the similar arguments and inequalities as in Lemma 2.3

$$\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\Big[\beta(R/(R-r))\Big]^{\rho(\alpha,\beta)}} \leq \frac{\alpha\Big(\log(\overline{M}(f,\sqrt{r.R}) - \log(1-\sqrt{r/R}))\Big)}{\Big[\alpha(R/(R-\sqrt{r.R}))\Big]^{\rho(\alpha,\beta)}} . \frac{\Big[\alpha(R/(R-\sqrt{r.R}))\Big]^{\rho(\alpha,\beta)}}{\Big[\alpha(R/(R-r))\Big]^{\rho(\alpha,\beta)}}.$$

We have $\limsup_{r\to R} \frac{\left[\alpha(R/(R-\sqrt{r.R}))\right]^{\rho(\alpha,\beta)}}{\left[\alpha(R/(R-r))\right]^{\rho(\alpha,\beta)}} = 1.$ □

Proceeding to the upper limit we get

$$(*) \quad \sigma(\alpha,\alpha) \leq \overline{\sigma_1}(\alpha,\beta).$$

Theorem 4.1. *Let E be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$. If f is of finite generalized (α,β)-order $\rho(\alpha,\beta)$, and*

$$\tau_E(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho(\alpha,\beta)}} < +\infty. \tag{4.2}$$

Then f is analytic in Ω_R, for some $R > 1$, and its (α,β)-type $\sigma(\alpha,\beta) = \tau_E(\alpha,\beta)$.

Proof. Put $\tau = \tau_E(\alpha,\beta)$, $\rho = \rho(\alpha,\beta)$, and $\sigma = \sigma(\alpha,\beta)$. The function is analytic by the definition $\tau_E(\alpha,\beta)$ and the arguments used in theorem 3.1.

1. Now we show that $\sigma(\alpha,\beta) \leq \tau_E(\alpha,\beta)$. If $\tau < \infty$, by the definition of τ, for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_\epsilon$ $\alpha(s_k) \leq (\tau+\epsilon).\left\{\beta\left(\frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho}$. A simple calculus gives for, $\overline{\tau} = \tau + \epsilon$.

$$\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) \leq \frac{s_k}{\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(s_k)\right)^{1/\rho}\right)}, \tag{4.3}$$

for every $k \geq k_\epsilon$for every $k \geq k_\epsilon$.
Since $\log\left(\mid f_k \mid .\tau_k^{s_k}.r^{s_k}\right) \leq s_k \log(r/R) + \log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)$. By (4.3), we get

$$\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq s_k \log(r/R) + \frac{s_k}{\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(s_k)\right)^{1/\rho}\right)}. \tag{4.4}$$

For every $r \in]1, R[$, and r and r sufficiently close to R, we put

$$\phi(x,r) = x\log(r/R) + \frac{x}{\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)}.$$

If we put $F = F(x,\overline{\tau},\frac{1}{\rho}) = \beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)$ then $\phi(x,r) = x\log(r/R) + \frac{x}{F}$, and the maximum of the function $x \to \phi(x,r)$ is reached for $x = x_r$ solution of the equation of

$$\frac{d\phi(x,r)}{dx} = \frac{\partial\phi}{\partial x}(x,r) = \log(r/R) + \frac{d}{dx}\left\{\frac{x}{F}\right\} = 0.$$

We have $\dfrac{\phi(x,r)}{dx} = 0 \Leftrightarrow \log(\dfrac{r}{R}) + \dfrac{F - x.\dfrac{dF}{dx}}{(F)^2} = 0$, or $F = \dfrac{1 - \dfrac{x}{F}.\dfrac{dF}{dx}}{\log(R/r)}$. Since $\dfrac{dF}{dx} = \dfrac{dF}{d\log(x)}.\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.\dfrac{dF}{d\log(x)}$, we get $F = \dfrac{1 - \dfrac{1}{F}.\dfrac{dF}{d\log(x)}}{\log(R/r)} = \dfrac{1 - \dfrac{d\log F}{d\log(x)}}{\log(R/r)}$, or

$$\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right) = \frac{1 - \dfrac{d\log\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)}{d\log(x)}}{\log(R/r)}.$$

We deduce $x = x_r = \alpha^{-1}\left\{\left[\overline{\tau}.\beta\left(\dfrac{1 - d\log\left(\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)\right)/d(\log(x))}{\log\left(R/r\right)}\right)\right]^{\rho}\right\}$. We have $\log\left(\dfrac{r}{R}\right) = \log\left(\dfrac{r-R}{R} + 1\right) \sim \dfrac{r-R}{R}$ $\left(\text{because } \dfrac{r-R}{R} \to 0\right)$ and $\left|\dfrac{d\left[\log\left(\beta^{-1}\left(\left(\alpha(x)\right)^{\rho}\right)\right)\right]}{d\log(x)}\right| \leq b$, where b is a positive constant. Then by the proprieties of α we get

$$x_r = (1 + o(1))\rho.\beta^{-1}\left(\overline{\tau}(\alpha(R/(R-r)))^{\rho}\right).$$

By (4.4), we have $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq \sup_{r\in\mathbb{N}}\phi(x,r) = \phi(x_r,r)$. Replacing s_k by x_r in this last relation we obtain $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq \dfrac{(1+o(1))\beta^{-1}\left(\overline{\tau}(\alpha(R/(R-r)))^{\rho}\right)}{R/(R-r)}$. Since $\dfrac{R}{R-r} > 1$ and $\dfrac{\rho - 1}{\rho} < 1$, then $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq C.\beta^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)$.

Then $\sup_{k\in\mathbb{N}}\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k} \leq C.\alpha^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)\right.$ or $\log(\overline{M}(f,r)) \leq C.\beta^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)$.

Therefore $\dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \overline{\tau}$.

Proceeding to the upper limit for $r \to R$, get $\overline{\sigma}_1(\alpha,\alpha) = \lim_{r\to R}\dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \tau$.

By the relations (*) of the proposition 4.1 we obtain $\sigma(\alpha,\alpha) = \lim_{r\to R}\dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \tau$.

Thus $\sigma(\alpha,\beta) \leq \tau_E(\alpha,\beta)$. The result is obviously holds for $\tau = +\infty$.

2. Now we show that $\sigma(\alpha,\beta) \geq \tau_E(\alpha,\beta)$. Put $\overline{\sigma} = \sigma(\alpha,\beta) + \epsilon$, $\rho = \rho(\alpha,\beta)$. Suppose that $\sigma < \infty$. By definition of $\sigma(\alpha,\beta)$, we have for every $\epsilon > 0$, there exist $r_\epsilon \in]1,R[$, such that for every

$r > r_\epsilon$ ($R > r > r_\epsilon > 1$) $\log\left(\|f\|_{\overline{\Omega}_r}\right) \leq \alpha^{-1}\left[\overline{\sigma}.\left(\beta(R/(R-r))\right)^{\rho}\right]$. Applying (3.3) and (2.6) we get, for every $k \in \mathbb{N}$ and r sufficiently close to R:

$$\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq -s_k \log(r) + \log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \log\left(\|f\|_{\overline{\Omega}_r}\right).$$

As for every $r \in]1, R[$ $\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) = -s_k \log\left(r/R\right) + \log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right)$ then $\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) \leq -s_k \log\left(r/R\right) + \log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \log\left(\|f\|_{\overline{\Omega}_r}\right)$. or $\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) \leq -s_k \log\left(r/R\right) + \log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \alpha^{-1}\left[\overline{\sigma}.\left(\beta(R/(R-r))\right)^{\rho}\right]$.

Since $s_k \geq 1$, we obtain, for k sufficiently large, $\frac{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}{s_k} \leq \omega(r,k)$ where $\omega(r,k) = -\log\left(r/R\right) + \frac{1}{s_k}\log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \frac{1}{s_k}\alpha^{-1}\left[\overline{\sigma}.\left(\beta(R/(R-r))\right)^{\rho}\right]$.

Since $\lim\limits_{k\to+\infty} \frac{1}{s_k}\log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \frac{1}{s_k}\alpha^{-1}\left[\overline{\sigma}.\left(\beta(R/(R-r))\right)^{\rho}\right] = 0$ we get, for r sufficiently close to R, $\lim\limits_{k\to+\infty} \omega(r,k) = -\log\left(r/R\right) = \log\left(R/r\right)$.

Then for k sufficiently large and r sufficiently close to R, we have $\omega(r,k) = (1+o(1))\log\left(R/r\right)$, $k \to +\infty$, then

$$\frac{1}{s_k}\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) \leq (1+o(1))\log\left(R/r\right). \tag{4.5}$$

Choose $r_k = R.\frac{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho}}{1+\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho}}$. Using the relation (4.5) and the proprieties of the function $t \to \log(t)$, we obtain, for r sufficiently close to R $\frac{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}{s_k} \leq (1+o(1))\left(\frac{R}{r}-1\right)$.

because $\log\left(\frac{R}{r}\right) = \log\left(\frac{R-r+r}{r}\right) = \log\left(1+\frac{R-r}{r}\right) \sim \frac{R-r}{r}$ $(r \to R)$.

Replacing r by the chosen r_k in this last relation we obtain $\frac{R-r_k}{r_k} = \frac{1}{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho}}$.

Then, for r sufficiently close to R and k r sufficiently large we get $\frac{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}{s_k} \leq \frac{1}{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho}}$, thus $\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho} \leq \frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}$ or $\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho} \leq \beta\left(\frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}\right)$.

Therefore $\frac{1}{\overline{\sigma}}\alpha(s_k) \leq \left\{\beta\left(\frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right)}\right)\right\}^{\rho}$ or $\frac{\alpha(s_k)}{\left\{\beta\left(\frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right)}\right)\right\}^{\rho}} \leq \overline{\sigma} = \sigma + \epsilon$.

Proceeding to the upper limit we obtain $\sigma(\alpha,\beta) \geq \tau_E(\alpha,\beta)$. The result is obviously holds for $\sigma(\alpha,\beta) = +\infty$.

□

5. Generalized (α,β)-growth and best polynomial approximation of analytic functions in L^p-norm.

Let E a L-regular compact of $\mathbb{C}^n$. The purpose of this paragraph is to give the relationship between the generalized order of an analytic function and speed of convergence to 0 in the best polynomial in L_p-norm on E. We need the following lemma.

Lemma 5.1. . *Let* $f = \sum_{k\geq 0} f_k.A_k$ *an element of* $L^p(E,\mu)$, *for* $p \geq 0$, *and*

$$\pi_k^p(E,f) = \inf\left\{\left\|f - P\right\|_{L^p(E,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}.$$

Then

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} \tag{5.1}$$

and

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho(\alpha,\beta)}} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\left\{\beta\left(\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{5.2}$$

Proof. Assume that $p \geq 2$. If $f \in L^p(E,\mu)$ where $p \geq 2$, then $f = \sum_{k=0}^{+\infty} f_k.A_k$ with convergence in $L^2(E,\mu)$, hence for $k \geq 0$, $f_k = \dfrac{1}{{\nu_k}^2}\displaystyle\int_E f.\overline{A}_k d\mu$ and therefore $f_k = \dfrac{1}{{\nu_k}^2}\displaystyle\int_E (f - P_{k-1}).\overline{A}_k d\mu$ (because $deg(A_k) = s_k$). Since the relation, $\left|f_k\right| \leq \dfrac{1}{{\nu_k}^2}\displaystyle\int_E \left|f - P_{k-1}\right|.\left|\overline{A}_k\right|\mu$ is satisfied, is easily verified by using inequalities Bernstein-walsh and Holder that we have for all $\varepsilon > 0$

$$\left|f_k\right|.\nu_k \leq C_\varepsilon.(1+\varepsilon)^{s_k}.\pi_{s_k-1}^p(E,f). \tag{5.3}$$

for all $k \geq 0$.

If $1 \leq p < 2$, let p' such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have $p' \geq 2$. According to the inequality of Hölder we have: $\left|f_k\right|.\nu_k^2 \leq \left\|f - P_{k-1}\right\|_{L^p(E,\mu)}.\left\||A_k\right\|_{L^{p'}(E,\mu)}$. But $\left\|A_k\right\|_{L^{p'}(E,\mu)} \leq C.\left\|A_k\right\|_E = C.a_k(E)$. This shows, according to inequality (BM), that: $\left|f_k\right|.\nu_k^2 \leq C.C_\varepsilon.(1+\varepsilon)^{s_k}.\left\|f - P_{s_k-1}\right\|_{L^p(E,\mu)}$.

Hence the result $\left|f_k\right|.\nu_k^2 \leq C'_\varepsilon.(1+\varepsilon)^{s_k}.\pi_k^{s_k-1}(E,f)$. In both cases we have therefore

$$\left|f_k\right|.\nu_k^2 \leq A_\varepsilon.(1+\varepsilon)^{s_k}.\pi^p_{s_k-1}(E,f) \tag{5.4}$$

where A_ε is a constant which depends only on ε.
After passing to the upper limit in the relation (5.4) and applying the relation (3.3) we get

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} \leq \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}.$$

To prove the other inequality we consider the polynomial of degree s_k, $P_k(z) = \sum_{s_j=0}^{k} f_j.A_j$ then $\pi^p_{s_k-1}(E,f) \leq \sum_{s_j=s_k}^{+\infty} |f_j|.\left\|A_j\right\|_{L^p(E,\mu)} \leq C_0 \sum_{s_j=s_k}^{+\infty} |f_j|.\left\|A_j\right\|_E$. By Bernstein-Walsh inequality we have $\pi^p_k(E,f) \leq C_\epsilon \sum_{s_j=s_k}^{+\infty} (1+\epsilon)^{s_j}\left|f_j\right|.\nu_j$ for $k \geq 0$ and $p \geq 1$. If we take as a common factor $(1+\varepsilon)^{s_k}.\left|f_k\right|.\nu_k$ the other factor is convergent thus we have $\pi^p_k(K,f) \leq C(1+\epsilon)^{s_k}.\left|f_k\right|.\nu_k$ and by (3.3) we have, then

$$\pi^p_k(E,f) \leq C(1+\epsilon)^{2s_k}.\left|f_k\right|.\tau_k^{s_k}. \tag{5.5}$$

We deduce $\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} \geq \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}$. □

Applying this Lemma 5.1 we get the following main result:

Theorem 5.1. *Let $f \in L^p(E,\mu)$, then f is μ-almost-surely the restriction to E of an analytic function in $\mathbb{C}^n$ of finite generalized order $\rho(\alpha,\beta)$ if and only if*

$$\rho(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} + \infty. \tag{5.6}$$

Theorem 5.2. *Let $f \in L^p(E,\mu)$, then f is μ-almost-surely the restriction to E of an analytic function in $\mathbb{C}^n$ of finite generalized order $\rho(\alpha,\beta)$ and finite generalized type $\sigma(\alpha,\beta)$ if and only if*

$$\sigma(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\left\{\beta\left(\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{5.7}$$

Proof. We prove only the first Theorem 5.1, the second is proved by the same arguments.

Suppose that f is μ-almost-surely the restriction to E of an entire function g of general order ρ $(0 < \rho < +\infty)$ and show that $\rho = \rho(\alpha,\beta)$.

We have $g \in L^p(E,\mu)$, $p \geq 2$ and $g = \sum_{k\geq 0} g_k.A_k$ in $L^2(E,\mu)$ Since g is an element of $L^2(E,\mu)$ then $g = \sum_{k=0}^{+\infty} g_k.A_k$ and according to the Theorem 3.1 $\rho(g,\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]}$ and with the Lemma 5.1 (relation(5.1)) we have $\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}$.

But $g = f$ on E hence $\rho = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$.

Now suppose that f is a function of $L^p(E,\mu)$ such that the relation (5.6) is verified. The proof is done in three steps $p \geq 2$, $1 \leq p < 2$ and $0 < p < 1$.

Step.1. Let $p \geq 2$, then $f = \sum_{k=0}^{+\infty} f_k.A_k$, because f is an element of $L^2(E,\mu)$ $\left((L^p(E,\mu))_{p\geq 1}\right.$ is decreasing sequence$\left.\right)$. Consider in $\mathbb{C}^n$ the series $\sum f_k.A_k$, $k \geq 0$. By the relation (5.6) and the inequality (BW) we have the inequality on coefficients $|A_k|$ (2.4), it can be seen that this series converges normally on all compact of $\mathbb{C}^n$, to an analytic function denoted f_1. We have $f_1 = f$, obviously, μ-almost surly on E.

We verify easily that this series converges normally on all compact of $\mathbb{C}^n$ to an analytic function denoted f_1. We have $f_1 = f$, obviously, μ-almost surly on E, and by Theorem 3.1 we have

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\frac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty.$$

According to the Lemma 5.1 we get $\rho(f_1) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$.

Let $f_1 = \sum_{k\geq 0} f_k.A_k$, then $f_1(z) = f(z)$ μ-almost surely for every z in E. Therefore the (α,β)-order of f_1 is: $\rho(f_1,\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\frac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$ (see Theorem 3.1). By Lemma 5.1 we check $\rho(f_1) = \rho$ so the proof is completed.

Step.2. Now let $p \in [1,2[$ and $f \in L^p(E,\mu)$. By (BM) inequality and Hölder inequality we have again the inequality the relation (5.4) and by the previous arguments we obtain the result.

Step.3. Let $0 < p < 1$, of course, for $0 < p < 1$ the L_p-norm does not satisfy the triangle inequality. But our relations (5.3) and relation (5.4) are also satisfied for $0 < p < 1$ (see Kumar (2011)), because using Holder's inequality we have, for some $M > 0$ and all $r > p$ (p fixed)

$$\|\|\| f \|_{L^p(E,\mu)} \leq M. \| f \|_{L^r(E,\mu)} .$$

Using the inequality $\int_E | f |^p \; d\mu \leq \| f \|_E^{p-r} . \int_E | f |^r \; d\mu$ we get $\| f \|_{L^p(E,\mu)} \leq \| f \|_E^{1-(r/p)} . \| f \|_{L^r(E,\mu)}^{r/p}$. We deduce that (E, μ) satisfies the Bernstein.Markov inequality. For $\epsilon > 0$ there is a constant $C = C(\epsilon, p) > 0$ such that, for all (analytic) polynomials P we have

$$\|\|\| P \|_E \leq C(1+\epsilon)_{deg(P)}. \; \| P \|_{L^p(E,\mu)} .$$

Thus if (E, μ) satisfies the Bernstein-Markov inequality for one $p > 0$ then (5.4) and (5.5) are satisfied for all$p > 0$.

The rest of proof is easily deduced using the same reasoning as in step 1 and step 2. □

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Common Fixed Points of Hardy and Rogers Type Fuzzy Mappings on Closed Balls in a Complete Metric Space

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Abstract

In this paper we obtain some common fixed point theorems for Hardy and Rogers type fuzzy mappings on closed balls in a complete metric space. Our investigation is based on the fact that fuzzy fixed point results can be obtained simply from the fixed point theorem of multi-valued mappings with closed values.In real world problems there are various mathematical models in which the mappings are contractive on the subset of a space under consideration but not on the whole space itself. Our results generalize several results of literature.

Keywords: Fuzzy fixed point, Hardy and Rogers mapping, contraction, closed balls, continuous mapping.
2010 MSC: 47H10, 54H25, 54A40.

1. Introduction

It is a well-known fact that the results of fixed points are very useful for determining the existence and uniqueness of solutions to various mathematical models.In 1922,Banach a Polish mathematician proved a theorem under appropriate of a fixed point this result is called Banach fixed point theorem.This theorem is also applied to prove the existence and uniqueness of the solutions of differential equations. Many authors have made different generalization of Banach fixed point theorem. The study of fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity, and it has a wide range of applications in different areas such as nonlinear and adoptive control systems, parameterize estimation problems, fractal image decoding,computing magneto static fields in a nonlinear medium and convergence of recurrent networks.

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The notion of fixed points for fuzzy mappings was introduced by Weiss (Weiss, 1975) and Butnariu (Butnariu, 1982). Fixed point theorems for fuzzy set valued mappings have been studied by Heilpern (Heilpern, 1981) who introduced the concept of fuzzy contraction mappings and established Banach contraction principle for fuzzy mappings in complete metric linear spaces which is a fuzzy extension of Banach fixed point theorem and Nadlers (Nadler, 1969) theorem for multi-valued mappings. Park and Jeong (Park & Jeong, 1997) proved some common fixed point theorems for fuzzy mappings satisfying in complete metric space which are fuzzy extensions of some theorems in (Azam, 1992; Park & Jeong, 1997). In this paper we obtain some common fixed point theorems of Hardy and Rogers type fuzzy mappings on closed balls.

2. Basic concepts

Let (X, d) be a metric space, then we use the following notations: Let

$$2^X = \{A : A \text{ is a subset of } X\},$$

$$CL(2^X) = \{A \in 2^X : A \text{ is nonempty and closed}\},$$

$$C(2^X) = \{A \in 2^X : A \text{ is nonempty and compact}\},$$

$$CB(2^X) = \{A \in 2^X : A \text{ is nonempty, closed and bounded}\},$$

For $A, B \in CB(2^X)$, $d(x, A) = \inf_{y \in A} d(x, y)$, $d(A, B) = \inf_{x \in A, y \in B} d(x, y)$ then the Hausdroff metric d_H on $CB(2^X)$ induced by d is defined as: $d_H(A, B) = \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}$.
A fuzzy set in X is a function with domain X and values in $[0, 1]$ and I^X is the collection of all fuzzy sets in X . If A is a fuzzy set and $x \in X$ then the function values $A(x)$ is called the grade of membership of x in A. The α-level set of a fuzzy set A, is denoted by $[A]_\alpha$, and is defined as:

$$[A]_\alpha = \{x : A(x) \geq \alpha \text{ if } \alpha \in (0, 1]\} \text{ and } [A]_0 = \overline{\{x : A(x) \geq 0\}}.$$

For $x \in X$, we denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where χ_A is the characteristic function of the crisp set A. Now we define a sub-collection of I^X as follows: $\tau(X) = \{A \in I^X : [A]_1 \text{is nonempty and closed}\}$, for A,B $\in I^X$, $A \subset B$ means $A(x) \leq B(x)$ for each $x, y \in X$. For $A, B \in \tau(X)$ then define $D_1\{A, B\} = d_H([A]_1, [B]_1)$.
A point $x^* \in X$ is called a fixed point of a fuzzy mappings $T : X \to I^X$ if $x^* \subset Tx^*$ see (Heilpern, 1981)

Lemma 2.1. *(Nadler, 1969) Let A and B be nonempty closed and bounded subsets of a metric space* (X, d) *. If* $a \in A$ *, then* $d(a, B) \leq d_H(A, B)$.

Lemma 2.2. *(Nadler, 1969) Let A and B be nonempty closed and bounded subsets of a metric space* (X, d) *and* $0 < \xi \in \Re$ *then for* $a \in A$ *there exists* $b \in B$ *such that* $d(a, B) \leq d_H(A, B) + \xi$.

Lemma 2.3. *(Nadler, 1969) The completeness of* (X, d) *implies that* $(CB(2^X), d_H)$ *is complete.*

Theorem 2.1. *(Hardy & Rogers, 1973) Let (X, d) be a complete metric space and a mapping $T: X \to X$ suppose there exists non-negative constants a_1, a_2, a_3, a_4, a_5 satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ such that for each $x, y \in X$*

$$d(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, Fx) + a_3 d(y, Fy) + a_4 d(x, Fy) + a_5 d(y, Fx)$$

holds then F has a unique fixed point in X.

3. Main Results

The mapping satisfies the contractive condition in Theorem (2.1) is called Hardy and Rogers type mapping. It is mentioned that Hardy and Rogers contractive condition does not implies that the mapping T is continuous, which differentiates it from Banach contractive condition for $c \in X$ and $0 < r < R$. Let $S_r(c) = \{x \in X/d(c, x) < r\}$ be the ball of radius r centered at c, the closure of $S_r(c)$ is denoted by $\overline{S_r(c)}$. We present a result regarding the existence of common fixed point for fuzzy mappings satisfying Hardy and Rogers type contractive condition on closed balls. The theorem is as follows:

Theorem 3.1. *Let (X, d) be a complete metric space $x_0 \in X$ and mapping $F, T: \overline{S_r(x_0)} \to \tau(X)$. Suppose there exist a constants a_1, a_2, a_3, a_4, a_5 satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Ty) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Ty]_1) + a_4 d(x, [Ty]_1) + a_5 d(y, [Fx]_1) \quad (3.1)$$

for all $x, y \in \overline{S_r(x_0)}$ and

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{(1 - a_3 - a_4)} \quad (3.2)$$

holds. Then F and T has a common fuzzy fixed point in $\overline{S_r(x_0)}$ that is there exists $x^* \in \overline{S_r(x_0)}$ with $\{x^*\} \subseteq Fx^* \cap Tx^*$.

Proof. Choose $x_1 \in X$ such that $\{x_1\} \subseteq Fx_0$ and

$$d(x_0, x_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{(1 - a_3 - a_4)} \quad (3.3)$$

since $[Fx_0]_1 \neq \phi$ for the sake of simplicity chooses $\lambda = \frac{(a_1+a_2+a_4)}{(1-a_3-a_4)}$ this gives us $d(x_0, x_1) < (1 - \lambda)r$ which implies that $x_1 \in \overline{S_r(x_0)}$. Now choose $\varepsilon > 0$ such that

$$\lambda d(x_0, x_1) + \frac{\varepsilon}{(1 - a_3 - a_4)} < \lambda(1 - \lambda)r. \quad (3.4)$$

Then choose $\varepsilon > 0$ such that $\{x_2\} \subseteq Tx_1$ and by using inequality (3.1) and Lemma 2.1 we have

$$\begin{aligned} d(x_1, x_2) &\leq D_1(Fx_0, Tx_1) + \varepsilon \\ &\leq a_1 d(x_0, x_1) + a_2 d(x_0, [Fx_0]_1) + a_3 d(x_1, [Tx_1]_1) + a_4 d(x_0, [Tx_1]_1) + a_5 d(x_1, [Fx_0]_1 + \varepsilon \\ &\leq a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + a_5 d(x_1, x_1) + \varepsilon \\ &= (a_1 + a_2) d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + \varepsilon \end{aligned}$$

i.e. $d(x_1, x_2) \leq \lambda d(x_0, x_1) + \frac{\varepsilon}{(1-a_3-a_4)}$ where $\lambda = \frac{(a_1+a_2+a_4)}{(1-a_3-a_4)}$. Now by inequality (3.4) we get $d(x_1, x_2) < \lambda(1-\lambda)r$. Note that $x_2 \in \overline{S_r(x_0)}$ since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) < (1-\lambda)r + \lambda(1-\lambda)r = (1-\lambda)r(1+\lambda) \\ &< (1-\lambda)(1+\lambda+\lambda^2+\lambda^3+...)r = r \end{aligned}$$

continue this process and having chosen $\{x_n\}$ in X such that $\{x_{2k+1}\} \subseteq Fx_{2k}$ and $\{x_{2k+2}\} \subseteq Tx_{2k+1}$ with $d(x_{2k+1}, x_{2k+2}) < \lambda^{2k+1}(1-\lambda)r$ where $k = 0, 1, 2, \ldots$

Notice that $\{x_n\}$ is cauchy sequence in $\overline{S_r(x_0)}$ which is complete. Therefore a point $x^* \in \overline{S_r(x_0)}$ exists with $lim_{n\longrightarrow\infty} x_n = x^*$. It remains to show that $\{x^*\} \subseteq Tx^*$ and $\{x^*\} \subseteq Fx^*$. Now by using Lemma 2.1 and inequality (3.1) we get

$$\begin{aligned} d(x^*, [Tx^*]_1) &\leq d(x^*, x_{2n+1}) + d(x_{2n+1}, [Tx^*]_1) \\ &\leq d(x^*, x_{2n+1}) + D_1(Fx_{2n+2}, Tx^*) \\ &\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, [Fx_{2n+2}]_1) + a_3 d(x^*, [Tx^*]_1) \\ &\quad + a_4 d(x_{2n+2}, [Tx^*]_1) + a_5 d(x^*, [Fx_{2n+2}]_1) \\ &\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) + a_3 d(x^*, [Tx^*]_1) \\ &\quad + a_4 d(x_{2n+2}, [Tx^*]_1) + a_5 d(x^*, x_{2n+1}) \\ &\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) + a_4 d(x_{2n+2}, x^*) \\ &\quad + a_4 d(x^*, [Tx^*]_1) + a_5 d(x^*, x_{2n+1}) \\ &\leq d(x^*, x_{2n+1}) + a_1 d(x_{2n+2}, x^*) + a_2 d(x_{2n+2}, x_{2n+1}) \\ &\quad + a_4 d(x_{2n+2}, x^*) + a_5 d(x^*, x_{2n+1}) \\ &\longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

This implies that $d(x^*, [Tx^*]_1) = 0$, which implies that $\{x^*\} \subseteq Tx^*$. Similarly consider that $d(x^*, [Fx^*]_1) \leq d(x^*, x_{2n+2}) + d(x_{2n+2}, [Fx^*]_1)$ to show that $\{x^*\} \subseteq Fx^*$. This implies that the mappings **F** and **T** have a common fixed point $\overline{S_r(x_0)}$, i.e. $\{x^*\} \subseteq Fx^* \cap Tx^*$. □

Corollary 3.1. *Let (X, d) be a complete metric space $x_0 \in X$ and mapping $F : \overline{S_r(x_0)} \rightarrow \tau(X)$. Suppose there exist a non negative constants a_1, a_2, a_3, a_4, a_5 satisfying $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ with*

$$D_1(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Fy]_1) + a_4 d(x, [Fy]_1) + a_5 d(y, [Fx]_1)$$

for all $x, y \in \overline{S_r(x_0)}$ and

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

holds. Then F has a common fuzzy fixed point in $\overline{S_r(x_0)}$ that is there exists $x^* \in \overline{S_r(x_0)}$ with

$$\{x^*\} \subseteq Fx^*.$$

Proof. Put $T = F$ in Theorem 3.1 we get $x^* \in \overline{S_r(x_0)}$ such that $\{x^*\} \subseteq Fx^*$. □

Theorem 3.2. *Let* (X, d) *be a complete metric space* $x_0 \in X$ *and mapping* $F, T : X : \to \tau(X)$*.Suppose there exist a constants* a_1, a_2, a_3, a_4, a_5 *satisfying* $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ *with*

$$D_1(Fx, Ty) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Ty]_1) + a_4 d(x, [Ty]_1) + a_5 d(y, [Fx]_1)$$

for all $x, y \in X$ *and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

holds.Then F *and* T *has a common fuzzy fixed point in* X *that is there exists* $x^* \in X$ *with*

$$\{x^*\} \subseteq Fx^* \cap Tx^*.$$

Proof: Fix $x_0 \in X$ and choose $r > 0$ such that

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

Now Theorem 3.1 guarantees that there exists $x^* \in X$ with

$$\{x^*\} \subseteq Fx^* \cap Tx^*.$$

Corollary 3.2. *Let* (X, d) *be a complete metric space* $x_0 \in X$ *and mapping* $F : X \to \tau(X)$*.Suppose there exist a constants* a_1, a_2, a_3, a_4, a_5 *satisfying* $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ *with*

$$D_1(Fx, Fy) \leq a_1 d(x, y) + a_2 d(x, [Fx]_1) + a_3 d(y, [Fy]_1) + a_4 d(x, [Fy]_1) + a_5 d(y, [Fx]_1)$$

for all $x, y \in X$ *and*

$$d(x_0, [Fx_0]_1) < \frac{(1 - a_1 - a_2 - a_3 - 2a_4)r}{1 - a_3 - a_4}$$

holds.Then F *has a common fuzzy fixed point in* X *that is there exists* $x^* \in X$ *with*

$$\{x^*\} \subseteq Fx^*.$$

Proof: In Theorem 3.2 take T=F we get $x^* \in X$ *such that* $\{x^*\} \subseteq Fx^*$.

4. The importance and future of this theory:

Fuzzy sets and mappings play important roles in the fuzzification of systems. In particular, in the recent years the fixed point theory for fuzzy mappings and for a family of these mappings obtained via implicit functions named Hardy and Rogers type mappings. In this article can further be used in the process of finding the solution of functional equations in fuzzy systems. As far as the application of contraction mapping is concerned the situation is not fully exploited. It is quite possible that a contraction T is defined on the whole space X but it is contractive on the subset Y of the subset of the space rather on the whole space X. Moreover the contraction mapping under consideration may not be continues. If Y is closed, then it is complete,so that a mapping T has a fixed point x in Y,and $x_n \to x$ as in the case of whole space X provided we improve a simple restriction on the choice of x_0, so that $x_n's$ remains in Y. In this paper, we have discussed this concept for fuzzy Hardy and Rogers mappings on a complete metric space x which generalize/improves several classical tendon with (Azam *et al.*, 2013) will become the foundation of fuzzy theory on closed balls.

An example of a fuzzy mapping which is contractive on the subset of a space but not on the whole space is as follows:

Example 4.1. Let $X = \mathbb{R}$ and $d : XxX \to \mathbb{R}$ is defined by $d(x, y) = |x - y|$ where $x, y \in X$ consider the mapping $F : X \to \tau(X)$ is defined by

$$F(x) = \begin{cases} \chi_{(1-x)}, & \text{if } x \text{ is irrational;} \\ \chi_{(\frac{1+x}{3})}, & \text{if } x \text{ is rational.} \end{cases}$$

then F is Hardy and Rogers type fuzzy mapping on the closed balls $\overline{S_{(\frac{1}{2})}(\frac{1}{2})} = [0, 1]$ but not on the whole space X.

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Some Fixed Point Theorems for Ordered F-generalized Contractions in 0-f-orbitally Complete Partial Metric Spaces

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Abstract

We prove some fixed point theorems for ordered F-generalized contractions in ordered 0-f-orbitally complete partial metric spaces. Our results generalize some well-known results in the literature, in particular the recent result of Wardowski [Fixed Point Theory Appl. 2012:94 (2012)] from metric spaces to ordered 0-f-orbitally complete partial metric spaces. Some examples are given which illustrate the new results.

Keywords: Partial metric space, partial order, fixed point, F-generalized contraction, 0-f-orbitally complete space.
2010 MSC: 47H10, 54H25.

1. Introduction

In 1994, Matthews (Matthews, 1994) introduced the notion of a partial metric space, as a part of the study of denotational semantics of dataflow networks. In a partial metric space, the usual distance was replaced by partial metric, with an interesting property of "nonzero self distance" of points. Also, the convergence of sequences in this space was defined in such a way that the limit of a convergent sequence need not be unique. Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verifications. Later on, several authors generalized the results of Matthews (see, for example, (Ahmad *et al.*, 2012; Bari *et al.*, 2012; Kadelburg *et al.*, 2013; Nashine *et al.*, 2012; Vetro & Radenović, 2012)). O'Neill (O'Neill, 1996) generalized the concept of partial metric space a bit further by admitting negative distances. The partial metric defined by O'Neill is called dualistic partial metric. Heckmann (Heckmann,

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1999) generalized it by omitting small self-distance axiom. The partial metric defined by Heckmann is called the weak partial metric. Romaguera (Romaguera, 2010) introduced the notions of 0-Cauchy sequences and 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

The existence of fixed points of mappings in partially ordered sets was investigated by Ran and Reurings (Ran & Reurings, 2004) and then by Nieto and Rodríguez-Lopez (Nieto & Lopez, 2005, 2007). In these papers, some results on the existence of a unique fixed point for nondecreasing mappings were applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Later on, these results were generalized by several authors in different spaces.

Recently, Wardowski (Wardowski, 2012) has introduced a new concept of F-contraction and proved a fixed point theorem which generalizes Banach contraction principle in a different direction than in the known results from the literature in complete metric spaces.

In this paper, we prove some fixed point theorems for ordered F-generalized contractions in ordered 0-f-orbitally complete partial metric spaces. The results of this paper generalize and extend the results of Wardowski (Wardowski, 2012), Ran and Reurings (Ran & Reurings, 2004), Nieto and Rodríguez-Lopez (Nieto & Lopez, 2005, 2007), Altun et al. (Altun *et al.*, 2010), Ćirić (Ćirić, 1971, 1972) and some other well-known results in the literature. Some examples are given which illustrate our results.

2. Preliminaries

First we recall some definitions and properties of partial metric spaces (see, e.g., (Matthews, 1994; Oltra & Valero, 2004; O'Neill, 1996; Romaguera, 2010, 2011)).

Definition 2.1. A partial metric on a nonempty set X is a function $p\colon X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Also, every metric space is a partial metric space, with zero self distance.

Example 2.1. If $p\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x, y) = \max\{x, y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.

For some more examples of partial metric spaces, we refer to (Aydi *et al.*, 2012) and the references therein.

Each partial metric on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \epsilon)\colon x \in X,\ \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X :\ p(x, y) < p(x, x) + \epsilon\}$ for all

$x \in X$ and $\epsilon > 0$. A mapping $f : X \to X$ is continuous if and only if, whenever a sequence $\{x_n\}$ in X converging with respect to τ_p to a point $x \in X$, the sequence $\{fx_n\}$ converges with respect to τ_p to $fx \in X$.

Theorem 2.1. *(Matthews, 1994) For each partial metric $p : X \times X \to \mathbb{R}^+$ the pair (X, d) where, $d(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, is a metric space.*

Here (X, d) is called the induced metric space and d is the induced metric. In further discussion, unless something else is specified, (X, d) will represent the induced metric space.

Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.
(2) A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.
(3) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(4) A sequence $\{x_n\}$ in (X, p) is called a 0-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 2.1. *(Matthews, 1994; Oltra & Valero, 2004; Romaguera, 2010, 2011) Let (X, p) be a partial metric space and $\{x_n\}$ be any sequence in X.*

(i) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d).
(ii) (X, p) is complete if and only if the metric space (X, d) is complete. Furthermore, $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(iii) Every 0-Cauchy sequence in (X, p) is Cauchy in (X, d).
(iv) If (X, p) is complete then it is 0-complete.

The converse assertions of (iii) and (iv) do not hold. Indeed, the partial metric space $(\mathbb{Q} \cap \mathbb{R}^+, p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x, y) = \max\{x, y\}$, provides an easy example of a 0-complete partial metric space which is not complete. Also, it is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

The proof of the following lemma is easy and for details we refer to (Karapınar, 2012) and the references therein.

Lemma 2.2. *Assume $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for all $y \in X$.*

The notion of orbital completeness of metric spaces was introduced in (Ćirić, 1971) and adapted to partial metric spaces in (Karapınar, 2012) as follows:

Let (X, p) be a partial metric space and $f : X \to X$ be a mapping. For any $x \in X$, the set $O(x) = \{x, fx, f^2x, \dots\}$ is called the orbit of f at point x. (X, p) is called f-orbitally complete if every Cauchy sequence in $O(x)$ converges in (X, p).

Now, we define 0-f-orbital completeness of a partial metric space.

Definition 2.2. Let (X, p) be a partial metric space and $f: X \to X$ be a mapping. (X, p) is said to be 0-f-orbitally complete, if every 0-Cauchy sequence in $O(x) = \{x, fx, f^2x, \ldots\}$, $x \in X$, converges with respect to τ_p to a point $u \in X$ such that $p(u, u) = 0$.

Note that every complete partial metric space is 0-complete, and every 0-complete partial metric space is 0-f-orbitally complete for every $f: X \to X$. But, the converse assertions need not hold as shown by the following example.

Example 2.2. Let $X = \mathbb{R}^+ \cap (\mathbb{Q} \setminus \{1\})$ and $p: X \times X \to \mathbb{R}^+$ be defined by

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1); \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

Define $f: X \to X$ by $fx = \frac{x}{2}$ for all $x \in X$. Then (X, p) is a partial metric space. Note that (X, p) is not complete because the induced metric space (X, d), where

$$d(x, y) = \begin{cases} 2|x - y|, & \text{if } x, y \in [0, 1); \\ |x - y|, & \text{otherwise,} \end{cases}$$

is not complete. Also (X, p) is not 0-complete. Indeed, for $x_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$, we observe that, $p(x_n, x_m) = |\frac{1}{n} - \frac{1}{m}| \to 0$ as $n \to \infty$. But, there is no $u \in X$ such that $\lim_{n\to\infty} p(x_n, u) = p(u, u) = 0$. Now, it is easy to see that (X, p) is 0-f-orbitally complete.

Consider, together with Wardowski in (Wardowski, 2012), the following properties for a mapping $F: \mathbb{R}^+ \to \mathbb{R}$:

(F1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$, $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;
(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote the set of all functions satisfying properties (F1)–(F3), by $\mathcal{F}$.

For examples of functions $F \in \mathcal{F}$, we refer to (Wardowski, 2012). Wardowski defined in (Wardowski, 2012) F-contractions as follows:

Let (X, ρ) be a metric space. A mapping $T: X \to X$ is said to be an F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$, $\rho(Tx, Ty) > 0$ we have

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)).$$

Similarly, we adopt the following definitions.

Definition 2.3. Let X be a nonempty set, $\preceq$ a partial order relation defined on X and p be a partial metric on X (then, $(X, \preceq, p)$ is called an ordered partial metric space). A map $f: X \to X$ is called:

1. an ordered F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \preceq y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \leq F(p(x, y)). \tag{2.1}$$

2. an ordered F-weak contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \preceq y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \leq F(\max\{p(x, y), p(x, fx), p(y, fy)\}). \tag{2.2}$$

If inequality (2.2) is satisfied for all $x, y \in X$, then f is called an F-weak contraction;

3. an ordered F-generalized contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \preceq y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \leq F(\max\{p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2}\}). \tag{2.3}$$

If inequality (2.3) is satisfied for all $x, y \in X$, then f is called an F-generalized contraction.

The following example shows that the class of F-contractions in partial metric spaces is more general than that in metric spaces.

Example 2.3. Let $X = \mathbb{R}^+$ and $p\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Note that the metric induced by p (as well as the usual metric) on X is given by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $f\colon X \to X$ by

$$fx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1); \\ 0, & \text{if } x = 1. \end{cases}$$

Then for $x = 1, y = \frac{9}{10}$ there is no $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)).$$

On the other hand, for $\tau = \log 2$ and $F(\alpha) = \log \alpha + \alpha$, it is easy to see that f is an F-contraction in (X, p).

3. Main results

The following is our first main result.

Theorem 3.1. *Let $(X, \preceq, p)$ be an ordered partial metric space and $f\colon X \to X$ be an ordered F-generalized contraction for some $F \in \mathcal{F}$. If (X, p) is 0-f-orbitally complete and the following conditions hold:*

(i) f is nondecreasing with respect to "$\preceq$", that is, if $x \preceq y$ then $fx \preceq fy$;
(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;
(iii) (a) f is continuous, or
(b) F is continuous and for every nondecreasing sequence $\{x_n\}$, $x_n \to u \in X$ as $n \to \infty$ implies $x_n \preceq u$ for all $n \in \mathbb{N}$.

Then f has a fixed point $u \in X$. Furthermore, the fixed point of f is unique if and only if the set of all fixed points of f is well-ordered.

Proof. First, we shall show the existence of fixed point of f. Let $x_0 \in X$ be the point given by (ii). We define a sequence $\{x_n\}$ in X by $x_{n+1} = fx_n$ for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then x_{n_0} is a fixed point of f. Therefore, assume that $x_{n+1} \neq x_n$ for all $n \geq 0$. As, $x_0 \preceq fx_0$ we have $x_0 \preceq x_1$, and since f is nondecreasing with respect to $\preceq$, we have $fx_0 \preceq fx_1$ that is $x_1 \preceq x_2$. Similarly, we obtain $x_n \preceq x_{n+1}$ for all $n \geq 0$. Also, f is an ordered F-generalized contraction therefore, for any $n \in \mathbb{N}$ it follows from (2.3) and symmetry of p that

$$\begin{aligned}\tau + F(p(fx_n, fx_{n-1})) &= \tau + F(p(fx_{n-1}, fx_n))\\ &\leq F(\max\{p(x_n, x_{n-1}), p(x_n, fx_n), p(x_{n-1}, fx_{n-1}),\\ &\qquad \frac{p(x_n, fx_{n-1}) + p(x_{n-1}, fx_n)}{2}\})\\ &= F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), p(x_{n-1}, x_n),\\ &\qquad \frac{p(x_n, x_n) + p(x_{n-1}, x_{n+1})}{2}\})\\ &\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1}),\\ &\qquad \frac{p(x_{n-1}, x_n) + p(x_{n+1}, x_n)}{2}\}).\end{aligned}$$

Note that, for any $a, b \in \mathbb{R}^+$ we have $\max\{a, b, \frac{a+b}{2}\} = \max\{a, b\}$, therefore it follows from the above inequality that

$$\begin{aligned}\tau + F(p(x_{n+1}, x_n)) &\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\})\\ F(p(x_{n+1}, x_n)) &\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}) - \tau.\end{aligned} \tag{3.1}$$

If, $\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$ then from (3.1) we have

$$F(p(x_{n+1}, x_n)) \leq F(p(x_n, x_{n+1})) - \tau < F(p(x_n, x_{n+1})),$$

a contradiction. Therefore, $\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n-1})$ and from (3.1) we have

$$F(p(x_{n+1}, x_n)) \leq F(p(x_n, x_{n-1})) - \tau \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

Setting $p_n = p(x_{n+1}, x_n)$ it follows by successive applications of (3.2) that

$$F(p_n) \leq F(p_{n-1}) - \tau \leq F(p_{n-2}) - 2\tau \leq \cdots \leq F(p_0) - n\tau. \tag{3.3}$$

From (3.3) we have $\lim_{n\to\infty} F(p_n) = -\infty$, and since $F \in \mathcal{F}$ we must have

$$\lim_{n\to\infty} p_n = 0. \tag{3.4}$$

Again, as $F \in \mathcal{F}$ there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (p_n)^k F(p_n) = 0. \tag{3.5}$$

From (3.3) we have

$$(p_n)^k[F(p_n) - F(p_0)] \leq -n\tau(p_n)^k \leq 0.$$

Letting $n \to \infty$ in the above inequality and using (3.4) and (3.5) we obtain

$$\lim_{n\to\infty} n(p_n)^k = 0. \tag{3.6}$$

It follows from (3.6) that there exists $n_1 \in \mathbb{N}$ such that $n(p_n)^k < 1$ for all $n > n_1$, that is,

$$p_n \leq \frac{1}{n^{1/k}} \quad \text{for all} \quad n > n_1. \tag{3.7}$$

Let $m, n \in \mathbb{N}$ with $m > n > n_1$. Then it follows from (3.7) that

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &\qquad - [p(x_n, x_n) + p(x_{n+1}, x_{n+1}) + \cdots + p(x_{m-1}, x_{m-1})] \\ &\leq p_n + p_{n+1} + \cdots \\ &\leq \frac{1}{n^{1/k}} + \frac{1}{(n+1)^{1/k}} + \cdots \\ &= \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

As $k \in (0,1)$, the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$ converges, so it follows from the above inequality that $\lim_{n\to\infty} p(x_n, x_m) = 0$, that is, the sequence $\{x_n\}$ is a 0-Cauchy sequence in $O(x_0) = \{x_0, fx_0, f^2x_0, \ldots\}$. Therefore, by 0-$f$-orbital completeness of (X, p), there exists $u \in X$ such that

$$\lim_{n\to\infty} p(x_n, u) = \lim_{n,m\to\infty} p(x_n, x_m) = p(u, u) = 0. \tag{3.8}$$

We shall show that u is a fixed point of f. For this, we consider two cases.

Case I: Suppose (a) is satisfied, that is, f is continuous. Then using (3.8) and Lemma 2.2, we obtain

$$p(u, fu) = \lim_{n\to\infty} p(x_n, fu) = \lim_{n\to\infty} p(fx_{n-1}, fu) = p(fu, fu).$$

Suppose that $p(fu, fu) > 0$. Then as $u \preceq u$, using (2.3) we obtain

$$\begin{aligned} \tau + F(p(fu, fu)) &\leq F(\max\{p(u,u), p(u,fu), p(u,fu), \frac{p(u,fu) + p(u,fu)}{2}\}) \\ &= F(\max\{p(u,u), p(u,fu)\}) \\ &= F(p(u,fu)), \end{aligned}$$

that is, $F(p(fu, fu)) < F(p(u, fu))$ and from $F \in \mathcal{F}$ we have $p(fu, fu) < p(u, fu) = p(fu, fu)$, a contradiction. Therefore, $p(fu, fu) = p(u, fu) = 0$, that is, $fu = u$, so u is a fixed point of f.

Case II: Suppose that (b) is satisfied. Then we consider two subcases.

(i): For each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $p(x_{k_n+1}, fu) = 0$ and $k_n > k_{n-1}$, where $k_0 = 1$. Then, using Lemma 2.2, we have

$$p(u, fu) = \lim_{n\to\infty} p(x_{k_n+1}, fu) = 0.$$

Therefore, $fu = u$, that is, u is a fixed point of f.

(ii): There exists $n_2 \in \mathbb{N}$ such that $p(x_n, fu) \neq 0$ for all $n > n_2$. In this case, since $\{x_n\}$ is a nondecreasing sequence and $x_n \to u$ as $n \to \infty$, we have $x_n \preceq u$ for all $n \in \mathbb{N}$. Therefore, using (2.3) we obtain

$$\begin{aligned}\tau + F(p(x_{n+1}, fu)) &= \tau + F(p(fx_n, fu)) \\ &\leq F(\max\{p(x_n, u), p(x_n, fx_n), p(u, fu), \frac{p(x_n, fu) + p(u, fx_n)}{2}\}) \\ &\leq F(\max\{p(x_n, u), p(x_n, x_{n+1}), p(u, fu), \\ &\quad \frac{p(x_n, u) + p(u, fu) + p(u, x_{n+1})}{2}\}).\end{aligned}$$

From (3.8), there exists $n_3 \in \mathbb{N}$ such that, for all $n > n_3$ we have

$$\max\{p(x_n, u), p(x_n, x_{n+1}), p(u, fu), \frac{p(x_n, u) + p(u, fu) + p(u, x_{n+1})}{2}\} = p(u, fu),$$

so, for $n > \max\{n_2, n_3\}$ we obtain

$$\tau + F(p(x_{n+1}, fu)) \leq F(p(u, fu)).$$

As F is continuous, letting $n \to \infty$ in the above inequality and using (3.8) and Lemma 2.2 we obtain

$$\tau + F(p(u, fu)) \leq F(p(u, fu)),$$

a contradiction. Therefore, we must have $p(u, fu) = 0$ that is $fu = u$. Thus u is a fixed point of f.

Suppose that the set of fixed points of f is well-ordered and $u, v \in F_f$ with $p(u, v) > 0$, where F_f denotes the set of all fixed points of f. As F_f is well-ordered, let $u \preceq v$. Then from (2.3) we obtain

$$\begin{aligned}\tau + F(p(u, v)) &= \tau + F(p(fu, fv)) \\ &\leq F(\max\{p(u, v), p(u, fu), p(v, fv), \frac{p(u, fv) + p(v, fu)}{2}\}) \\ &\leq F(\max\{p(u, v), p(u, u), p(v, v), p(v, u)\}) \\ &\leq F(p(u, v)),\end{aligned}$$

a contradiction. Similarly, for $v \preceq u$ we get a contradiction. Therefore, the fixed point of f is unique. For the converse, if the fixed point of f is unique then F_f, being a singleton, is well-ordered. □

The following corollaries are immediate consequences of the above theorem.

Corollary 3.1. *Let $(X, \preceq, p)$ be an ordered partial metric space and $f\colon X \to X$ be an ordered F-contraction. Let (X, p) is 0-f-orbitally complete and the following conditions hold:*

(i) f is nondecreasing with respect to "$\preceq$", that is, if $x \preceq y$ then $fx \preceq fy$;

(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(iii) (a) f is continuous, or

(b) F is continuous and for every nondecreasing sequence $\{x_n\}$ such that $x_n \to u \in X$ as $n \to \infty$ it follows that $x_n \preceq u$ for all $n \in \mathbb{N}$.

Then f has a fixed point $u \in X$. Furthermore, the fixed point of f is unique if and only if the set of all fixed points of f is well-ordered.

Corollary 3.2. *Let $(X, \preceq, p)$ be an ordered partial metric space and $f: X \to X$ be an ordered F-weak contraction. If (X, p) is 0-f-orbitally complete and the following conditions hold:*

(i) f is nondecreasing with respect to "$\preceq$", that is, if $x \preceq y$ then $fx \preceq fy$;

(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(iii) (a) f is continuous, or

(b) F is continuous and for every nondecreasing sequence $\{x_n\}$ such that $x_n \to u \in X$ as $n \to \infty$ it follows that $x_n \preceq u$ for all $n \in \mathbb{N}$.

Then f has a fixed point $u \in X$. Furthermore, the fixed point of f is unique if and only if the set of all fixed points of f is well-ordered.

Remark. We note that every metric space is a partial metric space with zero self distance. Therefore we can replace the partial metric p by a metric ρ in Theorem 3.1. Also, after this replacement, the 0-f-orbital completeness reduces to orbital completeness of the metric space. Therefore, by this replacement in Theorem 3.1, we obtain the fixed point result for ordered F-generalized contraction in orbitally complete metric spaces.

In the above theorems the fixed point of the self map f is the limit of a 0-Cauchy sequence and due to 0-f-orbital completeness of the space this limit has zero self distance. The next theorem shows that, if an ordered F-generalized contraction has a fixed point then its self distance must be zero, that is, it does not depend on the properties of space such as completeness etc.

Theorem 3.2. *Let $(X, \preceq, p)$ be an ordered partial metric space and $f: X \to X$ be an ordered F-generalized contraction. If f has a fixed point u then $p(u, u) = 0$.*

Proof. Suppose that $u \in F_f$ and $p(u, u) > 0$. Then, it follows from (2.3) that

$$\begin{aligned}\tau + F(p(u,u)) &= \tau + F(p(fu, fu)) \\ &\leq F(\max\{p(u,u), p(u,fu), p(u,fu), \frac{p(u,fu)+p(u,fu)}{2}\}) \\ &= F(p(u,u)).\end{aligned}$$

As $\tau > 0$, the above inequality yields a contradiction. Therefore, we have $p(u, u) = 0$ for all $u \in F_f$. □

The following example illustrates our results.

Example 3.1. Let $X = [0,2] \cap (\mathbb{Q} \setminus \{1\})$ and define $p\colon X \times X \to \mathbb{R}^+$ by

$$p(x,y) = \begin{cases} |x-y|, & \text{if } x,y \in [0,1); \\ 0, & \text{if } x = y = 2; \\ \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Then (X,p) is a partial metric space. Define a partial order relation "$\preceq$" on X by

$$\preceq \; = \; \{(x,y)\colon x,y \in [0,1), y \leq x\} \cup \{(x,y)\colon x,y \in (1,2), y \leq x\} \cup \{(2,2)\}.$$

Define $f\colon X \to X$ by

$$fx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1); \\ \frac{1}{4}, & \text{if } x \in (1,2); \\ 2, & \text{if } x = 2. \end{cases}$$

Then it is easy to see that (X,p) is a 0-f-orbitally complete partial metric space. Let $F(\alpha) = \log \alpha$ for all $\alpha \in \mathbb{R}^+$. Then f satisfies all the conditions of Corollary 3.1 (except that the set of fixed points of f is well-ordered) with $\tau \leq \log 2$. Note that, $F_f = \{0,2\}$ with $p(0,0) = p(2,2) = 0$ and $(2,0),(0,2) \notin \preceq$. Now, the metric d induced by p is given by

$$d(x,y) = \begin{cases} 2|x-y|, & \text{if } x,y \in [0,1); \\ |x-y|, & \text{otherwise,} \end{cases}$$

and (X,d) is not complete. Similarly, if ρ is the usual metric on X then (X,ρ) is not complete, therefore the results from metric cases are not applicable here. This example shows also that an ordered F-contraction may not be an F-contraction (not even an F-generalized contraction). Indeed, for $x \in [0,1)$, $y = 2$ there exists no $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(p(fx,fy)) \leq F(\max\{p(x,y), p(x,fx), p(y,fy), \frac{p(x,fy)+p(y,fx)}{2}\}).$$

Therefore, f is not an F-generalized contraction in (X,p). Similarly, for $x = 0$, $y = 2$ one can see that f is not an F-generalized contraction in (X,d) and (X,ρ).

In the following theorem the conditions on self map f, "nondecreasing", continuous and 0-f-orbital completeness of space, are replaced by another condition on self map f.

Theorem 3.3. *Let $(X,\preceq,p)$ be an ordered partial metric space and $f\colon X \to X$ be an ordered F-generalized contraction. Let there exists $u \in X$ such that $u \preceq fu$ and $p(u,fu) \leq p(x,fx)$ for all $x \in X$. Then f has a fixed point $u \in X$. Furthermore, the fixed point of f is unique if and only if the set of all fixed points of f is well-ordered.*

Proof. Let $G(x) = p(x,fx)$ for all $x \in X$. Then by assumption we have

$$G(u) \leq G(x) \text{ for all } x \in X. \tag{3.9}$$

We shall show that $fu = u$. Suppose that $G(u) = p(u, fu) > 0$. Then since $u \preceq fu$, it follows from (2.3) that

$$\begin{aligned} F(G(fu)) &= F(p(fu, ffu)) \\ &\leq F(\max\{p(u, fu), p(u, fu), p(fu, ffu), \frac{p(u, ffu) + p(fu, fu)}{2}\}) - \tau \\ &\leq F(\max\{p(u, fu), p(fu, ffu), \frac{p(u, fu) + p(fu, ffu)}{2}\}) - \tau \\ &= F(\max\{G(u), G(fu), \frac{G(u) + G(fu)}{2}\}) - \tau \\ &= F(\max\{G(u), G(fu)\}) - \tau. \end{aligned}$$

If $\max\{G(u), G(fu)\} = G(fu)$, then it follows from the above inequality that $F(G(fu)) < F(G(fu))$, a contradiction. If $\max\{G(u), G(fu)\} = G(u)$, then again we obtain $F(G(fu)) < F(G(u))$ and $F \in \mathcal{F}$ so $G(fu) < G(u)$, a contradiction. Thus, we must have $G(u) = p(u, fu) = 0$, that is $fu = u$ and so u is a fixed point of f.

The necessary and sufficient condition for the uniqueness of fixed point follows from a similar process as used in Theorem 3.1. □

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On Some Generalized I-Convergent Sequence Spaces Defined by a Sequence of Moduli

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Abstract

In this article we introduce the sequence spaces $c_0^I(F,p)$, $c^I(F,p)$ and $l_\infty^I(F,p)$ for $F=(f_k)$ a sequence of moduli and $p=(p_k)$ sequence of positive reals and study some of the properties and inclusion relation on these spaces.

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1. Introduction

Throughout the article $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and ω denotes the set of natural,real,complex numbers and the class of all sequences respectively.

The notion of the statistical convergence was introduced by H. Fast (Fast, 1951). Later on it was studied by J. A. Fridy (Fridy, 1985, 1993) from the sequence space point of view and linked it with the summability theory. The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát and Wilczyński (Kostyrko & Šalát and W. Wilczyński, 2000). Later on it was studied by Šalát, Tripathy and Ziman (Šalát *et al.*, 1963) and Demirci (Demirci, 2001). Recently it was studied by V. A. Khan and K. Ebadullah (Khan & Ebadullah, 2011; Khan *et al.*, 2011; Khan & Ebadullah, 2012; Khan *et al.*, 2012) and Tripathy and Hazarika (Tripathy & Hazarika, 2009, 2011).

Here we give some preliminaries about the notion of I-convergence.

Let N be a non empty set. Then a family of sets $I \subseteq 2^N$ (2^N denoting the power set of N) is said to be an ideal if I is additive i.e $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $\pounds(I) \subseteq 2^N$ is said to be filter on N if and only if $\phi \notin \pounds(I)$,for $A, B \in \pounds(I)$ we have $A \cap B \in \pounds(I)$ and for each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$.

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An Ideal $I \subseteq 2^N$ is called non-trivial if $I \neq 2^N$. A non-trivial ideal $I \subseteq 2^N$ is called admissible if $\{\{x\} : x \in N\} \subseteq I$.

A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

For each ideal I, there is a filter £(I) corresponding to I. i.e £(I) $= \{K \subseteq N : K^c \in I\}$,where $K^c = N - K$.

Definition 1.1. A sequence (x_k) is said to be I-convergent to a number L if for every $\epsilon > 0$. $\{k \in N : |x_k - L| \geq \epsilon\} \in$ I. In this case we write I-lim $x_k = L$. The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_k) : \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I, \text{for some L} \in \mathbb{C}\}.$$

Definition 1.2. A sequence (x_k) is said to be I-null if L = 0. In this case we write I-lim $x_k = 0$.

Definition 1.3. A sequence (x_k) is said to be I-cauchy if for every $\epsilon > 0$ there exists a number m = m(ϵ) such that $\{k \in N : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.4. A sequence (x_k) is said to be I-bounded if there exists M >0 such that $\{k \in N : |x_k| > M\} \in I$.

Definition 1.5. Let $(x_k), (y_k)$ be two sequences. We say that $(x_k) = (y_k)$ for *almost all k relative to I (a.a.k.r.I)*, if $\{k \in \mathbb{N} : x_k \neq y_k\} \in I$.

Definition 1.6. For any set E of sequences the space of multipliers of E, denoted by $M(E)$ is given by

$$M(E) = \{a \in \omega : ax \in E \text{ for all } x \in E\}).$$

Definition 1.7. The concept of paranorm(See (Maddox, 1969)) is closely related to linear metric spaces. It is a generalization of that of absolute value.

Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y, z \in X$,

(PI) $g(x) = 0 \; if \; x = \theta$,

(P2) $g(-x) = g(x)$,

(P3) $g(x + y) \leq g(x) + g(y)$,

(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a \;\; (n \to \infty)$, in the sense that $g(x_n - a) \to 0 \;\; (n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0 \;\; (n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on X, and the pair (X, g) is called a totally paranormed space.

The idea of modulus was structured in 1953 by Nakano. (See (Nakano, 1953)).

A function $f : [0,\infty) \longrightarrow [0,\infty)$ is called a modulus if

(1)f(t) = 0 if and only if t = 0,

(2) f(t+u)$\leq$ f(t)+ f(u) for all t,u$\geq$0,

(3) f is increasing, and

(4) f is continuous from the right at zero.

Ruckle (Ruckle, 1968, 1967, 1973) used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space,and Ruckle (Ruckle, 1968, 1967, 1973) proved that that the intersecton of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$.Thus Ruckle (Ruckle, 1968, 1967, 1973) proved that, for any modulus f.

$$X(f) \subset l_1 \text{ and } X(f)^{\alpha} = l_{\infty}.$$

The space $X(f)$ is a Banach space with respect to the norm $\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty$.

Spaces of the type $X(f)$ are a special case of the spaces structured by B. Gramsch in (Gramsch, 1967). From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by D. J. H Garling (Garling, 1966, 1968)and W. H. Ruckle (Ruckle, 1968, 1967, 1973).

After then E. Kolk (Kolk, 1993, 1994) gave an extension of $X(f)$ by considering a sequence of modulii $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.\text{(See (Kolk, 1993, 1994)).}$$

The following subspaces of ω were first introduced and discussed by Maddox (Maddox, 1986, 1970, 1969). $l(p) = \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\}$, $l_{\infty}(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$, $c(p) = \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \text{ for some } l \in \mathbb{C}\}$, $c_0(p) = \{x \in \omega : \lim_k |x_k|^{p_k} = 0, \}$, where $p = (p_k)$ is a sequence of strictly positive real numbers.

After then Lascarides (Lascarides, 1971, 1983) defined the following sequence spaces:
$l_{\infty}\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sup_k |x_k r|^{p_k} t_k < \infty\}$,
$c_0\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \lim_k |x_k r|^{p_k} t_k = 0, \}$,
$l\{p\} = \{x \in \omega : \text{ there exists } r > 0 \text{ such that } \sum_{k=1}^{\infty} |x_k r|^{p_k} t_k < \infty\}$,
Where $t_k = p_k^{-1}$, for all $k \in \mathbb{N}$.

We need the following lemmas in order to establish some results of this article.

Lemma 1.1. *Let* $h = \inf_k p_k$ *and* $H = \sup_k p_k$. *Then the following conditions are equivalent.(See[28]).*
(a) $H < \infty$ *and* $h > 0$.
(b) $c_0(p) = c_0$ *or* $l_{\infty}(p) = l_{\infty}$.
(c) $l_{\infty}\{p\} = l_{\infty}(p)$.
(d) $c_0\{p\} = c_0(p)$.
(e) $l\{p\} = l(p)$.

Lemma 1.2. *Let K∈ £(I) and M⊆N. If M∉I, then M∩K ∉ I.(See (Tripathy & Hazarika, 2009, 2011)). (c.f (Dems, 2005; Gurdal, 2004; Khan & Ebadullah, 2011, 2012; Kolk, 1993; Lascarides, 1971; Tripathy & Hazarika, 2011)).*

2. Main Results

Throughout the article $l_\infty, c^I, c_0^I, m^I$ and m_0^I represent the bounded , I-convergent, I-null, bounded I-convergent and bounded I-null sequence spaces respectively.

In this article we introduce the following classes of sequence spaces.

$$c^I(F,p) = \{(x_k) \in \omega : f_k(|x_k - L|^{p_k}) \geq \epsilon \text{ for some L}\} \in I$$

$$c_0^I(F,p) = \{(x_k) \in \omega : f_k(|x_k|^{p_k}) \geq \epsilon\} \in I.$$

$$l_\infty^I(F,p) = \{(x_k) \in \omega : \sup_k f_k(|x_k|^{p_k}) < \infty\} \in I.$$

Also we denote by $m^I(F,p) = c^I(F,p) \cap l_\infty(F,p)$ and $m_0^I(F,p) = c_0^I(F,p) \cap l_\infty(F,p)$.

Theorem 2.1. *Let $(p_k) \in l_\infty$. Then $c^I(F,p), c_0^I(F,p), m^I(F,p)$ and $m_0^I(F,p)$ are linear spaces.*

Proof. Let $(x_k), (y_k) \in c^I(F,p)$ and α, β be two scalars. Then for a given $\epsilon > 0$ we have

$$\{k \in \mathbb{N} : f_k(|x_k - L_1|^{p_k}) \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C}\} \in I$$

$$\{k \in \mathbb{N} : f_k(|y_k - L_2|^{p_k}) \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C}\} \in I$$

where $M_1 = D.max\{1, \sup_k |\alpha|^{p_k}\}$, $M_2 = D.max\{1, \sup_k |\beta|^{p_k}\}$ and $D = max\{1, 2^{H-1}\}$ where $H = \sup_k p_k \geq 0$. Let $A_1 = \{k \in \mathbb{N} : f_k(|x_k - L_1|^{p_k}) < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathbb{C}\} \in £(I)$, $A_2 = \{k \in \mathbb{N} : f_k(|y_k - L_2|^{p_k}) < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C}\} \in £(I)$ be such that $A_1^c, A_2^c \in I$. Then

$$A_3 = \{k \in \mathbb{N} : f_k(|(\alpha x_k + \beta y_k) - f_k(\alpha L_1 + \beta L_2)|^{p_k}) < \epsilon\} \supseteq \{k \in \mathbb{N} : |\alpha|^{p_k} f_k(|x_k - L_1|^{p_k}) < \frac{\epsilon}{2M_1}|\alpha|^{p_k}.D\}$$

$$\cap \{k \in \mathbb{N} : |\beta|^{p_k} f_k(|y_k - L_2|^{p_k}) < \frac{\epsilon}{2M_2}|\beta|^{p_k}.D\}.$$

Thus $A_3^c = A_1^c \cap A_2^c \in I$. Hence $(\alpha x_k + \beta y_k) \in c^I(F,p)$. Therefore $c^I(F,p)$ is a linear space. The rest of the result follows similarly. □

Theorem 2.2. *Let $(p_k) \in l_\infty$. Then $m^I(F,p)$ and $m_0^I(F,p)$ are paranormed spaces, paranormed by $g(x_k) = \sup_k f_k(|x_k|^{\frac{p_k}{M}})$ where $M = max\{1, \sup_k p_k\}$.*

Proof. Let $x = (x_k), y = (y_k) \in m^I(F,p)$. (1) Clearly, $g(x) = 0$ if and only if $x = 0$. (2) $g(x) = g(-x)$ is obvious. (3) Since $\frac{p_k}{M} \leq 1$ and $M > 1$, using Minkowski's inequality and the definition of f we have $\sup_k f_k(|x_k + y_k|^{\frac{p_k}{M}}) \leq \sup_k f_k(|x_k|^{\frac{p_k}{M}}) + \sup_k f(|y_k|^{\frac{p_k}{M}})$ (4) Now for any complex λ we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$. Let $x_k \in m^I(f,p)$ such that $f_k(|x_k - L|^{p_k}) \geq \epsilon$. Therefore, $g(x_k - L) = \sup_k f_k(|x_k - L|^{\frac{p_k}{M}}) \leq \sup_k f_k(|x_k|^{\frac{p_k}{M}}) + \sup_k f_k(|L|^{\frac{p_k}{M}})$. Hence $g(\lambda_n x_k - \lambda L) \leq g(\lambda_n x_k) + g(\lambda L) = \lambda_n g(x_k) + \lambda g(L)$ as $(k \to \infty)$. Hence $m^I(F,p)$ is a paranormed space. The rest of the result follows similarly. □

Theorem 2.3. *A sequence $x = (x_k) \in m^I(F, p)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(F, p)$.*

Proof. Suppose that $L = I - \lim x$. Then $B_\epsilon = \{k \in \mathbb{N} : |x_k - L|^{p_k} < \frac{\epsilon}{2}\} \in m^I(F, p)$. For all $\epsilon > 0$. Fix an $N_\epsilon \in B_\epsilon$. Then we have $|x_{N_\epsilon} - x_k|^{p_k} \leq |x_{N_\epsilon} - L|^{p_k} + |L - x_k|^{p_k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ which holds for all $k \in B_\epsilon$. Hence $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(F, p)$.

Conversely, suppose that $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(F, p)$. That is $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|^{p_k}) < \epsilon\} \in m^I(F, p)$ for all $\epsilon > 0$. Then the set $C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m^I(F, p)$ for all $\epsilon > 0$. Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m^I(F, p)$ as well as $C_{\frac{\epsilon}{2}} \in m^I(f, p)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m^I(F, p)$. This implies that $J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ that is $\{k \in \mathbb{N} : x_k \in J\} \in m^I(F, p)$ that is $diamJ \leq diamJ_\epsilon$ where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals $J_\epsilon = I_0 \supseteq I_1 \supseteq ... \supseteq I_k \supseteq ...$ with the property that $diamI_k \leq \frac{1}{2} diamI_{k-1}$ for $(k = 2, 3, 4, ...)$ and $\{k \in \mathbb{N} : x_k \in I_k\} \in m^I(F, p)$ for $(k = 1, 2, 3, ...)$. Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi = I - \lim x$. So that $f_k(\xi) = I - \lim f_k(x)$, that is $L = I - \lim f_k(x)$. □

Theorem 2.4. *Let $H = \sup_k p_k < \infty$ and I an admissible ideal. Then the following are equivalent.*
(a) $(x_k) \in c^I(F, p)$;
(b) there exists$(y_k) \in c(F, p)$ such that $x_k = y_k$, for a.a.k.r.I; (c) there exists$(y_k) \in c(F, p)$ and $(x_k) \in c_0^I(F, p)$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : f_k(|y_k - L|^{p_k}) \geq \epsilon\} \in I$; (d) there exists a subset $K = \{k_1 < k_2 ....\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n\to\infty} f_k(|x_{k_n} - L|^{p_{k_n}}) = 0$.

Proof. (a) implies (b). Let $(x_k) \in c^I(F, p)$. Then there exists $L \in \mathbb{C}$ such that $\{k \in \mathbb{N} : f_k(|x_k - L|^{p_k}) \geq \epsilon\} \in I$. Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that $\{k \leq m_t : f_k(|x_k - L|^{p_k}) \geq \epsilon\} \in I$. Define a sequence (y_k) as $y_k = x_k$, for all $k \leq m_1$. For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.
$y_k = \begin{cases} x_k, & \text{if } |x_k - L|^{p_k} < t^{-1}, \\ L, otherwise. \end{cases}$ Then $(y_k) \in c(F, p)$ and form the following inclusion $\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : f_k(|x_k - L|^{p_k}) \geq \epsilon\} \in I$. We get $x_k = y_k$, for a.a.k.r.I.

(b) implies (c).For $(x_k) \in c^I(F, p)$. Then there exists $(y_k) \in c(F, p)$ such that $x_k = y_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $k \in I$. Define a sequence (z_k) as $z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, otherwise. \end{cases}$ Then $z_k \in c_0^I(F, p)$ and $y_k \in c(F, p)$.

(c) implies (d).Let $P_1 = \{k \in \mathbb{N} : f_k(|x_k|^{p_k}) \geq \epsilon\} \in I$ and $K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$. Then we have $\lim_{n\to\infty} f_k(|x_{k_n} - L|^{p_{k_n}}) = 0$.

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and $\lim_{n\to\infty} f_k(|x_{k_n} - L|^{p_{k_n}}) = 0$. Then for any $\epsilon > 0$, and Lemma 1.9, we have $\{k \in \mathbb{N} : f_k(|x_k - L|^{p_k}) \geq \epsilon\} \subseteq K^c \cup \{k \in K : f_k(|x_k - L|^{p_k}) \geq \epsilon\}$. Thus $(x_k) \in c^I(F, p)$. □

Theorem 2.5. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(F, p) \supseteq m_0^I(F, q)$ if and only if $\lim_{k\in K} \inf \frac{p_k}{q_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

Proof. Let $\liminf_{k\in K} \frac{p_k}{q_k} > 0$. and $(x_k) \in m_0^I(F,q)$. Then there exists $\beta > 0$ such that $p_k > \beta q_k$, for all sufficiently large $k \in K$. Since $(x_k) \in m_0^I(F,q)$, for a given $\epsilon > 0$, we have $B_0 = \{k \in \mathbb{N} : f_k(|x_k|^{q_k}) \geq \epsilon\} \in I$ Let $G_0 = K^c \cup B_0$. Then $G_0 \in I$. Then for all sufficiently large $k \in G_0$, $\{k \in \mathbb{N} : f_k(|x_k|^{p_k}) \geq \epsilon\} \subseteq \{k \in \mathbb{N} : f_k(|x_k|^{\beta q_k}) \geq \epsilon\} \in I$. Therefore $(x_k) \in m_0^I(F,p)$. □

Theorem 2.6. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(F,q) \supseteq m_0^I(F,p)$ if and only if $\liminf_{k\in K} \frac{q_k}{p_k} > 0$, where $K^c \subseteq \mathbb{N}$ such that $K \in I$.*

Proof. The proof follows similarly as the proof of Theorem 2.5. □

Theorem 2.7. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(F,q) = m_0^I(F,p)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k} > 0$, and $\liminf_{k\in K} \frac{q_k}{p_k} > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.*

Proof. On combining Theorem 2.5 and 2.6 we get the required result. □

Theorem 2.8. *Let $h = \inf_k p_k$ and $H = \sup_k p_k$. Then the following results are equivalent.*
(a) $H < \infty$ and $h > 0$. (b) $c_0^I(F,p) = c_0^I$.

Proof. Suppose that $H < \infty$ and $h > 0$,then the inequalities $min\{1, s^h\} \leq s^{p_k} \leq max\{1, s^H\}$ hold for any $s > 0$ and for all $k \in \mathbb{N}$. Therefore the equivalent of (a) and (b) is obvious. □

Theorem 2.9. *Let $F = (f_k)$ be a sequence of modulii. Then $c_0^I(F,p) \subset c^I(F,p) \subset l_\infty^I(F,p)$ and the inclusions are proper.*

Proof. Let $(x_k) \in c^I(F,p)$. Then there exists $L \in \mathbb{C}$ such that $I - \lim f_k(|x_k - L|^{p_k}) = 0$. We have $f_k(|x_k|^{p_k}) \leq \frac{1}{2} f_k(|x_k - L|^{p_k}) + \frac{1}{2} f_k(|L|^{p_k})$. Taking supremum over k both sides we get $(x_k) \in l_\infty^I(F,p)$. The inclusion $c_0^I(F,p) \subset c^I(F,p)$ is obvious. Hence $c_0^I(F,p) \subset c^I(F,p) \subset l_\infty^I(F,p)$. □

Theorem 2.10. *If $H = \sup_k p_k < \infty$, then for a sequence of moduli F, we have $l_\infty^I \subset M(m^I(F,p))$, where the inclusion may be proper.*

Proof. Let $a \in l_\infty^I$. This implies that $\sup_k |a_k| < 1 + K$. for some $K > 0$ and all k. Therefore $x \in m^I(F,p)$ implies $\sup_k f_k(|a_k x_k|^{p_k}) \leq (1+K)^H \sup_k f_k(|x_k|^{p_k}) < \infty$. which gives $l_\infty^I \subset M(m^I(F,p))$. To show that the inclusion may be proper, consider the case when $p_k = \frac{1}{k}$ for all k. Take $a_k = k$ for all k. Therefore $x \in m^I(F,p)$ implies $\sup_k f_k(|a_k x_k|^{p_k}) \leq \sup_k f_k(|k|^{\frac{1}{k}}) \sup_k f_k(|x_k|^{p_k}) < \infty$. Thus in this case $a = (a_k) \in M(m^I(F,p))$ while $a \notin l_\infty^I$. □

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Some Inequalities Involving Fuzzy Complex Numbers

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Abstract

In this paper we wish to establish a few inequalities related to fuzzy complex numbers which extend some standard results.

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1. Introduction, Definitions and Notations

The idea of fuzzy subset μ of a set X was primarily introduced by L.A. Zadeh (Zadeh, 1965) as a function $\mu : X \to [0, 1]$. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Among the various types of fuzzy sets, those which are defined on the universal set of complex numbers are of particular importance. They may, under certain conditions, be viewed as fuzzy complex numbers.

A fuzzy set z_f is defined by its membership function $\mu\left(z \mid z_f\right)$ which is a mapping from the complex numbers $\mathbb{C}$ into [0, 1] where z is a regular complex number as $z = x + iy$, is called a fuzzy complex number if it satisfies the following conditions :

1. $\mu\left(z \mid z_f\right)$ is continuous;

2. An α-cut of z_f which is defined as $z_f^\alpha = \left\{z \mid \mu\left(z \mid z_f\right) > \alpha\right\}$, where $0 \le \alpha < 1$, is open, bounded, connected and simply connected; and

3. $z_f^1 = \left\{z \mid \mu\left(z \mid z_f\right) = 1\right\}$ is non-empty, compact, arcwise connected and simply connected.

(For detail on the set z_f as mentioned above, one may see (Buckley, 1989)).

Using this concept of fuzzy complex numbers, J. J. Buckley (Buckley, 1989) shown that fuzzy complex numbers is closed under the basic arithmetic operations. In paper (Buckley, 1989) we

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also see the development of fuzzy complex numbers by defining addition and multiplication from the extension principle which has been shown in terms of α-cuts.

We now review some definitions used in this paper.

Definition 1.1. (Buckley, 1989)The complex conjugate $\bar{z}_f$ of z_f is defined as

$$\mu\left(z \mid \bar{z}_f\right) = \mu\left(\bar{z} \mid z_f\right),$$

where $\bar{z} = x - iy$ is the complex conjugate of $z = x + iy$. The complex conjugate $\bar{z}_f$ of a fuzzy complex number z_f is also a fuzzy complex number because the mapping $z = x + iy \to \bar{z} = x - iy$ is continuous.

Definition 1.2. (Buckley, 1989)The modulus $\left|z_f\right|$ of a fuzzy complex number z_f is defined as

$$\mu\left(r \mid \left|z_f\right|\right) = \sup\left\{\mu\left(z \mid z_f\right) \mid |z| = r\right\},$$

where r is the modulus of z.

Similarly we may define the modulus of a real fuzzy number R_f as follows:

$$\mu\left(|a| \mid \left|R_f\right|\right) = \sup\{\mu\left(a \mid R_f\right) \mid |a| = a \text{ if } a > 0, |a| = 0 \text{ if } a = 0 \text{ and } |a| = -a \text{ if } a < 0\}.$$

Now in the following, we define two special types of fuzzy complex numbers z_f^n and nz_f of the fuzzy complex number z_f, for any complex number $z \in z_f$ and $n \in R$.

Definition 1.3. Fuzzy complex numbers z_f^n and nz_f are defined as

$$\mu(z \mid z_f^n) = \mu(z^n \mid z_f)$$

and

$$\mu(z \mid nz_f) = \mu(n.z \mid z_f).$$

In particular when $n = 2$, we have

$$\mu(z \mid z_f^2) = \mu(z^2 \mid z_f) \quad \text{and} \quad \mu(z \mid 2z_f) = \mu(2.z \mid z_f)\ .$$

It can be easily verified that

$$z_f^2 \neq z_f.z_f \qquad \text{and} \qquad 2z_f \neq z_f + z_f \text{ but } 2(z_{f_1} + z_{f_2}) = 2z_{f_1} + 2z_{f_2}.$$

From the definition of fuzzy complex number one may easily verify that z_f^n and nz_f are also fuzzy complex numbers when z_f is a fuzzy complex number. It should be noted that the significance of Definition 1.3 is completely different from the definitions of additions and multiplications of fuzzy complex numbers as mentioned in (Buckley, 1989).

In this paper we wish to establish a few standard inequalities related to fuzzy complex numbers.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *(Buckley, 1989) Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers. Suppose $A = z_{f_1} + z_{f_2}$ and $M = z_{f_1}.z_{f_2}$ respectively. Then for $0 \leq \alpha \leq 1$, $A^{\alpha} = S^{\alpha}$ and $M^{\alpha} = P^{\alpha}$ holds where*

$$S^{\alpha} = \left\{ z_{f_1} + z_{f_2} \mid (z_1, z_2) \in z_{f_1}^{\alpha} \times z_{f_2}^{\alpha} \right\}$$

and

$$P^{\alpha} = \left\{ z_{f_1}.z_{f_2} \mid (z_1, z_2) \in z_{f_1}^{\alpha} \times z_{f_2}^{\alpha} \right\}.$$

Also $z_{f_1} + z_{f_2}$ and $z_{f_1}.z_{f_2}$ are fuzzy complex numbers.

The following lemma may be deduced in the line of Lemma 2.1 and so its proof is omitted.

Lemma 2.2. *Let $z_{f_1}, z_{f_2}, z_{f_3}, ..., z_{f_n}$ be any n number of fuzzy complex numbers. Also let $A = z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}$ and $M = z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}$ respectively. Then for $0 \leq \alpha \leq 1$, $A^{\alpha} = S^{\alpha}$ and $M^{\alpha} = P^{\alpha}$ holds where*

$$S^{\alpha} = \left\{ z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n} \mid (z_1, z_2, z_3, ..., z_n) \in z_{f_1}^{\alpha} \times z_{f_2}^{\alpha} \times z_{f_3}^{\alpha} \times ... \times z_{f_n}^{\alpha} \right\}$$

and

$$P^{\alpha} = \left\{ z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n} \mid (z_1, z_2, z_3, ..., z_n) \in z_{f_1}^{\alpha} \times z_{f_2}^{\alpha} \times z_{f_3}^{\alpha} \times ... \times z_{f_n}^{\alpha} \right\}.$$

Lemma 2.3. *(Buckley, 1989) If z_f is any fuzzy complex number then*

$$\left| z_f \right|^{\alpha} = \left| z_f^{\alpha} \right|$$

where $0 \leq \alpha \leq 1$ and $\left| z_f \right|$ is a truncated real fuzzy number.

Lemma 2.4. *(Kaufmann & Gupta, 1985) If M and N be any two real fuzzy numbers then*

$$(M + N)^{\alpha} = M^{\alpha} + N^{\alpha}$$

and if $M \geq 0, N \geq 0$ then

$$(M \cdot N)^{\alpha} = M^{\alpha} \cdot N^{\alpha}.$$

Lemma 2.5. *(Buckley, 1989) Let $\bar{z}_f$ be a fuzzy complex conjugate number of a fuzzy complex number z_f. Then*

$$\bar{z}_f^{\alpha} = \overline{z_f^{\alpha}}$$

where $0 \leq \alpha \leq 1$.

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers. Then*

$$\left|z_{f_1} - z_{f_2}\right| \geq \left|z_{f_1}\right| - \left|z_{f_2}\right| .$$

Proof. The meaning of the inequality is that the interval $\left|z_{f_1} - z_{f_2}\right|^{\alpha}$ is greater that or equal to the interval $\left(\left|z_{f_1}\right| - \left|z_{f_2}\right|\right)^{\alpha}$ for $0 \leq \alpha \leq 1$.

Now from Lemma 2.1 and Lemma 2.3, we get that

$$\left|z_{f_1} - z_{f_2}\right|^{\alpha} = \left|\left(z_{f_1} - z_{f_2}\right)^{\alpha}\right| = \left|z_{f_1}^{\alpha} - z_{f_2}^{\alpha}\right| = \left\{|z_1 - z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.1}$$

Again in view of Lemma 2.4, we obtain from Lemma 2.3 that

$$\left(\left|z_{f_1}\right| - \left|z_{f_2}\right|\right)^{\alpha} = \left|z_{f_1}\right|^{\alpha} - \left|z_{f_2}\right|^{\alpha} = \left|z_{f_1}^{\alpha}\right| - \left|z_{f_2}^{\alpha}\right| = \left\{|z_1| - |z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.2}$$

Hence the result follows from (3.1) and (3.2) and in view of

$$|z_1 - z_2| \geq |z_1| - |z_2| .$$

This proves the theorem. □

J. J. Buckley (Buckley, 1989) proved the following results:

Theorem A (Buckley, 1989) *Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers. Then*

$$(1).\quad \left|z_{f_1} - z_{f_2}\right| \leq \left|z_{f_1}\right| + \left|z_{f_2}\right| \quad \textit{and} \quad (2).\quad \left|z_{f_1}.z_{f_2}\right| = \left|z_{f_1}\right| \left|z_{f_2}\right| .$$

But he (Buckley, 1989) remained silent about the question when the equality holds in the inequality (1) of Theorem A. In the next two theorems, we wish to generalise the results of Theorem A and find out the condition for which $\left|z_{f_1} - z_{f_2}\right| = \left|z_{f_1}\right| + \left|z_{f_2}\right|$ holds respectively.

Theorem 3.2. *Let $z_{f_1}, z_{f_2}, z_{f_3}, ..., z_{f_n}$ be any n number of fuzzy complex numbers. Then*

$$(i).\quad \left|z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}\right| \leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + ... + \left|z_{f_n}\right| \ \textit{and}$$

$$(ii).\quad \left|z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}\right| = \left|z_{f_1}\right| \left|z_{f_2}\right| \left|z_{f_3}\right| ... \left|z_{f_n}\right| .$$

Proof. In view of Lemma 2.1, it follows from Theorem A that

$$\begin{aligned} \left|z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}\right| &\leq \left|z_{f_1}\right| + \left|z_{f_2} + z_{f_3} + ... + z_{f_n}\right| \\ &\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3} + ... + z_{f_n}\right| \\ &\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + \left|z_{f_4} + ... + z_{f_n}\right| \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + ... + \left|z_{f_n}\right| . \end{aligned}$$

This proves the first part of the theorem.

Similarly with the help of Lemma 2.1 and the equality $\left|z_{f_1}.z_{f_2}\right| = \left|z_{f_1}\right| \left|z_{f_2}\right|$, one can easily establish the second part of the theorem. □

Remark. In view of Lemma 2.2, Lemma 2.3 and Lemma 2.4 it can also be shown that the intervals $\left|z_{f_1}+z_{f_2}+z_{f_3}+...+z_{f_n}\right|^{\alpha}$ and $\left|z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}\right|^{\alpha}$ are less than or equal to the intervals $\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|+\left|z_{f_3}\right|+...+\left|z_{f_n}\right|\right)^{\alpha}$ and $\left(\left|z_{f_1}\right|\left|z_{f_2}\right|\left|z_{f_3}\right|...\left|z_{f_n}\right|\right)^{\alpha}$ respectively in Theorem 3.2 for $0\le\alpha\le 1$.

Theorem 3.3. *Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers such that $\left|z_{f_1}+z_{f_2}\right|=\left|z_{f_1}\right|+\left|z_{f_2}\right|$ then either $\arg z_1-\arg z_2$ is an even multiple of π or $\frac{z_1}{z_2}$ is a positive real number where z_1 and z_2 are any two members of z_{f_1} and z_{f_2} respectively.*

Proof. The meaning of the equality is that the interval $\left|z_{f_1}+z_{f_2}\right|^{\alpha}$ is equal to the interval $\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)^{\alpha}$ for $0\le\alpha\le 1$.
Thus $\left|z_{f_1}+z_{f_2}\right|=\left|z_{f_1}\right|+\left|z_{f_2}\right|$ i.e, $\left|z_{f_1}+z_{f_2}\right|^{\alpha}=\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)^{\alpha}$ i.e., $\left|z_{f_1}+z_{f_2}\right|^{\alpha}=\left(\left|z_{f_1}\right|^{\alpha}+\left|z_{f_2}\right|^{\alpha}\right)$ i.e, $\left|z_{f_1}^{\alpha}+z_{f_2}^{\alpha}\right|=\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|$ i.e., $|z_1+z_2|=|z_1|+|z_2|\mid z_i\in z_{f_i}^{\alpha}, i=1,2$; which is only possible when either $\arg z_1-\arg z_2$ is an even multiple of π or $\frac{z_1}{z_2}$ is a positive real number. Hence the theorem follows. □

Theorem 3.4. *If z_{f_1} and z_{f_2} are any two fuzzy complex numbers with $\left|z_{f_1}+z_{f_2}\right|=\left|z_{f_1}-z_{f_2}\right|$, then $\arg z_1$ and $\arg z_2$ differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ where z_1 and z_2 are any two members of z_{f_1} and z_{f_2} respectively.*

Proof. The meaning of the equality is that the α-cuts of $\left|z_{f_1}+z_{f_2}\right|$ is equal to the corresponding α-cuts of $\left|z_{f_1}-z_{f_2}\right|$ for $0\le\alpha\le 1$.
Now in view of Lemma 2.1 and Lemma 2.3, we obtain that

$$\left|z_{f_1}+z_{f_2}\right|^{\alpha}=\left|\left(z_{f_1}+z_{f_2}\right)^{\alpha}\right|=\left|z_{f_1}^{\alpha}+z_{f_2}^{\alpha}\right|=\left\{|z_1+z_2|\mid z_i\in z_{f_i}^{\alpha}, i=1,2\right\}. \tag{3.3}$$

Similarly,

$$\left|z_{f_1}-z_{f_2}\right|^{\alpha}=\left|\left(z_{f_1}-z_{f_2}\right)^{\alpha}\right|=\left|z_{f_1}^{\alpha}-z_{f_2}^{\alpha}\right|=\left\{|z_1-z_2|\mid z_i\in z_{f_i}^{\alpha}, i=1,2\right\}. \tag{3.4}$$

Therefore from (3.3) and (3.4) it follows that $\left|z_{f_1}+z_{f_2}\right|=\left|z_{f_1}-z_{f_2}\right|$ which implies that $|z_1+z_2|=|z_1-z_2|\mid z_i\in z_{f_i}^{\alpha}, i=1,2$ which is only possible when $\arg z_1$ and $\arg z_2$ differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.
Thus the theorem is established. □

Theorem 3.5. *Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers. Then*

$$\left|z_{f_1}\pm z_{f_2}\right|\ge\left|\left|z_{f_1}\right|-\left|z_{f_2}\right|\right|.$$

Proof. For $0\le\alpha\le 1$, we have

$$\left|z_{f_1}\pm z_{f_2}\right|^{\alpha}=\left|\left(z_{f_1}\pm z_{f_2}\right)^{\alpha}\right|=\left|z_{f_1}^{\alpha}\pm z_{f_2}^{\alpha}\right|=\left\{|z_1\pm z_2|\mid z_i\in z_{f_i}^{\alpha}, i=1,2\right\}. \tag{3.5}$$

We also deduce that

$$\left|\left|z_{f_1}\right|-\left|z_{f_2}\right|\right|^{\alpha}=\left|\left(\left|z_{f_1}\right|-\left|z_{f_2}\right|\right)^{\alpha}\right|=\left|\left|z_{f_1}\right|^{\alpha}-\left|z_{f_2}\right|^{\alpha}\right|=\left|\left|z_{f_1}^{\alpha}\right|-\left|z_{f_2}^{\alpha}\right|\right|=\left\{||z_1|-|z_2||\mid z_i\in z_{f_i}^{\alpha}, i=1,2\right\}. \tag{3.6}$$

Hence the theorem follows from (3.5) and (3.6) and in view of the following inequality :

$$|z_1\pm z_2|\ge||z_1|-|z_2||.$$

□

Theorem 3.6. *If z_{f_1} and z_{f_2} are any two fuzzy complex numbers, then*

$$2\left|z_{f_1}+z_{f_2}\right| \geq \left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|.$$

Proof. In order to prove this theorem, we wish to show that the interval $\left(2\left|z_{f_1}+z_{f_2}\right|\right)^{\alpha}$ is greater than or equal to the interval $\left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|\right\}^{\alpha}$ for $0 \leq \alpha \leq 1$.

From Lemma 2.3 and Lemma 2.4, we get that

$$\left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|\right\}^{\alpha} = \left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)^{\alpha}\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|^{\alpha}\right\} = \left\{\left(\left|z_{f_1}\right|^{\alpha}+\left|z_{f_2}\right|^{\alpha}\right)\left|\left(\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right)^{\alpha}\right|\right\}$$

$$= \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(\frac{z_{f_1}}{\left|z_{f_1}\right|}\right)^{\alpha}+\left(\frac{z_{f_2}}{\left|z_{f_2}\right|}\right)^{\alpha}\right|\right\} = \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(z_{f_1}\left|z_{f_1}\right|^{-1}\right)^{\alpha}+\left(z_{f_2}\left|z_{f_2}\right|^{-1}\right)^{\alpha}\right|\right\}$$

$$= \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(z_{f_1}^{\alpha}\left|z_{f_1}^{\alpha}\right|^{-1}\right)+\left(z_{f_2}^{\alpha}\left|z_{f_2}^{\alpha}\right|^{-1}\right)\right|\right\} = \left\{\left(|z_1|+|z_2|\right)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}. \quad (3.7)$$

Since

$$2\left|z_1+z_2\right| \geq \left(|z_1|+|z_2|\right)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|,$$

in view of Definition 1.2 and Definition 1.3, the theorem follows from (3.3) and (3.7). □

Theorem 3.7. *Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers. Then*

$$\left|\left(z_{f_1}+z_{f_2}\right)^2\right|+\left|\left(z_{f_1}-z_{f_2}\right)^2\right| = \left(2\left|z_{f_1}^2\right|-2\left|z_{f_2}^2\right|\right).$$

Proof. In view of Lemma 2.1, Lemma 2.3 and Lemma 2.4, we get for $0 \leq \alpha \leq 1$ that

$$\left(\left|\left(z_{f_1}+z_{f_2}\right)^2\right|+\left|\left(z_{f_1}-z_{f_2}\right)^2\right|\right)^{\alpha} = \left(\left|\left(z_{f_1}+z_{f_2}\right)^2\right|^{\alpha}+\left|\left(z_{f_1}-z_{f_2}\right)^2\right|^{\alpha}\right) = \left|\left(\left(z_{f_1}+z_{f_2}\right)^2\right)^{\alpha}\right|+\left|\left(\left(z_{f_1}-z_{f_2}\right)^2\right)^{\alpha}\right|$$

$$= \left(\left|\left(z_{f_1}^{\alpha}+z_{f_2}^{\alpha}\right)^2\right|+\left|\left(z_{f_1}^{\alpha}-z_{f_2}^{\alpha}\right)^2\right|\right) = \left\{\left|(z_1+z_2)^2\right|+\left|(z_1-z_2)^2\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}$$

$$= \left\{|z_1+z_2|^2+|z_1-z_2|^2 \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}. \quad (3.8)$$

Analogously we also see that

$$\left(2\left|z_{f_1}^2\right|-2\left|z_{f_2}^2\right|\right)^{\alpha} = \left(2\left|z_{f_1}^2\right|^{\alpha}-2\left|z_{f_2}^2\right|^{\alpha}\right) = \left(2\left|\left(z_{f_1}^2\right)^{\alpha}\right|-2\left|\left(z_{f_2}^2\right)^{\alpha}\right|\right) = \left(2\left|\left(z_{f_1}^{\alpha}\right)^2\right|-2\left|\left(z_{f_2}^{\alpha}\right)^2\right|\right)$$

$$= \left\{2\left|z_1^2\right|-2\left|z_2^2\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\} = \left\{2|z_1|^2-2|z_2|^2 \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}. \quad (3.9)$$

Now in the line of Definition 1.3, it follows from (3.8)and (3.9) that the corresponding α-cuts are equal. Hence the theorem follows as we obtain the equality of the two real fuzzy numbers. □

In the next theorem we establish a few properties of fuzzy complex conjugate numbers depending on the concept of it.

Theorem 3.8. *Let $\bar{z}_f$ be a fuzzy complex conjugate number of a fuzzy complex number z_f. Then*
(1) . $\bar{\bar{z}}_f = z_f$, (2) . $\overline{\left(z_{f_1} \pm z_{f_2}\right)} = \bar{z}_{f_1} \pm \bar{z}_{f_2}$, (3). $\overline{\left(z_{f_1}.z_{f_2}\right)} = \bar{z}_{f_1} \cdot \bar{z}_{f_2}$,
(4). $\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)} = \frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}$ *and* (5) . $\left|z_f\right| = \left|\bar{z}_f\right|$.

Proof. In view of Lemma 2.5 and for $0 \le \alpha \le 1$, we obtain that

$$\left(\bar{\bar{z}}_f\right)^{\alpha} = \overline{\left(\bar{z}_f\right)^{\alpha}} = \overline{\overline{z_f^{\alpha}}} = \left\{\bar{\bar{z}} \mid \text{ for all } z \in z_f^{\alpha}\right\}.$$

Again

$$z_f^{\alpha} = \left\{z \mid \mu\left(z \mid z_f\right) > \alpha\right\} = \left\{z \mid \text{ for all } z \in z_f^{\alpha}\right\}.$$

Since $\bar{\bar{z}} = z$, the first part of the theorem follows from above.

For the second part of the theorem, we have to prove that the α-cuts of $\overline{\left(z_{f_1} \pm z_{f_2}\right)}$ are equal to the corresponding α-cuts of $\bar{z}_{f_1} \pm \bar{z}_{f_2}$.
Now it follows from Lemma 2.1 and Lemma 2.5 that

$$\left(\overline{\left(z_{f_1} \pm z_{f_2}\right)}\right)^{\alpha} = \overline{\left(z_{f_1} \pm z_{f_2}\right)^{\alpha}} = \overline{\left(z_{f_1}^{\alpha} \pm z_{f_2}^{\alpha}\right)} = \left\{\overline{z_1 \pm z_2} \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}$$

and

$$\left(\bar{z}_{f_1} \pm \bar{z}_{f_2}\right)^{\alpha} = \left(\bar{z}_{f_1}\right)^{\alpha} \pm \left(\bar{z}_{f_2}\right)^{\alpha} = \left(\overline{z_{f_1}^{\alpha}}\right) \pm \left(\overline{z_{f_2}^{\alpha}}\right) = \left\{\bar{z}_1 \pm \bar{z}_2 \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}.$$

Thus the second part of the theorem is established in view of $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

We also observe that

$$\overline{\left(z_{f_1}.z_{f_2}\right)}^{\alpha} = \overline{\left(z_{f_1}.z_{f_2}\right)^{\alpha}} = \overline{\left(z_{f_1}^{\alpha}.z_{f_2}^{\alpha}\right)} = \left\{\overline{z_1.z_2} \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.10}$$

We may also see that

$$\left(\bar{z}_{f_1}.\bar{z}_{f_2}\right)^{\alpha} = \left(\bar{z}_{f_1}\right)^{\alpha}.\left(\bar{z}_{f_2}\right)^{\alpha} = \overline{z_{f_1}^{\alpha}}.\overline{z_{f_2}^{\alpha}} = \left\{\bar{z}_1.\bar{z}_2 \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.11}$$

Now from (3.10) and (3.11), we obtain that the corresponding α-cuts are equal. This proves the third part of the theorem.

For the fourth part of the theorem, we deduce that

$$\left(\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)}\right)^{\alpha} = \overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)^{\alpha}} = \overline{\left(z_{f_1}.z_{f_2}^{-1}\right)^{\alpha}} = \overline{z_{f_1}^{\alpha}.\left(z_{f_2}^{-1}\right)^{\alpha}} = \overline{\left(z_{f_1}^{\alpha}.\left(z_{f_2}^{\alpha}\right)^{-1}\right)} = \left\{\overline{\frac{z_1}{z_2}} \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}$$

and

$$\left(\frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}\right)^{\alpha} = \left(\bar{z}_{f_1}.\bar{z}_{f_2}^{-1}\right)^{\alpha} = \left(\bar{z}_{f_1}\right)^{\alpha}.\left(\bar{z}_{f_2}^{-1}\right)^{\alpha} = \overline{z_{f_1}^{\alpha}}.\left(\left(\overline{z_{f_2}}\right)^{\alpha}\right)^{-1} = \overline{z_{f_1}^{\alpha}} \cdot \left(\left(\overline{z_{f_2}^{\alpha}}\right)\right)^{-1} = \left\{\frac{\bar{z}_1}{\bar{z}_2} \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}.$$

Hence the α-cuts of $\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)}$ are equal to the corresponding α-cuts of $\frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}$ which implies that the two fuzzy complex numbers are equal. Thus the fourth part of the theorem follows.
Again we have from Lemma 2.3 and Lemma 2.5 that

$$\left|z_f\right|^{\alpha} = \left|z_f^{\alpha}\right| = \left\{|z| \mid \text{ for all } z \in z_f\right\}$$

and

$$\left(\left|\bar{z}_f\right|\right)^{\alpha} = \left|\left(\bar{z}_f\right)^{\alpha}\right| = \left|\bar{z}_f^{\alpha}\right| = \left\{\left|\bar{z}\right| \mid \text{ for all } z \in z_f\right\}.$$

Consequently the last part of the theorem follows in view of $|z| = \left|\bar{z}\right|$. □

4. Open Problem

As open problems, there are several scopes to investigate the theory of analyticity and singularity in case of functions of fuzzy complex variables; and analogously entire or meromorphic functions of fuzzy complex variables may be defined. Naturally, the theory of different aspects of growth properties of entire and meromorphic functions, comparative growth estimates of iterated entire functions, results related to exponent of convergence of zeros of entire functions of fuzzy complex variables etc. may also be studied afresh.

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Banach Frames, Double Infinite Matrices and Wavelet Coefficients

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Abstract

In this paper we study the action of a double infinite matrix A on $f \in H_v^p$ (weighted Banach space, $1 \le p \le \infty$) and on its wavelet coefficients. Also, we find the frame condition for $A-$transform of $f \in H_v^p$ whose wavelet series expansion is known.

Keywords: Frames, Riesz basis, wavelet coefficients, Banach space and frame operators.
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1. Introduction

The mathematical background for today's signal processing applications are Gabor (Feichtinger & Strohmer, 1998), wavelet (Daubechies, 1992) and sampling theory (Benedetto & Ferreira, 2001). Without signal processing methods several modern technologies would not be possible, like mobile phone, UMTS, xDSL or digital television. In other words, we can say that any advance in signal processing sciences directly leads to an application in technology and information processing. A signal is sampled and then analyzed using a Gabor respectively wavelet system. Many applications use a modification on the coefficients obtained from the analysis and synthesis operations. If the coefficients are not changed, the result of synthesis should be the original signal, i.e., perfect reconstruction is needed. One way is to analyze the signal using orthonormal basis. For practical point of view it is noted that the concept of an orthonormal basis is not always useful. Sometimes it is more important for a decomposing set to have other special properties rather than guaranteing unique coefficients. This led to the concept of frames introduced by Duffin and Schaeffer (Duffin & Schaeffer, 1952). Now a days it is one of most important foundations of Gabor (Moricz & Rhoades, 1989), wavelet (S.T. Ali & Gazeau, 2000) and sampling theory (Aldroubi & Gröchenig, 2001). In signal processing applications frames have received more and more attention

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(H. Bölcskei & Feichtinger, 1998; Kronland-Martinet & Grossmann, 1991; Munch, 1992; Sheikh & Mursaleen, 2004).

Frame provide stable expansions in Hilbert spaces, but they may be over complete and the coefficients in the frame expansion need not be unique unlike in orthogonal expansions. This redundancy is useful for the application point of view that is to noise reduction or for the reconstruction from lossy data (Daubechies, 1992; Duffin & Schaeffer, 1952; Matz & Hlawatsch, 2002). The construction of stable orthonormal basis are often difficult in a numerical efficient way than the construction of frames which are more flexible. Sometimes it is reasonable to use the frames to analyze additional properties of functions beyond the Hilbert space. These properties are encoded in the frame coefficients. Wavelet frames encode information on the smoothness and decay properties or phase space localization of functions by means of the magnitudes of the frame coefficients. The aim is to study these properties in Banach space norms. Moreover, to characterize an associated family of Banach spaces of functions by the values of the frame coefficients which play an important role in non-linear approximation and in compression algorithms (DeVore & Temlyakov, 1996). However, in (Gröchenig, 2004) Gröchenig showed that certain frames for Hilbert spaces extend automatically to Banach frames. Using this theory he derived some results on the construction of non-uniform Gabor frames and solved a problem about non-uniform sampling in shift-invariant spaces. Recently, Kumar (Kumar, 2013) studied the convergence of wavelet expansions associated with dilation matrix in the variable L^p spaces using the approximate identity. In an another paper Kumar (Kumar, 2009) studied the convergence of non-orthogonal wavelet expansions in $L^p(R), 1 < p < \infty$.

The space $L^2(R)$ of measurable function f is defined on the real line R, that satisfies $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$. The inner product of two square integrable functions $f, g \in L^2(R)$ is defined as

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx, \;\; \|f\|^2 =< f, f >^{1/2} .$$

Every function $f \in L^2(R)$ can be written as $f(x) = \sum_{j,k\in z} c_{j,k}\varphi_{j,k}(x)$ (z is the set of integers).

This series representation of f is called wavelet series. Analogous to the notation of Fourier coefficients, the wavelet coefficients $c_{j,k}$ are given by $c_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\varphi_{j,k}(x)}dx =< f, \varphi_{j,k} >, \;\; \varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)$.

Now, if we define continuous wavelet transform as $(W_\varphi(f))(b, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x)\overline{\varphi\left(\frac{x-b}{a}\right)}dx, f \in L^2(R)$ then the wavelet coefficients are given by $c_{j,k} = (W_\varphi(f))\left(\frac{k}{2^j}, \frac{1}{2^j}\right)$.

A sequence $\{x_n\}$ in a Hilbert space H is a frame if there exist constant c_1 and $c_2, 0 < c_1 \leq c_2 < \infty$, such that $c_1\|f\|^2 \leq \sum_{n\in z} | < f, x_n > |^2 \leq c_2\|f\|^2$, for all $f \in H$. The supremum of all such numbers c_1 and infimum of all such numbers c_2 are called the frame bounds of the frame. The frame is called tight frame when $c_1 = c_2 = 1$. Any orthonormal basis in a Hilbert space H is a normalized tight frame. The connection between frames and numerically stable reconstruction from discretized wavelet was pointed out by (Grossmann *et al.*, 1985). In 1985, they defined that a wavelet function $\varphi \in L^2(R)$, constitutes a frame with frame bounds c_1 and c_2, if any $f \in L^2(R)$ such that $c_1\|f\|^2 \leq \sum_{j,k\in z} | < f, \varphi_{j,k} > |^2 \leq c_2\|f\|^2$. Again, it is said to be tight if $c_1 = c_2$ and is said to be exact if it ceases to be frame by removing any of its element. For more details see (Chui, 1992; Daubechies *et al.*, 1986).

2. Notations and Auxiliary Results

Let N and χ be countable index sets in some R^2 and both χ and N are separated i.e., $\inf_{m,n\in\chi; m\neq n} |m-n| \geq \delta > 0$, and likewise for N.

Weight Functions of Polynomial Growth. A weight is a non-negative continuous function on R^d. An $s-$moderate weight ν is called polynomially grows, if there are constants $C, s \geq 0$ such that $\nu(t) \leq C(1+|t|)^s$.

Lemma 2.1. *If $f(x) = \sum_{j,k\in N} c_{j,k}\varphi_{j,k}(x)$ is a wavelet expansion of $f \in L^2(R^d)$ with wavelet coefficients $c_{j,k} = \int_{-\infty}^{\infty} f(x)\overline{\varphi_{j,k}(x)}dx =< f, \varphi_{j,k} >$ and $A(a_{mnjk}) = [(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}$ for some $\varepsilon > 0$ and $j,k \in N, m,n \in \chi$, then the operator A defined on finite sequences $(c_{j,k})_{j,k\in N}$ by matrix multiplication $(Ac)_{m,n} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk}c_{j,k}$ extends to a bonded operator from $l_\nu^p(N)$ to $l_\nu^p(\chi)$ for all $p \in [1,\infty]$ and all $s-$moderate weights ν.*

Proof. To prove the result we have to show the boundedness of A from $l_\nu^1(N)$ to $l_\nu^1(\chi)$ and from $l_\nu^\infty(N)$ to $l_\nu^\infty(\chi)$. Then using the interpolation technique of [4] for weighted L^p-space, the lemma holds for all $p \in [1,\infty]$.

First we consider

$$\begin{aligned}
||Ac_{j,k}||_{l_\nu^1(\chi)} &= \sum_{m,n\in\chi}\left|\sum_{j,k\in N} a_{mnjk}c_{j,k}\right|\nu(m,n) \leq \sum_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}|c_{j,k}|\nu(m,n)\\
&\leq \sup_{j,k\in N}\left(\sum_{m,n\in\chi}[(1+|m-j|)(1+|n-k|)]^{-d-\varepsilon}\right)\times\\
&\quad \left(\sup_{m,n\in\chi; j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s}[\nu(j,k)]^{-1}\nu(m,n)\right)\times\sum_{j,k\in N}|c_{j,k}|\nu(j,k).
\end{aligned}$$

Using (Gröchenig, 2004), Lemma 2.2 in above inequality we obtain

$$\begin{aligned}
&\leq \sup_{j,k\in N}\left(C(1+|j-k|)^{-d-\varepsilon}\right)\left(\sup_{j,k\in N} C(1+|j-k|)^{-s}\right)\times\\
&\quad [\nu(j,k)]^{-1}\nu(m,n)\times\sum_{j,k\in N}|c_{j,k}|\nu(j,k) = C||c_{j,k}||_{l_\nu^1(N)}.
\end{aligned}$$

The first supremum in right hand side of above inequality is finite by (Gröchenig, 2004), Lemma 2.1] and second supremum in finite due to $s-$moderate and sub multiplicavity of the weights. Now we have

$$\begin{aligned}
||Ac_{j,k}||_{l_\nu^\infty(\chi)} &= \sup_{m,n\in\chi}\left|\sum_{j,k\in N} a_{mnjk}c_{j,k}\right|\nu(m,n)\\
&\leq \sup_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}|c_{j,k}|\nu(m,n)\\
&\leq \left(\sup_{m,n\in\chi}\sum_{j,k\in N}[(1+|m-j|)(1+|n-k)]^{-d-\varepsilon}\right)\times
\end{aligned}$$

$$\left(\sup_{\substack{m,n\in\chi \\ j,k\in N}} [(1+|m-j|)(1+|n-k|)]^{-s}.\nu(m,n)\nu(j,k)^{-1} \right) \times \left(\sup_{j,k\in N} |c_{j,k}|\nu(j,k) \right).$$

Again, using (Gröchenig, 2004), Lemma 2.2 in above inequality we get

$$\leq \left(\left(C \sup \sum_{j,k\in N} (1+|j-k|)^{-d-\varepsilon} \right) \left(\sup (1+|j-k|)^{-s} \nu(m,n)\nu(j,k)^{-1} \right) \times \left(\sup_{j,k\in N} |c_{j,k}|\nu(j,k) \right) \leq CC' \|c_{j,k}\|_{l^{\infty}_{\nu}(N)}. \right.$$

Let $\{\phi_{j,k} : j,k \in N\}$ be a Riesz basis of H with dual basis $\{\widetilde{\phi}_{j,k} : j,k \in N\}$ and ν be a weight function on R^d of polynomial type. □

Definition 2.1. Assume that $l^p_\nu(N) \subseteq l^2_\nu(N)$. Then the Banach space H^p_ν is defined to be

$$H^p_\nu = \{f \in H : f = \sum_{j,k\in N} c_{j,k}\phi_{j,k} \ \texttt{for} \ c_{j,k} \in l^p_\nu(N)\}$$

with norm $\|f\|_{H^p_\nu} = \|c_{j,k}\|_{l^p_\nu}$. It should be noted that $c_{j,k}$ is uniquely determined, in fact, $c_{j,k} =< f, \widetilde{\phi}_{j,k} >$.

By assumption $l^p_\nu(N) \subseteq l^2_\nu(N)$, it means H^p_ν is a (dense) subset of H. On the other hand, if $l^p_\nu \nsubseteq l^2_\nu$ and $p < \infty$, we define H^p_ν to be the completion of the subspace H_0 of finite linear combinations, i.e., $H_0 = \{f = \sum_{j,k\in N} c_{j,k}\phi_{j,k} : supp$ c is finite $\}$, with respect to the norm $\|f\|_{H^p_\nu} = \|c\|_{l^p_\nu}$. If $p = \infty$ and $l^p_\nu \nsubseteq l^2$, we take the weak completion of H_0 to define H^∞_ν. In these cases $H^p_\nu \nsubseteq H$.

Frame Operators and Localization of Frames. Let $\mathcal{F} = \{\varphi_{m,n} : m,n \in \chi\}$ be a frame for H and $Sf = \sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n}$ be the corresponding frame operator. Each frame element has a natural expansion with respect to the given Riesz basis as

$$\varphi_{m,n} = \sum_{j,k\in N} \left\langle \varphi_{m,n}, \widetilde{\phi}_{j,k} \right\rangle \phi_{j,k} = \sum_{j,k\in N} < \varphi_{m,n}, \phi_{j,k} > \widetilde{\phi}_{j,k}.$$

The frame operator S is invertible on H. Our problem is how to extend the mapping properties of S on Banach spaces H^p_ν. For this purpose we take $f = \sum_{j,k} f_{j,k}\phi_{j,k}$ such that

$$\begin{aligned} Sf &= \sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n} = \sum_{m,n\in\chi} \sum_{j,k\in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} > \varphi_{m,n} \\ &= \sum_{m,n\in\chi} \sum_{i,l\in N} \sum_{j,k\in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle \phi_{i,l} \\ &= \sum_{i,l} \left(\sum_{j,k} \left(\sum_{m,n} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle \right) f_{j,k} \right) \phi_{i,l}. \end{aligned}$$

Now let $A = a_{iljk}$ be infinite matrix defined as

$$a_{iljk} = \sum_{m,n\in\chi} < \phi_{j,k}, \varphi_{m,n} > \left\langle \varphi_{m,n}, \widetilde{\phi}_{i,l} \right\rangle = \left\langle S\phi_{j,k}, \widetilde{\phi}_{i,l} \right\rangle. \tag{2.1}$$

Define a mapping Γ such that $\Gamma : H \to l^2(N), (\Gamma f)_{j,k} = \left\langle f, \widetilde{\phi}_{j,k} \right\rangle$.

Since $\{\phi_{j,k}\}$ is a Riesz basis, Γ is invertible and an isometric isomorphism between H^p_ν and $l^p_\nu(N)$. Therefore, $S = \Gamma^{-1}A\Gamma$ carries over to the Banach spaces H^p_ν. To study the behavior of frame operator S on H^p_ν, it is sufficient to study the infinite matrix A on sequence space $l^p_\nu(N)$. For this purpose we will use the following fundamental theorem of Jaffard [14].

Theorem A. Assume that the matrix $G = (G_{k,l})_{k,l\in N}$ satisfies the following properties:

(a) G is invertible as an operator on $l^2(N)$, and
(b) $|G_{kl}| \le C(1 + |k - l|)^{-l}, k, l \in N$ for some constant $C > 0$ and some $r > d$. Then the inverse matrix $H = G^{-1}$ satisfies the same off-diagonal decay, that is

$$|H_{kl}| \le C'(1 + |k - l|)^{-r}, k, l \in N.$$

Using above theorem we can prove:

Theorem 2.1. *Assume that the matrix $A = (a_{iljk})_{i,l,j,k\in N}$ satisfies the following properties:*

(a) A is invertible as an operator on $l^2(N)$, and
(b) $|a_{iljk}| \le C[(1 + |i - j|)(1 + |l - k|)]^{-r}, i, l, j, k \in N$ for some constant $C > 0$ and some $r > d$.

Then the inverse matrix $T = A^{-1}$ satisfies the same off-diagonal decay, i.e.,

$$\left|T_{iljk}\right| \le C'[(1 + |i - j|)(1 + |l - k|)]^{-r}, i, l, j, k \in N.$$

Definition 2.2. The frame $\mathcal{F} = \{\varphi_{m,n} : m, n \in \chi\}$ is said to be polynomially localized with respect to Riesz basis $\{\phi_{j,k}\}$ with decay $s > 0$ (or simply s−localized), if

$$| < \varphi_{m,n}, \phi_{j,k} > | \le C[(1 + |m - j|)(1 + |n - k|)]^{-s}$$

and

$$| < \varphi_{m,n}, \widetilde{\phi}_{j,k} > | \le C[(1 + |m - j|)(1 + |n - k|)]^{-s} \forall i, k \in N \text{ and } m, n \in \chi.$$

Now we prove:

Proposition 2.1. *Let $\mathcal{F} = (\varphi_{m,n} : m, n \in \chi)$ is an $(s + d + \varepsilon)$−localized frame for $\varepsilon > 0, r \ge 0$ and $1 \le p \le \infty$.Then*

(i) the analysis operator defined by $C_\varepsilon f = (< f, \varphi_{m,n} >)_{m,n\in\chi}$ is bounded from H^p_ν to $l^p_\nu(\chi)$.
(ii) the synthesis operator defined on finite sequences by $D_\varepsilon c = \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}$ extends to a bounded mapping from $l^p_\nu(\chi)$ to H^p_ν.

(iii) the frame operator $S = S_\varepsilon = D_\varepsilon C_\varepsilon = \sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n}$ *maps* H_ν^p *into* H_ν^p, *and the series converges unconditionally for* $1 \le p \le \infty$.

Proof. (i) Assume that $f = \sum_{j,k\in N} f_{j,k}\phi_{j,k}$, $| < f, \varphi_{m,n} > | = \left|\sum_{j,k\in N} f_{j,k} < \phi_{j,k}, \varphi_{m,n} >\right|$. In view of Definition 2.4, we get $\le C\sum_{j,k\in N}\left|f_{j,k}\right|[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon} \le CC'\sum_{j,k\in N}|f_{j,k}|(1+|j-k|)^{-s-d-\varepsilon}$.

If $f \in H_\nu^p$, then $\|f\|_{H_\nu^p} = \left\|(f_{j,k})_{j,k\in N}\right\|_{l_\nu^p(N)}$ and Lemma 2.1 gives that $\|C_\varepsilon f\|_{l_\nu^p(\chi)} \le CC'\left\|(f_{j,k})_{j,k\in N}\right\|_{l_\nu^p(N)} = CC'\|f\|_{H_\nu^p}$.

(ii) Now we have $(D_\varepsilon c)_{j,k\in N} = \left\langle \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}, \widetilde{\phi}_{j,k}\right\rangle$ or

$$
\begin{aligned}
|(D_\varepsilon c)_{j,k\in N}| &\le \sum_{m,n\in\chi} |c_{m,n}|\left|\left\langle\varphi_{m,n}, \widetilde{\phi}_{j,k}\right\rangle\right| \le C\sum_{m,n\in\chi}|c_{m,n}|[(1+|m-j|)(1+|n-k|)]^{-s-d-\varepsilon}\\
&\le CC'\sum_{m,n\in\chi}|c_{m,n}|(1+|j-k|)^{-s-d-e} = CC'(A^*|c|)_{j,k}.
\end{aligned}
$$

Now Lemma 2.1 (by interchanging N and χ) gives $\|D_\varepsilon c\|_{H_\nu^p} = \|A^*|c|\|_{l_\nu^p(N)} \le \|A^*\|_{op}\|c\|_{l_\nu^p(\chi)}$.

(iii) The boundlessness of frame operator S follows by combining (1) and (ii). For unconditional convergence of the series defining S, let $\varepsilon > 0$, choose $N_0 = N_0(\varepsilon)$, such that $\| < f, \varphi_{m,n} >_{m,n(not\in N_0)} \|_{l_\nu^p} \le \varepsilon$. Then for any finite set $N_1 \supseteq N_0$, from (i) and (ii), we obtain

$$
\left\|Sf - \sum_{m,n\in N} < f, \varphi_{m,n} > \varphi_{m,n}\right\|_{H_\nu^p} \le \|C_\varepsilon\|_{op}\| < f, \varphi_{m,n} > \| \le \|C_\varepsilon\|_{op}.\varepsilon.
$$

Which implies that the series $\sum_{m,n\in\chi} < f, \varphi_{m,n} > \varphi_{m,n}$ converges unconditionally in H_ν^p. □

Proposition 2.2. *Assume that* $\mathcal{F} = \{\varphi_{m,n} : m, n \in \chi\}$ *is polynomially localized with respect to the Riesz basis* $\{\phi_{j,k}\}$ *with decay* $s > d$. *Then*

$$
|A| = |a_{iljk}| \le C(1+|j-k|)^{-s} \;\; for \;\; i, l, j, k \in N.
$$

Proposition 2.3. *From (2.1) we get*

$$
\begin{aligned}
|a_{iljk}| &\le C\sum_{m,n\in\chi}[(1+|m-j|)(1+|n-k|)(1+|i-m|)(1+|l-n|)]^{-s}\\
&\le CC'\sum_{i,l\in N}[(1+|i-j|)(1+|l-k|)]^{-s} \le CC'C''(1+|j-k|)^{-s}.
\end{aligned}
$$

3. Main Results

The following definition is due to Moricz and Rhoades (Moricz & Rhoades, 1989).

Definition 3.1. Let $A = (a_{iljk})$ be a double non-negative infinite matrix of real numbers. Then, $A-$transform of a double sequence $x = \{x_{j,k}\}$ is $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{mnjk}x_{j,k}$ which is called $A-$means or $A-$transform of the sequence $x = \{x_{j,k}\}$.

Sheikh and Mursaleen (Sheikh & Mursaleen, 2004) study the frame condition by using the action of frame operator A on non-negative infinite matrix in Hilbert space. In this paper our aim is to extend these results on weighted Banach space in R^d.

Now we prove our main results:

Theorem 3.1. *Let $A = (a_{iljk})$ be a double non-negative infinite matrix. If $f(x) = \sum_{m,n\in\chi} c_{m,n}\varphi_{m,n}(x)$ is a wavelet expansion of $f \in H^p_\nu$ with wavelet coefficients $c_{m,n} =< f, \varphi_{m,n} >$, then the frame condition for $A-$transform of $f \in H^p_\nu$ is*

$$c_1\|f\|_{H^p_\nu} \leq \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \leq c_2\|f\|_{H^p_\nu}$$

where $\{\varphi_{m,n} : m, n \in \chi\}$ is an $(s + d + \varepsilon)-$localized frame for $\varepsilon > 0$, $s \geq 0$ and $1 \leq p \leq \infty$.

Proof. We take $f = \sum_{j,k\in N} f_{j,k}\phi_{j,k}$, then

$$\sum_{m,n\in\chi} | < Af, \varphi_{m,n} > | \leq \left| \sum_{j,k\in N} \sum_{m,n\in\chi} Af_{j,k} < \phi_{j,k}, \varphi_{m,n} > \right| \leq \sum_{j,k\in N} |Af_{jk}|| < \phi_{j,k}, \varphi_{m,n} > |$$
$$\leq c \sum_{j,k\in N} |Af_{j,k}|((1 + |m - j|)(1 + |n - k|))^{-s-d-\varepsilon} \leq CC' \sum_{j,k\in N} |Af_{j,k}|(1 + |j - k|)^{-s-d-\varepsilon}.$$

If $f \in H^p_\nu$, then $\|f\|_{H^p_\nu} = \|(f_{j,k})_{j,k\in N}\|_{l^p_\nu}$. Hence we get

$$\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \leq CC'\|A\|_{op}\|f\|_{H^p_\nu} \leq c_2\|f\|_{H^p_\nu}.$$

Now, for any $f \in H^p_\nu$, define

$$\widetilde{f} \;=\; \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|^{-1}_{l^p_\nu} f < A\widetilde{f}, \varphi_{m,n} >= \left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|^{-1}_{l^p_\nu} < Af, \varphi_{m,n} >$$

then

$$\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \leq 1.$$

Hence, if there exists a positive constant α, such that

$$\|Ac_{m,n}\|_{l^p_\nu} \leq \alpha \left[\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \right]^{-1} \|Ac_{m,n}\|_{l^p_\nu} \leq \alpha \left[\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \right]^{-1} \|f\|_{H^p_\nu} \leq \left(\frac{\alpha}{\|A\|_{op}} \right)$$

it follows that $\left[\left\| \sum_{m,n\in\chi} < Af, \varphi_{m,n} > \right\|_{l^p_\nu} \right] \geq c_1\|f\|_{H^p_\nu}$.

Hence the proof is completed. □

Theorem 3.2. *If* $f = \sum_{j,k \in N} c_{j,k} \phi_{j,k}$ *and* $\{\varphi_{m,n} : m, n \in \chi\}$ *forms a frame with respect to Riesz basis* $\{\phi_{j,k}\}$*, then the* $\alpha_{j,k}$ *are the wavelet coefficients of* Af*, where* $\{d_{i,l}\}$ *is defined as the* $A-$*transform of* $\{c_{j,k}\}$ *such that*

$$\begin{aligned} d_{i,l} &= \sum_{j,k \in N} a_{iljk} c_{j,k}, \\ \alpha_{j,k} &= \sum_{i,l \in \chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > . \end{aligned}$$

Proof. Using the definition of $A-$transform of $f = \sum_{il \in \chi} c_{i,l} \varphi_{i,l}$ by assumption we get

$$< Af, \varphi_{i,l} >= \sum_{j,k \in N} a_{iljk} c_{j,k} < \phi_{j,k}, \varphi_{i,l} >$$

or

$$\sum_{i,l \in \chi} < Af, \varphi_{i,l} >= \sum_{i,l \in \chi} (Ac)_{i,l} < \phi_{j,k}, \varphi_{i,l} >= \sum_{i,l \in \chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > .$$

Therefore, the wavelet coefficients of Af with respect to Riesz basis $\{\phi_{j,k}\}$ are given by

$$\alpha_{j,k} = \sum_{i,l \in \chi} d_{i,l} < \phi_{j,k}, \varphi_{i,l} > .$$

Hence the proof is completed. □

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