Theory and Applications of
Mathematics & Computer Science

ISSN 2067-2764, e-ISSN 2247-6202
https://www.uav.ro/jour/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 10 (1) (2025) 32–36

A Study of Kalecki's Model of Business Cycle Using Weakly Picard Operators Technique

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Abstract

Kalecki's 1935 work introduced the first precise macro-dynamic model and emphasized the implementation lag between investment decisions and productive capacity. This paper aims to establish conditions for the models solution to exist and its continuous data dependence.

Keywords: Integral equations, Picard operators, fixed points, data dependence, Gronwall lemma.

2020 MSC: 45G10, 47H10, 45D05.

1. Introduction

1.1. Weakly Picard operators

I.A. Rus initiated and developed in Rus (2001) the theory of weakly Picard operators with applications in the study of existence and data dependence of fixed point of different operators.

Let us consider (X, d) a metric space and $A : X \to X$ an operator. Next we shall use the following notations:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\},$$

$$F_A := \{x \in X \mid A(x) = x\},$$

$$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\},$$

$$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in \mathbb{N}.$$

Definition 1.1. Rus (2001) The operator A is said to be weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit is a fixed point of A.

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Definition 1.2. Rus (2001) If A is an weakly Picard operator and $F_A = \{x^*\}$ then A is a Picard operator (briefly PO).

We have the following characterization of the WPOs.

Theorem 1.1. *Rus (2001) Let us consider (X, d) a metric space and $A : X \to X$ an operator. Then A is WPO if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

such that :

(a) $X_\lambda \in I(A)$

(b) $A \mid X_\lambda : X_\lambda \to X_\lambda$ *is PO for all* $\lambda \in \Lambda$.

1.2. Formulation of Kalecki's model

Kalecki's model, see Kalecki (1935), highlights that productive capacity cannot be created instantaneously: investment projects require a fixed implementation lag, or gestation period, denoted by $\theta > 0$. Let $K(t)$ be the capital stock and $I(t-\theta)$ the net investment decided at time $t-\theta$. Capital accumulation is therefore

$$\dot{K}(t) = I(t-\theta). \tag{1.1}$$

Replacement of depreciated capital also experiences the same lag and is represented by a constant $U > 0$. Assuming continuous market clearing with no inventories, government, or international trade, consumption depends on a constant saving propensity $s \in (0, 1)$. Investment decisions are modeled as a linear function of output and capital:

$$I(t) = a \cdot Y(t) - b \cdot K(t), \tag{1.2}$$

where output is given by

$$Y(t) = \frac{1}{s} \cdot U + \frac{1}{s \cdot \theta}[K(t+\theta) - K(t)]. \tag{1.3}$$

Substituting these relationships yields a mixed differential–difference equation in the single variable $K(t)$:

$$\dot{K}(t) = \frac{a}{s} \cdot U + \frac{a}{s \cdot \theta}[K(t) - K(t-\theta)] - b \cdot K(t-\theta). \tag{1.4}$$

A solution $K(t)$ to (1.4) guarantees the existence of solutions to (1.2) and (1.3), thus fully determining the model. The next step is to identify the parameter conditions under which (1.4) admits at least one continuous solution. The model was studied from fixed point point of view in many papers. In Olaru *et al.* (2009) there was proved an results of existence and uniqueness for Cauchy problem associated to the above model by using a Bielecki norm on class of continuous functions defined on $[-\theta, T]$. Further in Olaru (2025) there was studied the existence and uniqueness in regards Chebyshev norm.

2. Existence result

Our proposal on current section is to study the above model by using weakly Picard technique. More exactly based on characterization of weakly Picard operators we prove that the Kalecki model has at least a solution. Further our study will be done on the class of continuous functions $K : [-\theta, T] \to \mathbb{R}$ denoted by $C([-\theta, T], \mathbb{R})$ endowed with Chebyshev norm defined by

$$\|K\|_\infty = \sup_{t\in[-\theta,T]} |K(t)|.$$

Let us consider the following partition of $C([-\theta, T], \mathbb{R})$

$$C([-\theta, T], \mathbb{R}) = \bigcup_{\varphi\in C([-\theta,0])} X_\varphi$$

where

$$X_\varphi = \{x \in C([-\theta, T]) \mid x(t) = \varphi(t), (\forall)t \in [-\theta, 0]\}.$$

Then (1.4) is equivalent with

$$K(t) = \begin{cases} K(0) + \int_0^t [\frac{a}{s} \cdot U + \frac{a}{s\cdot\theta}[K(u) - K(u-\theta)] - b \cdot K(u-\theta)]du & , \quad t \in [0, T] \\ K(t) & , \quad t \in [-\theta, 0] \end{cases} \tag{2.1}$$

Therefore we reduced the existence of solution for (1.4) to a fixed point problem for the operator $A : C[-\theta, T] \to C[-\theta, T]$ defined by:

$$A(K)(t) = \begin{cases} K(0) + \int_0^t [\frac{a}{s} \cdot U + \frac{a}{s\cdot\theta}[K(u) - K(u-\theta)] - b \cdot K(u-\theta)]du & , \quad t \in [0, T] \\ K(t) & , \quad t \in [-\theta, 0] \end{cases}$$

Thus we got the following result for model's solution existence

Theorem 2.1. *The Kalecki model (1.4) has at least a solution $K \in C[-\theta, T]$ which can be approximated by the sequence $\{A^n(K_0)\}_{n\in\mathbb{N}}$, $K_0 \in C[-\theta, T]$ being arbitrarily chosen.*

Proof. First of all we remark that $X_\varphi \in I(A)$. On the other side we claim that $A \mid X_\varphi$ is a Picard operator. Indeed, let us consider $K_1, K_2 \in C[-\theta, T]$. Then

$$|A(K_1)(t) - A(K_2)(t)| \le \int_0^t [\frac{a}{s\cdot\theta} \cdot |K_1(u) - K_2(u)| + (\frac{a}{s\cdot\theta} - b)|K_1(u-\theta) - K_2(u-\theta)|]du \le$$

$$\le (2 \cdot \frac{a}{s\cdot\theta} + b) \cdot t \cdot \|K_1 - K_2\|_\infty.$$

By using induction arguments we get that for any iteration A^k we have

$$|A^n(K_1)(t) - A^n(K_2)(t)| \le$$

$$\int_0^t [\frac{a}{s\cdot\theta} \cdot |A^{n-1}(K_1)(u) - A^{n-1}(K_2)(u)| + (\frac{a}{s\cdot\theta} - b)|A^{n-1}(K_1)(u-\theta) - A^{n-1}(K_2)(u-\theta)|]du \le$$

$$\leq (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot \frac{t^n}{n!} \cdot \|K_1 - K_2\|_\infty \leq (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot \frac{T^n}{n!} \cdot \|K_1 - K_2\|_\infty.$$

Consequently for $n \geq$ we have

$$\|A^n(K_1) - A^n(K_2)\|_\infty \leq (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot \frac{T^n}{n!} \cdot \|K_1 - K_2\|_\infty$$

and from here we get that there exists $N \in \mathbb{N}$ such that A^N is a contraction. Therefore $A \mid X_\varphi$ is a Picard operator and now the conclusion follows from Theorem 1.1, □

3. Data dependence: continuity with respect to data

Further let us consider the equation (1.4) which satisfies the initial Cauchy conditions

$$K(t) = \varphi_1(t), t \in [-\theta, 0]. \tag{3.1}$$

$$K(t) = \varphi_2(t), t \in [-\theta, 0]. \tag{3.2}$$

Then we have the following data dependence result:

Theorem 3.1. *(a) There exists $K(\cdot, \varphi_1), K(\cdot, \varphi_2) \in C[-\theta, T]$ unique solutions for (1.4) + (3.1) respectively (1.4) + (3.2).*

(b) if there exists $\eta > 0$ such that

$$|\varphi_1(t) - \varphi_2(t), (\forall)t \in [-\theta, 0]$$

then

$$\|K(\cdot, \varphi_1) - K(\cdot, \varphi_2)\|_\infty \leq \eta \cdot (1 + \int_A (\frac{a}{s \cdot \theta} + b)du) \cdot exp(\int_{[0,t]\setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b)du.$$

Proof. (a) Let us consider the operator $A_i : C[-\theta, T] \to C[-\theta, T]$, $i = \overline{1, 2}$ defined by:

$$A_i(K)(t) = \begin{cases} \varphi_i(0) + \int_0^t [\frac{a}{s} \cdot U + \frac{a}{s \cdot \theta}[K(u) - K(u - \theta)] - b \cdot K(u - \theta)]du & , \quad t \in [0, T] \\ \varphi_i(t) & , \quad t \in [-\theta, 0] \end{cases}$$

By using the same approach like in the proof of Theorem 2.1 we get that A_1, A_2 are Picard operators and thus they have the unique fixed points $K(\cdot, \varphi_1)$ respectively $K(\cdot, \varphi_2)$.

(b) Let us consider $x : [-\theta, T] \to (0, \infty)$ defined by $x(v) =: |K(v, \varphi_1) - K(v, \varphi_2)|$. Then

$$x(t) \leq |\varphi_1(0) - \varphi_2(0)\| + \int_0^t [\frac{a}{s \cdot \theta} x(u) + (\frac{a}{s \cdot \theta} + b) \cdot x(u - \theta)]du, (\forall)t \in [0, T]$$

and

$$x(t) = |\varphi_1(t) - \varphi_2(t)|, (\forall)t \in [-\theta, 0]$$

Further, for each $v \in [0, T]$ let us denote

$$y(v) = |\varphi_1(0) - \varphi_2(0)| + \int_0^v [\frac{a}{s \cdot \theta} x(u) + (\frac{a}{s \cdot \theta} + b) \cdot x(u - \theta)] du.$$

From here we get that

$$y'(v) = \frac{a}{s \cdot \theta} x(v) + (\frac{a}{s \cdot \theta} + b) \cdot x(v - \theta)$$

$$\leq \frac{a}{s \cdot \theta} y(v) + (\frac{a}{s \cdot \theta} + b) \cdot \begin{cases} y(v - \theta) & , \quad v - \theta \geq 0 \\ |\varphi_1(v - \theta) - \varphi_2(v - \theta)| & , \quad v - \theta < 0 \end{cases}$$

$$\leq \frac{a}{s \cdot \theta} y(v) + (\frac{a}{s \cdot \theta} + b) \cdot \begin{cases} y(v) & , \quad v - \theta \geq 0 \\ |\varphi_1(v - \theta) - \varphi_2(v - \theta)| & , \quad v - \theta < 0 \end{cases}$$

By integrating on $[0, t]$ and considering $A := \{t \in [0, T] \mid t - \theta < 0\}$ we get that

$$y(t) \leq$$

$$|\varphi_1(0) - \varphi_2(0)| + \int_A (\frac{a}{s \cdot \theta} + b) \cdot |\varphi_1(u - \theta) - \varphi_2(u - \theta)| du + \int_{[0,T] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot y(u) du \leq$$

$$\eta \cdot (1 + \int_A (\frac{a}{s \cdot \theta} + b) du) + \int_{[0,t] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) \cdot y(u) du.$$

Now, by applying Gronwall lemma, we get that for all $t \in [0, T]$

$$x(t) \leq y(t) \leq \eta \cdot (1 + \int_A (\frac{a}{s \cdot \theta} + b) du) \cdot exp(\int_{[0,t] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) du)$$

and thus we have

$$||K(, \varphi_1) - K(t, \varphi_2)||_\infty \leq \eta \cdot (1 + \int_A (\frac{a}{s \cdot \theta} + b) du) \cdot exp(\int_{[0,t] \setminus A} (2 \cdot \frac{a}{s \cdot \theta} + b) du).$$

$\square$

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