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On Some Nonuniform Dichotomic Behaviors of Discrete Skew-product Semiflows

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Abstract

In this paper we approach concepts of nonuniform dichotomy for the case of discrete skew-product semiflows. Different characterizations of this properties are given from the point of view of invariant and strongly invariant projector families.

Keywords: discrete skew-product semiflow, nonuniform dichotomy, nonuniform exponential dichotomy.
2010 MSC: 34D05, 34D09.

1. Introduction

The (exponential) dichotomy is one of the most representative asymptotic properties studied for discrete dynamical systems in (Alonso *et al.*, 1999), (Babuţia & Megan, 2016), (Crai, 2016), (Elaydi & Janglajew, 1998), (Popa *et al.*, 2012), (Sasu & Sasu, 2013) from various perspectives.

In (Sasu, 2009) is approached the uniform exponential dichotomy for discrete skew-product flows and in (Biriş *et al.*, 2019) the authors investigate a generalization of the uniform exponential dichotomy property (the uniform exponential splitting) for discrete skew-product semiflows. Other significant results for the dichotomic behaviors of skew-product semiflows are obtained in (Biriş & Megan, 2016), (Chow & Leiva, 1996) and (Huy & Phi, 2010).

Regarding the nonuniform dichotomies, M. Megan, B. Sasu and A. L. Sasu ((Megan *et al.*, 2002)) prove interesting results for the nonuniform exponential dichotomy of evolution operators, using admissibility techniques. Also, different concepts of nonuniform exponential dichotomy and nonuniform polynomial dichotomy are studied in (Megan & Stoica, 2010) and (Stoica, 2016).

In this article, the properties of nonuniform dichotomy and nonuniform exponential dichotomy are treated for discrete variational systems, described through discrete skew-product semiflows. We prove criteria for the nonuniform exponential dichotomy, based on some results from (Przyluski & Rolewicz, 1984) and in particular we illustrate the characterizations for the nonuniform dichotomy.

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2. Preliminaries

In the following, we denote by Θ a metric space, by X a Banach space and by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. Let I be the identity operator on X and $\Gamma = \Theta \times X$.

Definition 2.1. A mapping $S : \mathbb{N} \times \Theta \to \Theta$ is called *discrete semiflow* on Θ, if:

(ds_1) $S(0, \theta) = \theta$, for all $\theta \in \Theta$;

(ds_2) $S(m, S(n, \theta)) = S(m + n, \theta)$, for all $(m, n, \theta) \in \mathbb{N}^2 \times \Theta$.

Example 1. We consider $\Theta = \mathbb{N}$ and $S : \mathbb{N} \times \Theta \to \Theta$, $S(n, \theta) = n + \theta$. It is immediate to see that S is a discrete semiflow on Θ.

Definition 2.2. We say that $C : \mathbb{N} \times \Theta \to \mathcal{B}(X)$ is *discrete cocycle* over the discrete semiflow $S : \mathbb{N} \times \Theta \to \Theta$ if:

(dc_1) $C(0, \theta) = I$, for all $\theta \in \Theta$;

(dc_2) $C(m, S(n, \theta))C(n, \theta) = C(m + n, \theta)$, for all $(m, n, \theta) \in \mathbb{N}^2 \times \Theta$.

Example 2. Let $U : \{(m, n) \in \mathbb{N}^2 : m \geq n\} \to \mathcal{B}(X)$ be a discrete evolution operator on the Banach space X and $\Theta = \mathbb{N}$. Then $C_U : \mathbb{N} \times \Theta \to \mathcal{B}(X)$, given by

$$C_U(n, \theta) = U(n + \theta, \theta), \quad \text{for all} \quad (n, \theta) \in \mathbb{N} \times \Theta$$

is a discrete cocycle over the discrete semiflow considered in Example 1.

Definition 2.3. The mapping $\pi : \mathbb{N} \times \Gamma \to \Gamma$, given by

$$\pi(n, \theta, x) = (S(n, \theta), C(n, \theta)x),$$

where C is a discrete cocycle over a discrete semiflow S, is called *discrete skew-product semiflow on* Γ.

Definition 2.4. A mapping $P : \Theta \to \mathcal{B}(X)$ is said to be *family of projectors* if:

$$P^2(\theta) = P(\theta), \text{ for all } \theta \in \Theta.$$

If $P : \Theta \to \mathcal{B}(X)$ is a family of projectors, then $Q : \Theta \to \mathcal{B}(X)$, defined by $Q(\theta) = I - P(\theta)$ represents the *complementary family of projectors of* P.

Definition 2.5. A family of projectors $P : \Theta \to \mathcal{B}(X)$ is called

- *invariant* for a discrete skew-product semiflow $\pi = (S, C)$ if:

$$P(S(n, \theta))C(n, \theta) = C(n, \theta)P(\theta), \quad \text{for all } (n, \theta) \in \mathbb{N} \times \Theta;$$

- *strongly invariant* for a discrete skew-product semiflow $\pi = (S, C)$ if it is invariant for π and for all $(n, \theta) \in \mathbb{N} \times \Theta$, the restriction $C(n, \theta)$ is an isomorphism from $Ker\ P(\theta)$ to $Ker\ P(S(n, \theta))$.

Remark 1. If $P : \Theta \to \mathcal{B}(X)$ is a strongly invariant family of projectors for $\pi = (S, C)$, then there exists the mapping $D : \mathbb{N} \times \Theta \to \mathcal{B}(X)$ such that for all $(n, \theta) \in \mathbb{N} \times \Theta$ the bounded linear operator $D(n, \theta)$ is an isomorphism from $Ker\ P(S(n, \theta))$ to $Ker\ P(\theta)$ and

(i) $C(n, \theta)D(n, \theta)Q(S(n, \theta)) = Q(S(n, \theta))$;

(ii) $D(n, \theta)C(n, \theta)Q(\theta) = Q(\theta)$;

(iii) $Q(\theta)D(n, \theta)Q(S(n, \theta)) = D(n, \theta)Q(S(n, \theta))$,

for all $(n, \theta) \in \mathbb{N} \times \Theta$.

3. Nonuniform dichotomic behaviors of discrete skew-product semiflows

Let $\pi = (S, C)$ be a discrete skew-product semiflow and $P : \Theta \to \mathcal{B}(X)$ an invariant family of projectors for π.

Definition 3.1. The pair (π, P) is called *nonuniformly dichotomic* if there exists a mapping $N : \Theta \to \mathbb{R}_+^*$ such that:

(nd_1) $||C(n, \theta)P(\theta)x|| \leq N(\theta)||P(\theta)x||$;

(nd_2) $||Q(\theta)x|| \leq N(\theta)||C(n, \theta)Q(\theta)x||$,

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$.

In particular, if N is a constant function, then (π, P) is named *uniformly dichotomic*.

Remark 2. The pair (π, P) admits a nonuniform dichotomy if and only if there exists $N : \Theta \to \mathbb{R}_+^*$ with:

(nd_1') $||C(m + n, \theta)P(\theta)x|| \leq N(\theta)||C(n, \theta)P(\theta)x||$;

(nd_2') $||C(n, \theta)Q(\theta)x|| \leq N(\theta)||C(m + n, \theta)Q(\theta)x||$,

for all $(m, n, \theta, x) \in \mathbb{N}^2 \times \Gamma$.

Definition 3.2. We say that (π, P) is *nonuniformly exponentially dichotomic* if there exist two functions $N, \nu : \Theta \to \mathbb{R}_+^*$ such that:

(ned_1) $||C(n, \theta)P(\theta)x|| \leq N(\theta)e^{-\nu(\theta)n}||P(\theta)x||$;

(ned_2) $e^{\nu(\theta)n}||Q(\theta)x|| \leq N(\theta)||C(n, \theta)Q(\theta)x||$,

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$.

Remark 3. We observe that, if

- ν is a constant function, then we have the concept of *nonuniform exponential dichotomy in the classical sense*;
- N and ν are constant functions, then obtain the property of *uniform exponential dichotomy*.

Remark 4. The pair (π, P) has a nonuniform exponential dichotomy if and only if there exist $N, \nu : \Theta \to \mathbb{R}_+^*$ with:

(ned_1') $||C(m + n, \theta)P(\theta)x|| \leq N(\theta)e^{-\nu(\theta)m}||C(n, \theta)P(\theta)x||$;

(ned_2') $e^{\nu(\theta)m}||C(n, \theta)Q(\theta)x|| \leq N(\theta)||C(m + n, \theta)Q(\theta)x||$,

for all $(m, n, \theta, x) \in \mathbb{N}^2 \times \Gamma$.

Remark 5. If the pair (π, P) admits nonuniform exponential dichotomy, then (π, P) has nonuniform dichotomy.

Theorem 3.1. *The pair* (π, P) *is nonuniformly exponentially dichotomic if and only if there exist the functions* $\delta,\ \Delta : \Theta \to \mathbb{R}_+^*$ *such that the following conditions hold:*

$$(dned_1)\quad \sum_{k=n}^{+\infty} e^{\delta(\theta)k}\|C(k,\theta)P(\theta)x\| \leq \Delta(\theta)\|P(\theta)x\|$$

$$(dned_2)\quad \sum_{k=0}^{n} e^{\delta(S(n,\theta))k}\|C(n-k, S(k,\theta))Q(S(k,\theta))x\| \leq \Delta(S(n,\theta))\|C(n, S(n,\theta))Q(S(n,\theta))x\|,$$

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$.

Proof. Necessity. We consider $\delta, \Delta : \Theta \to \mathbb{R}_+^*$, with $\delta(\theta) < \nu(\theta)$ and $\Delta(\theta) = \dfrac{N(\theta)}{1 - e^{\delta(\theta)-\nu(\theta)}}$, for all $\theta \in \Theta$. Thus, for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$ we have:

$(dned_1)$

$$\sum_{k=n}^{+\infty} e^{\delta(\theta)k}\|C(k,\theta)P(\theta)x\| \leq N(\theta)\sum_{k=0}^{+\infty} e^{\delta(\theta)k}e^{-\nu(\theta)k}\|P(\theta)x\| =$$

$$= N(\theta)\cdot\frac{1}{1 - e^{\delta(\theta)-\nu(\theta)}}\|P(\theta)x\| = \Delta(\theta)\|P(\theta)x\|;$$

$(dned_2)$

$$\sum_{k=0}^{n} e^{\delta(S(n,\theta))k}\|Q(S(n,\theta))C(n-k, S(k,\theta))x\| \leq$$

$$\leq N(S(n,\theta))\sum_{k=0}^{n} e^{(\delta(S(n,\theta))-\nu(S(n,\theta)))k}\|C(k, S(n,\theta))Q(S(n,\theta))C(n-k, S(k,\theta))x\| =$$

$$= N(S(n,\theta))\sum_{k=0}^{n} e^{(\delta(S(n,\theta))-\nu(S(n,\theta)))k}\|C(n, S(n,\theta))Q(S(n,\theta))x\| =$$

$$= N(S(n,\theta))\frac{1 - e^{(\delta(S(n,\theta))-\nu(S(n,\theta)))(n+1)}}{1 - e^{\delta(S(n,\theta))-\nu(S(n,\theta))}}\|C(n, S(n,\theta))Q(S(n,\theta))x\| \leq$$

$$\leq \Delta(S(n,\theta))\|C(n, S(n,\theta))Q(S(n,\theta))x\|.$$

Sufficiency. Considering $k = n$ in the relations $(dned_1)$, respectively $(dned_2)$, it follows that

$$e^{\delta(\theta)n}\|C(n,\theta)P(\theta)x\| \leq \Delta(\theta)\|P(\theta)x\|,$$

respectively

$$e^{\delta(S(n,\theta))n}\|Q(S(n,\theta))x\| \leq \Delta(S(n,\theta))\|C(n, S(n,\theta))Q(S(n,\theta))x\|,$$

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$, which implies that (π, P) has a nonuniform exponential dichotomy. □

Corollary 3.2. *The pair* (π, P) *is nonuniformly dichotomic if and only if there are* $\delta,\ \Delta : \Theta \to \mathbb{R}_+^*$ *such that the conditions* $(dned_1)$ *and* $(dned_2)$ *from Theorem 3.1 are verified.*

Proof. It yields from Theorem 3.1 and Remark 5. □

Proposition 1. *Let $P : \Theta \to \mathcal{B}(X)$ be a strongly invariant family of projectors for $\pi = (S, C)$. Then (π, P) is nonuniformly exponentially dichotomic if and only if there are two functions $N, \nu : \Theta \to \mathbb{R}_+^*$ such that:*

(ned_1) $\|C(n,\theta)P(\theta)x\| \le N(\theta)e^{-\nu(\theta)n}\|P(\theta)x\|$;

(ned_2') $\|D(n,\theta)Q(S(n,\theta))x\| \le N(\theta)e^{-\nu(\theta)n}\|Q(S(n,\theta))x\|$,

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$.

Proof. We show that (ned_2') is equivalent with (ned_2), using the relations from Remark 1.
For the implication $(ned_2') \Rightarrow (ned_2)$, we have

$$e^{\nu(\theta)n}\|Q(\theta)x\| = e^{\nu(\theta)n}\|D(n,\theta)C(n,\theta)Q(\theta)x\| =$$

$$= e^{\nu(\theta)n}\|D(n,\theta)Q(S(n,\theta))C(n,\theta)x\| \le N(\theta)\|Q(S(n,\theta))C(n,\theta)x\| = N(\theta)\|C(n,\theta)Q(\theta)x\|,$$

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$.
Similarly, for the converse implication $(ned_2) \Rightarrow (ned_2')$, we deduce

$$\|D(n,\theta)Q(S(n,\theta))x\| = \|Q(\theta)D(n,\theta)Q(S(n,\theta))x\| \le$$

$$\le N(\theta)e^{-\nu(\theta)n}\|C(n,\theta)Q(\theta)D(n,\theta)Q(S(n,\theta))x\| = N(\theta)e^{-\nu(\theta)n}\|Q(S(n,\theta))x\|,$$

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$. □

Proposition 2. *Let $P : \Theta \to \mathcal{B}(X)$ be a strongly invariant family of projectors for $\pi = (S, C)$. Then (π, P) admits nonuniform dichotomy if and only if there exists $N : \Theta \to \mathbb{R}_+^*$ such that:*

(nd_1) $\|C(n,\theta)P(\theta)x\| \le N(\theta)\|P(\theta)x\|$;

(nd_2') $\|D(n,\theta)Q(S(n,\theta))x\| \le N(\theta)\|Q(S(n,\theta))x\|$,

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$.

Proof. It is a consequence of Proposition 1. □

Theorem 3.3. *Let $P : \Theta \to \mathcal{B}(X)$ be a strongly invariant family of projectors for $\pi = (S, C)$. The pair (π, P) is nonuniformly exponentially dichotomic if and only if there exist the functions $\delta,\ \Delta : \Theta \to \mathbb{R}_+^*$ such that the following conditions are satisfied:*

$(dned_1)$ $$\sum_{k=n}^{+\infty} e^{\delta(\theta)k}\|C(k,\theta)P(\theta)x\| \le \Delta(\theta)\|P(\theta)x\|$$

$(dned_2')$ $$\sum_{k=0}^{n} e^{\delta(\theta)(n-k)}\|D(n-k, S(k,\theta))Q(S(n,\theta))x\| \le \Delta(\theta)\|Q(S(n,\theta))x\|,$$

for all $(n, \theta, x) \in \mathbb{N} \times \Gamma$.

Proof. Necessity. We consider $\delta, \Delta : \Theta \to \mathbb{R}_+^*$, with $\delta(\theta) < \nu(\theta)$ and $\Delta(\theta) = \dfrac{N(\theta)}{1 - e^{\delta(\theta)-\nu(\theta)}}$, for all $\theta \in \Theta$.
The condition $(dned_1)$ follows as in Theorem 3.1.

For $(dned_2')$ we use Proposition 1 and we obtain

$$\sum_{k=0}^{n} e^{\delta(\theta)(n-k)}\|D(n-k,S(k,\theta))Q(S(n,\theta))x\| \le N(\theta)\sum_{k=0}^{n} e^{(\delta(\theta)-\nu(\theta))(n-k)}\|Q(S(n,\theta))x\| \le$$

$$\le N(\theta)\frac{e^{\nu(\theta)-\delta(\theta)}-e^{(\delta(\theta)-\nu(\theta))n}}{e^{\nu(\theta)-\delta(\theta)}-1}\|Q(S(n,\theta))x\| \le \Delta(\theta)\|Q(S(n,\theta))x\|,$$

for all $(n,\theta,x)\in\mathbb{N}\times\Gamma$.

Sufficiency. Taking $k=n$ in the relation $(dned_1)$, it results

$$e^{\delta(\theta)n}\|C(n,\theta)P(\theta)x\| \le \Delta(\theta)\|P(\theta)x\|$$

and for $k=0$ in $(dned_2')$ we deduce

$$e^{\delta(\theta)n}\|D(n,\theta)Q(S(n,\theta))x\| \le \Delta(\theta)\|Q(S(n,\theta))x\|,$$

for all $(n,\theta,x)\in\mathbb{N}\times\Gamma$.
Hence, (π,P) is nonuniformly exponentially dichotomic. □

Corollary 3.4. *The pair (π,P) admits a nonuniform dichotomy if and only if there are $\delta,\ \Delta:\Theta\to\mathbb{R}_+^*$ such that the conditions $(dned_1)$ and $(dned_2')$ from Theorem 3.3 hold.*

Proof. It follows from Theorem 3.3 and Remark 5. □

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