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A Common Fixed Point Theorem in Cone Metric Spaces over Banach Algebras

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Abstract

In this paper, a common fixed point theorem for four mappings in cone metric spaces over Banach algebras is proved without assuming the normality of underlying cone. The results of this paper unify, generalize and extend some known results in cone metric spaces over Banach algebras. An example is presented which shows the significance of the result proved herein.

Keywords: Cone metric space, coincidence point, common fixed point.
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1. Introduction

The study of K-metric and K-normed spaces were introduced in the mid-20th century (Aliprantis & Tourky, 2007; Kantorovich, 1957; Vandergraft, 1967; Zabreǐko, 1997). In these papers, the set of real numbers was replaced by an ordered Banach space, as the codomain for a metric. In 2007, such spaces are reintroduced by (Huang & Zhang, 2007) under the name of cone metric space. (Huang & Zhang, 2007) defined convergent and Cauchy sequences in cone metric spaces in terms of interior points of underlying cone. Some basic versions of the fixed point theorems in cone metric spaces can be found in (Huang & Zhang, 2007). (Abbas & G.Jungck, 2008) proved some common fixed point results in these spaces. (Radenović, 2009) obtained a coincidence point theorem for two mappings in this new setting, which satisfy a new type of contractive condition. The result of (Radenović, 2009) was extended by (Rangamma & Prudhvi, 2012) for three mappings which satisfy a generalized contractive condition without exploiting the notion of continuity.

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In the papers (Radenović, 2009) and (Rangamma & Prudhvi, 2012) the contractive conditions were generalized by using the norm function. Notice that, the norm function is defined from the Banach space into the set of real numbers, hence, the results of (Huang & Zhang, 2007) can not be obtained by the results of (Rangamma & Prudhvi, 2012). Inspired by this fact, (Malhotra *et al.*, 2012) used a more competent function ϕ instead of the norm function $\|\cdot\|$. The benefit of using the function ϕ was that, the new results generalize and unify the results of (Rangamma & Prudhvi, 2012) as well as (Huang & Zhang, 2007) and (Vetro, 2007). (Malhotra *et al.*, 2012) defined a function ϕ from a normal cone into another normal cone, and so, their results cannot be applied if the cone is non-normal.

Some recent studies (see, (Çakallı *et al.*, 2012; Du, 2010; Feng & Mao, 2010; Kadelburg *et al.*, 2011)) show that the fixed point results proved in cone metric space are direct consequences of their usual metric versions. To overcome this drawback, recently, (Liu & Xu, 2013) improved the concept of cone metric spaces by defining the cone metric with values in a Banach algebra, instead of a Banach space, so that, the contractive conditions can involve the vector constants. The fixed point results thus obtained cannot be derived by their usual metric versions which was shown by an example in (Liu & Xu, 2013).

Inspired by the results of (Liu & Xu, 2013), in this paper, we prove some coincident and common fixed point results for four mappings in cone metric spaces over Banach algebras with solid cone which are not necessarily normal. We improve the definition of function ϕ used by (Malhotra *et al.*, 2012), by removing the normality condition from the domain and codomain cones of ϕ, as well as, we use a vector constant, instead the scalar in the contractive condition involving the function ϕ. Our result generalizes and unifies the results of (Liu & Xu, 2013), (Radenović, 2009) and (Rangamma & Prudhvi, 2012) and several other results, in cone metric spaces over Banach algebras.

2. Preliminaries

We first state some known definitions and facts which will be used throughout the paper.

Let E be a real Banach algebra with a unit e_E and a zero element 0_E. A nonempty closed subset P of E is called a cone if the following conditions hold:

(1) $\{0_E, e_E\} \subset P$;
(2) if $\alpha, \beta \in [0, \infty)$, then $\alpha P + \beta P \subseteq P$;
(3) $P^2 = PP \subset P$;
(4) $P \cap (-P) = \{0_E\}$.

A cone P is called a solid cone if P° is nonempty, where P° stands for the interior of P.

We can always define a partial ordering $\preceq_P$ with respect to P by $x \preceq_P y$ if and only if $y - x \in P$. We shall write $x \ll_P y$ to indicate that $y - x \in P^\circ$. We shall also write $\|\cdot\|$ as the norm on E. A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0_E \preceq_P x \preceq_P y$ implies $\|x\| \leq K\|y\|$.

Throughout the paper, we consider the real Banach algebras.

Definition 2.1. (Liu & Xu, 2013) Let X be a nonempty set and E be a Banach algebra. A mapping $d\colon X \times X \to E$ is called a cone metric if it satisfies:

(i) $0_E \preceq_P d(x,y)$, for all $x, y \in X$, $d(x,y) = 0_E$ if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(iii) $d(x,y) \preceq_P d(x,z) + d(z,y)$ for all $x, y \in X$.

In this case, the pair (X, d) is called a cone metric space over Banach algebra E. If the cone P is normal then (X, d) is called a normal cone metric space.

Definition 2.2. (Dordević *et al.*, 2011) A sequence $\{u_n\}$ in a P is said to be a c-sequence in P if for each $c \gg_P 0_E$ (i.e., $0_E \ll_P c$), there exists $N \in \mathbb{N}$ such that $u_n \ll_P c$ for all $n > N$.

Definition 2.3. (Huang *et al.*, 2017) Let (X, d) be a cone metric space over Banach algebra E and $\{x_n\}$ be a sequence in X. We say that

(i) $\{x_n\}$ converges to $x \in X$ if $\{d(x_n, x)\}$ is a c-sequence and in this case we write $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is a Cauchy sequence if $\{d(x_n, x_m)\}$ is a c-sequence for n, m, i.e., for each $c \gg_P 0_E$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll_P c$ for all $n, m > N$.

(iii) (X, d) is complete if every Cauchy sequence in X is convergent.

It is obvious that the limit of a convergent sequence in a cone metric space (X, d) over Banach algebra E is unique.

Lemma 2.1. *(Janković* et al., *2011) Let E be a Banach algebra and $u, v, w \in E$. Then*

(1) $u \ll_P w$ if $u \preceq_P v \ll_P w$ or $u \ll_P v \preceq_P w$;

(2) $u = 0_E$ if $0_E \preceq_P u \ll_P c$ for each $c \gg_P 0_E$.

The following results are well known and will be used in the sequel.

Lemma 2.2. *Let E be a Banach algebra and $u \in E$. Then the spectral radius of u is equal to $\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}}$.*

Lemma 2.3. *Let E be a Banach algebra and $k \in E$. If $\rho(k) < \lambda$, for some $\lambda > 0$ then $\lambda e_E - k$ is invertible in E, moreover, $(\lambda e_E - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{\lambda^{i+1}}$.*

Lemma 2.4. *(Huang* et al., *2016) Let P be a cone in a Banach algebra E, $\{u_n\}$ and $\{v_n\}$ be two c-sequences in P, and $\alpha, \beta \in P$ be vectors, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence in P.*

Lemma 2.5. *(Huang & Radenović, 2015) Let P be a cone and $k \in P$ with $\rho(k) < 1$. Then $\{k^n\}$ is a c-sequence in P.*

Let X be a nonempty set and f, g be two self-maps on X and $x, z \in X$. Then x is called coincidence point of pair (f, g) if $fx = gx$, and z is called point of coincidence of pair (f, g) if $fx = gx = z$. The pair (f, g) is called weakly compatible if f and g commutes at their coincidence point, i.e. $fgx = gfx$, whenever $fx = gx$ for some $x \in X$.

For results on weakly compatible mappings in cone metric spaces, see (Janković *et al.*, 2010; Jungck *et al.*, 2009).

Now we can state our main results.

3. Main Results

Let E, B be two real Banach algebras, P and C be solid cones in E and B respectively. Let "$\preceq_P$" and "$\preceq_C$" be the partial orderings induced by P and C in E and B respectively, 0_E and 0_B are the zero vectors of E and B respectively; and e_E and e_B are the units of E and B respectively.

The following definition of function ϕ is an improved version of the definition used by (Malhotra *et al.*, 2012) (see, also, (Khan *et al.*, 2015)). Let $\phi : P \to C$ be a function satisfying:

(i) if $a, b \in P$ with $a \preceq_P b$ then $\phi[a] \preceq_C k\phi[b]$, for some positive real k;
(ii) $\phi[a + b] \preceq_C \phi[a] + \phi[b]$ for all $a, b \in P$;
(iii) the sequence $\{\phi[a_n]\}$ is c-sequence in C if and only if the sequence $\{a_n\}$ is a c-sequence in P.

We denote the set of all such functions by $\Phi(P, C)$, i.e., $\phi \in \Phi(P, C)$ if ϕ satisfies all above properties. It is clear that $\phi[a] = 0_B$ if and only if $a = 0_E$.

Let (X, d) be a cone metric space with solid cone P and $\phi \in \Phi(P, C)$. Then $d(x, y) \preceq_P d(x, z) + d(z, y)$ for all $x, y, z \in X$, therefore

$$\phi[d(x, y)] \preceq_C k\phi[d(x, z)] + k\phi[d(z, y)]. \tag{3.1}$$

Example 3.1. Let $E = L[0, 1]$ be the real Banach algebra of integrable functions $f(x)$ such that $\int_0^1 |f(x)|dx < \infty$ with norm $\|f\| = \int_0^1 |f(x)|dx$, the point-wise multiplication and the unit 1. Let $P = \{f \in E\colon f(t) \geq 0, t \in [0, 1]\}$ be the solid cone in E. Let $B = C^1_{\mathbb{R}}[0, 1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, point-wise multiplication and unit 1. Let $C = \{f \in B\colon f(t) \geq 0, t \in [0, 1]\}$ be the solid cone in B. Define $\phi\colon P \to B$ by $\phi[f] = \int_0^t f(x)dx$, $t \in [0, 1]$ for all $f \in P$. Then $\phi \in \Phi(P, C)$ with $k = 1$.

Example 3.2. Let E be a Banach algebra with solid cone P. Define $\phi\colon P \to P$ by $\phi[a] = a$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $E = B, P = C$ and $k = 1$.

Example 3.3. Let $E = \mathbb{R}^2, P = \{(a, b) : a, b \in \mathbb{R}$ and $a, b \geq 0\}$ and $B = \mathbb{R}^3, C = \{(a, b, c) : a, b, c \in \mathbb{R}$ and $a, b, c \geq 0\}$, with coordinatewise multiplication and units $(1, 1)$ and $(1, 1, 1)$ respectively. Then P and C are solid cones. Define $\phi\colon P \to C$ by $\phi[(x, y)] = (x, y, ax + by)$, where a, b are positive constants, then $\phi \in \Phi(P, C)$ with $k = 1$.

Example 3.4. Let E be any real Banach algebra with normal cone P and normal constant K. Define $\phi : P \to [0, \infty)$ by $\phi[a] = \|a\|$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $B = \mathbb{R}, C = [0, \infty)$ and $k = K$.

Example 3.5. Let E be the real vector space defined by

$$E = \{ax + b\colon a, b \in \mathbb{R}, x \in [1/2, 1]\}$$

with supremum norm and $P = \{ax + b \in E\colon a \leq 0, b \geq 0\}$. Then E is a real Banach algebra with point-wise multiplication and unit $e_E = 1$ and P is a normal cone with normal constant $K > 2$ (see (Rezapour & Hamlbarani, 2008)). Let $B = \mathbb{R}^2$ be with Euclidean norm, coordinate-wise multiplication, unit $e_B = (1, 1)$ and $C = \{(a, b) : a \geq 0, b \geq 0\}$. Then C is a normal cone with normal constant $K = 1$. Define $\phi : P \to C$ by

$$\phi\,[ax + b] = (-a, b) \text{ for all } ax + b \in P.$$

Then, $\phi \in \Phi(P, C)$ with $k = 1$.

The following theorem is an improved version of the main results of (Radenović, 2009), (Rangamma & Prudhvi, 2012) and (Liu & Xu, 2013), and unifies and generalizes these results.

Theorem 3.1. *Let (X, d) be a complete cone metric space over Banach algebra E and P be a solid cone. Suppose that f, g, h, l be four self-maps of X, $f(X) \subset l(X), g(X) \subset h(X)$ and the following condition is satisfied: there exist $\phi \in \Phi(P, C)$ and $\alpha \in C$ such that $\rho(\alpha) < 1$ and*

$$\phi[d(fx, gy)] \preceq_C \alpha\phi[d(hx, ly)] \text{ for all } x, y \in X. \tag{3.2}$$

If $h(X)$, $l(x)$ are closed subsets of X, and the pairs (f, h), (g, l) are weakly compatible, then the mappings f, g, h and l have a unique common fixed point.

Proof. Suppose, x_0 be any arbitrary point of X. Since $f(X) \subset l(X)$, there exists $x_1 \in X$ such that $fx_0 = lx_1$. Again, as $g(X) \subset h(X)$, there exists $x_2 \in X$ such that $gx_1 = hx_2$. Continuing in this manner, we obtain a sequence $\{z_n\}$ such that

$$\begin{aligned} z_{2n} &= fx_{2n} = lx_{2n+1}, \\ z_{2n+1} &= gx_{2n+1} = hx_{2n+2} \text{ for all } n \geq 0. \end{aligned}$$

We shall prove that $\{z_n\}$ is a Cauchy sequence in X.

Note that, if there exists $n \in \mathbb{N}$ such that $z_n = z_{n+1}$, e.g., suppose, $z_{2n_0} = z_{2n_0+1}$, then it follows from (3.2) that

$$\begin{aligned} \phi[d(z_{2n_0+2}, z_{2n_0+1})] &= \phi[d(fx_{2n_0+2}, gx_{2n_0+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n_0+2}, lx_{2n_0+1})] \\ &= \alpha\phi[d(z_{2n_0+1}, z_{2n_0})]. \end{aligned}$$

As, $z_{2n_0} = z_{2n_0+1}$ the above inequality yields

$$\phi[d(z_{2n_0+2}, z_{2n_0+1})] = 0_B.$$

As, $\phi \in \Phi(P, C)$ therefore the above equality implies that $d(z_{2n_0+2}, z_{2n_0+1}) = 0_E$, i.e., $z_{2n_0+2} = z_{2n_0+1}$. Similarly, we obtain that

$$z_{2n_0} = z_{2n_0+1} = z_{2n_0+2} = z_{2n_0+3} = \cdots .$$

Therefore, $\{z_n\}$ is a Cauchy sequence.

Now, suppose that z_n and z_{n+1} are distinct for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$ we obtain from (3.2) that

$$\begin{aligned} \phi[d(z_{2n}, z_{2n+1})] &= \phi[d(fx_{2n}, gx_{2n+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n}, lx_{2n+1})] \\ &= \alpha\phi[d(z_{2n-1}, z_{2n})]. \end{aligned}$$

Writing $d_n = \phi[d(z_n, z_{n+1})]$, we obtain

$$d_{2n} \preceq_C \alpha d_{2n-1} \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

Again, for each $n \in \mathbb{N}$ we obtain from (3.2) that

$$\begin{aligned} \phi[d(z_{2n+2}, z_{2n+1})] &= \phi[d(fx_{2n+2}, gx_{2n+1})] \\ &\preceq_C \alpha\phi[d(hx_{2n+2}, lx_{2n+1})] \\ &= \alpha\phi[d(z_{2n+1}, z_{2n})]. \end{aligned}$$

It follows from the above inequality that

$$d_{2n+1} \preceq_C \alpha d_{2n} \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$d_n \preceq_C \alpha d_{n-1} \text{ for all } n \in \mathbb{N}. \tag{3.5}$$

Repeated use of (3.5) that

$$d_n \preceq_C \alpha d_{n-1} \preceq_C \alpha^2 d_{n-2} \preceq_C \cdots \preceq_C \alpha^n d_0 \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Let $n, m \in \mathbb{N}$ and $m > n$, then by (3.1) and (3.6) we obtain

$$\begin{aligned} \phi[d(z_n, z_m)] &\preceq_C k\phi[d(z_n, z_{n+1})] + k\phi[d(z_{n+1}, z_{n+2})] + \cdots + k\phi[d(z_{m-1}, z_m)] \\ &= k[d_n + d_{n+1} + \cdots + d_{m-1}] \\ &\preceq_C k[\alpha^n d_0 + \alpha^{n+1} d_0 + \cdots + \alpha^{m-1} d_0] \\ &= k\alpha^n [e_B + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}] d_0 \\ &\preceq_C k\alpha^n \left[\sum_{i=0}^{\infty} \alpha^i\right] d_0 \\ &= k\alpha^n (e_B - \alpha)^{-1} d_0. \end{aligned}$$

Since $\rho(\alpha) < 1$, therefore, by Lemma 2.4 and Lemma 2.5, the sequence $\{k\alpha^n (e_B - \alpha)^{-1} d_0\}$ is a c-sequence in C. Hence, by Lemma 2.1 and the fact that $\phi \in \Phi(P, C)$, we have, the sequence $\{z_n\} = \{hx_{n+1}\}$ is a Cauchy sequence.

Since X is complete, there exists $w \in X$ such that $z_n \to w$ as $n \to \infty$. Since, $z_{2n} = lx_{2n+1} \in l(X)$, $z_{2n+1} = hx_{2n+2} \in h(X)$ for all $n \in \mathbb{N}$ and $l(X)$, $h(X)$ are closed subsets of X, there exist $u, v \in X$ such that

$$w = lu = hv.$$

Therefore, from by (3.1) and (3.2) we obtain

$$\begin{aligned} \phi[d(fv, w)] &\preceq_C k\phi[d(fv, gx_{2n+1})] + k\phi[d(gx_{2n+1}, w)] \\ &\preceq_C k\alpha\phi[d(hv, lx_{2n+1})] + k\phi[d(gx_{2n+1}, w)] \\ &= k\alpha\phi[d(w, z_{2n})] + k\phi[d(z_{2n+1}, w)]. \end{aligned}$$

Since, $z_n \to w$ as $n \to \infty$, the sequences $\{d(w, z_{2n})\}$ and $\{d(z_{2n+1}, z)\}$ are c-sequences in P. As, $\phi \in \Phi(P, C)$, the sequences $\phi[\{d(w, z_{2n})\}]$ and $\phi[\{d(z_{2n+1}, z)\}]$ are c-sequences in C. By Lemma 2.1, Lemma 2.4 and the above inequality, the sequence $\{\phi[d(fv, w)]\}$ is a c-sequence in C. This shows that the sequence $\{d(fv, w)\}$ is a c-sequence in P, and so, we must have $d(fv, w) = 0_E$, i.e., $fv = w$. Therefore

$$\begin{aligned} \phi[d(w, gu)] &= \phi[d(fv, gu)] \\ &\preceq_C \alpha\phi[f(hv, lu)] \\ &= \alpha\phi[f(w, w)] \\ &= \theta_B. \end{aligned}$$

Hence, $d(w, gu) = 0_E$, i.e., $w = gu$. Thus

$$w = gu = lu = hv = fv. \tag{3.7}$$

As, the pairs (f, h), (g, l) are weakly compatible it follows from (3.7) that $fw = fhv = hfv = hw$ and $gw = glu = lgu = lw$. Hence

$$gw = lw = hw = fw. \tag{3.8}$$

Using (3.2), (3.7) and (3.8) we obtain

$$\begin{aligned} \phi[d(w, gw)] &= \phi[d(fv, gw)] \\ &\preceq_C \alpha\phi[d(hv, lw)] \\ &= \alpha\phi[d(w, gw)]. \end{aligned}$$

Thus, $\phi[d(w, gw)] \preceq_C \alpha\phi[d(w, gw)]$. Successive use of this inequality yields

$$\phi[d(w, gw)] \preceq_C \alpha\phi[d(w, gw)] \preceq_C \alpha^2\phi[d(w, gw)] \preceq_C \cdots \preceq_C \alpha^n\phi[d(w, gw)].$$

As, $\rho(\alpha) < 1$, the sequence $\{\alpha^n\}$ is a c-sequence in C, and by Lemma 2.1, Lemma 2.4 we obtain that the sequence $\{\phi[d(w, gw)]\}$ is a c-sequence in C. By definition, $d(w, gw)$ is a c-sequence in P, and so, $d(w, gw) = 0_E$, i.e., $gw = w$. Hence, we obtain from (3.8) that

$$w = gw = lw = hw = fw. \tag{3.9}$$

Thus, w is a common fixed point of the mappings f, g, h and l.

For uniqueness of fixed point, suppose w' is a common fixed point of the mappings f, g, h and l and w and w' are distinct. Then, we have

$$w' = gw' = lw' = hw' = fw'.$$

Using (3.2) we obtain

$$\begin{aligned} \phi[d(w, w')] &= \phi[d(fw, gw')] \\ &\preceq_C \alpha\phi[d(hw, lw')] \\ &= \alpha\phi[d(w, w')]. \end{aligned}$$

Again, since $\rho(\alpha) < 1$, the above inequality yields $\phi[d(w, w')] = 0_B$. This shows that $d(w, w') = 0_E$, i.e., $w = w'$. This contradiction proves the uniqueness of common fixed point. □

The following corollary is a generalized version of Theorems 2.1 of (Rangamma & Prudhvi, 2012) and Theorem 2.1 of (Radenović, 2009).

Corollary 3.1. *Let (X, d) be a complete cone metric space over a Banach algebra E and P be solid cone. Suppose that f, g, h are self-maps of X, $f(X) \cup g(X) \subset h(X)$ and the following condition is satisfied: there exists a number $a \in [0, 1)$ such that*

$$\|d(fx, gy)\| \leq a\|d(hx, hy)\| \text{ for all } x, y \in X.$$

If $h(X)$ is a closed subset of X, and the pairs (f, h), (g, h) are weakly compatible, then the mappings f, g and h have a unique common fixed point.

Proof. Take $B = \mathbb{R}, C = [0, \infty)$, $lx = hx$ for all $x \in X$, and $\phi[a] = \|a\|$ for all $a \in P$, in Theorem 3.1. Then $\phi \in \Phi(P, C)$ with $k = K$ =normal constant of P and the result follows from Theorem 3.1. □

The following corollary is an improved and generalized version of (Huang & Zhang, 2007) and (Liu & Xu, 2013).

Corollary 3.2. *Let (X, d) be a complete cone metric space over a Banach algebra E and P be solid cone. Suppose that f, g, h are self-maps of X, $f(X) \cup g(X) \subset h(X)$ and the following condition is satisfied: there exists $\alpha \in C$ such that $\rho(\alpha) < 1$ and*

$$d(fx, gy) \preceq \alpha d(hx, hy) \text{ for all } x, y \in X.$$

If $h(X)$ is a closed subset of X, and the pairs (f, h), (g, h) are weakly compatible, then the mappings f, g and h have a unique common fixed point.

Proof. Take $E = B, P = C$ and define $\phi : P \to P$ by $\phi[a] = a$, for all $a \in P$, in Theorem 3.1. Then $\phi \in \Phi(P, C)$ with $k = 1$ and the result follows from Theorem 3.1. □

Example 3.6. Let $E = \mathbb{R}^2$ be the Banach algebra with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$ and the multiplication defined by $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$. The unit of E is $e_E = (1, 0)$. Let $P = \{(x_1, x_2)\colon x_1, x_2 \geq 0\}$. Then, P is a solid cone. Define $\phi\colon P \to P$ by $\phi[a] = a$, for all $a \in P$. Then $\phi \in \Phi(P, C)$ with $E = B, P = C$ and $k = 1$.

Let $X = \mathbb{R}^2$ and let $d\colon X \times X \to \mathbb{R}$ be defined by

$$d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1|, |x_2 - y_2|)\,.$$

Then, (X, d) is a complete cone metric space over Banach algebra E. Define the mappings $f, g, h\colon X \to X$ by

$$f(x_1, x_2) = g(x_1, x_2) = (\ln(1 + |x_1|), \arctan(2 + |x_2|) + 2ax_1)\,,\ g(x_1, x_2) = h(x_1, x_2) = (2x_1, x_2)$$

for all $(x_1, x_2) \in X$, where $a > 0$. Then, it is easy to see that the condition (3.2) is satisfied with $\alpha = \left(\frac{1}{2}, a\right)$. Obviously, $h(X)$ is a closed subset of X and $f(X) \subset h(X)$. It is easy to see that f and h are weakly compatible. All the conditions of Theorem 3.1 are satisfied, hence f and h has a unique common fixed point.

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