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A Proof of a Generic Fibonacci Identity From Wolfram's MathWorld

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Abstract

The generic identity $F_{kn+c} = \sum_{i=0}^{k} C_k^i F_{c-i} F_n^i F_{n+1}^{k-i}$ involving the Fibonacci numbers F_n (where C_k^i denotes the binomial coefficient counting the number of choices of i elements from a set of k elements), is attributed on Wolfram's MathWorld website (Chandra & Weisstein, 2018) to a personal communication from Aleksandrs Mihailovs. In spite of a very thorough search, we have been unable to find a published proof. We present here a combinatorial proof of our own, using the methods of Benjamin, Eustis and Plott (Benjamin *et al.*, 2008).

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2010 MSC: 05A19, 11B39

1. Introduction

Pravin Chandra and Eric W. Weisstein on Wolfram's Mathworld website (Chandra & Weisstein, 2018) list over 100 identities involving the Fibonacci numbers, including

$$F_{kn+c} = \sum_{i=0}^{k} C_k^i F_{c-i} F_n^i F_{n+1}^{k-i} \tag{1.1}$$

as eq. (50). This is attributed to Aleksandrs Mihailovs (also known as Alec Mihailovs), through a personal communication from the 24th of January, 2003. Having been unable to find any published proof of this identity, we present here our own combinatorial proof, which uses the methods of (Benjamin *et al.*, 2008).

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2. Theory

Definition 2.1. The Fibonacci sequence $(F_n)_{n \geq 0}$ is defined recursively by the initial conditions $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_{n+2} = F_{n+1} + F_n$, for all $n \geq 0$. See (Sloane, 1964).

Definition 2.2. As in (Benjamin *et al.*, 2008), for each integer $L \geq 1$ denote by f_L the number of linear tilings of length L made from squares s of length 1 and dominoes d of length 2.

Then $f_1 = 1$, $f_2 = 2$, and for all $L \geq 1$, $f_{L+2} = f_{L+1} + f_L$, since there are f_{L+1} tilings of length $L + 2$ ending in a square, and f_L tilings of length $L + 2$ ending in a domino. Setting $f_0 = 1$, it follows that for all $L \geq 0$, $f_L = F_{L+1}$.

For example, Figure 1 shows one possible tiling of length $L = 25$, namely $ddsdssssddsssssdds$ (or $d^2sds^4d^2s^5d^2s$).

Figure 1. A tiling of length 25 made from squares (s) and dominoes (d).

There are $f_{25} = F_{26} = 121,393$ possible such tilings.

Definition 2.3. Let $l \geq 1$. We define an *l-block* to be either one of the following:

(i) a tiling of length l (of which there are f_l); or
(ii) a tiling of length $l + 1$ that ends in a domino (of which there are $f_{l+1-2} = f_{l-1}$),

and we define the *l-blocking* of a tiling to be the unique representation of that tiling as a sequence of consecutive l-blocks, followed by a "tail" of length t such that $0 \leq t \leq l - 1$.

The requirement, that an l-block of length $l + 1$ must end in a domino, is made in order to ensure the uniqueness of the l-blocking of a tiling. In our proof of the theorem below, this will prevent us from overcounting. Figure 2 shows the unique 4-blocking of the tiling in Figure 1. Note that there are five 4-blocks of length $l = 4$ or $l + 1 = 5$, and that the tail has length $t = 3$.

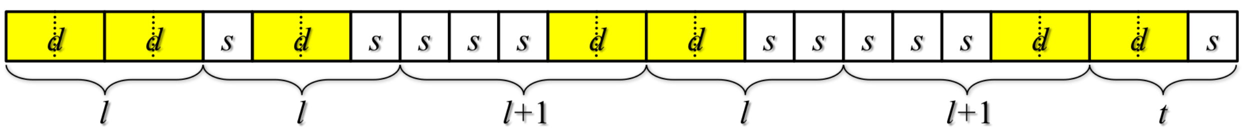

Figure 2. The unique 4-blocking of the tiling in Figure 1.

Theorem 2.1. *For all $k \geq 0, l \geq 1$ and $c \geq k+1$, the identity*

$$F_{kl+c} = \sum_{i=0}^{k} C_k^i F_{c-i} F_l^i F_{l+1}^{k-i}$$

holds, where C_k^i denotes the binomial coefficient counting the number of choices of i elements from a set of k elements.

Proof. Let S be a tiling of length $kl+c-1$, as in Definition 2.2. Since $kl+c-1 \geq k(l+1)$, S must begin with at least k consecutive l-blocks, as in Definition 2.3. Consider the prefix X of S comprising the first k such blocks, and note that X is uniquely determined by S, l and k, and conversely that any sequence of k many l-blocks may constitute X. Let $0 \leq i \leq k$, and suppose that X includes exactly i blocks of length $l+1$, hence $k-i$ blocks of length l. There are C_k^i possible such patterns of blocks, considering where in X the blocks of length $l+1$ may occur. Since X has length $i(l+1)+(k-i)l = i+kl$, the remainder of S, which is comprised of any l-blocks appearing after X, and the tail of the blocking, has length $kl+c-1-(i+kl) = c-i-1$, and can therefore be tiled in f_{c-i-1} different ways. It follows that

$$f_{kl+c-1} = \sum_{i=0}^{k} C_k^i f_{c-i-1} f_{l-1}^i f_l^{k-i},$$

hence

$$F_{kl+c} = \sum_{i=0}^{k} C_k^i F_{c-i} F_l^i F_{l+1}^{k-i},$$

as required. □

In the example in Figures 1 and 2, we have $L = 25$ and $l = 4$. There are five 4-blocks altogether, so we can choose any k such that $0 \leq k \leq 5$, and then let c equal $L+1-kl = 26-4k$. Let's suppose that $k = 3$, hence $c = 14$. Among the first $k = 3$ blocks in Figure 2 there is exactly $i = 1$ of length $l+1 = 5$. The lengths of those first 3 blocks form the pattern 4, 4, 5, which is one of $C_k^i = C_3^1 = 3$ possible such patterns. In general, we have

$$\begin{aligned}\sum_{i=0}^{3} C_3^i F_{14-i} F_4^i F_5^{3-i} &= F_{14}F_5^3 + 3 \cdot F_{13}F_4F_5^2 + 3 \cdot F_{12}F_4^2F_5 + F_{11}F_4^3 \\ &= 377 \cdot 5^3 + 3 \cdot 233 \cdot 3 \cdot 5^2 + 3 \cdot 144 \cdot 3^2 \cdot 5 + 89 \cdot 3^3 \\ &= 121,393 \\ &= F_{26}.\end{aligned}$$

Staying with $L = 25$ and $l = 4$, if instead we were to set $k = 5$, hence $c = 6$, we would obtain

$$\begin{aligned}\sum_{i=0}^{5} C_5^i F_{6-i} F_4^i F_5^{5-i} &= F_6F_5^5 + 5 \cdot F_5F_4F_5^4 + 10 \cdot F_4F_4^2F_5^3 + 10 \cdot F_3F_4^3F_5^2 + 5 \cdot F_2F_4^4F_5 + F_1F_4^5 \\ &= 8 \cdot 5^5 + 5 \cdot 5 \cdot 3 \cdot 5^4 + 10 \cdot 3 \cdot 3^2 \cdot 5^3 + 10 \cdot 2 \cdot 3^3 \cdot 5^2 + 5 \cdot 1 \cdot 3^4 \cdot 5 + 1 \cdot 3^5 \\ &= 121,393 \\ &= F_{26}.\end{aligned}$$

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