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On Comultisets and Factor Multigroups

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Abstract

In this paper, we vividly study the concept of comultisets of a multigroup and obtain some related results. The notion of Lagrange theorem in multigroup setting is proposed. Also, this paper proposes the notion of factor multigroups as an extension of factor groups in classical group setting and deduces some results. Subsequently, some homomorphic properties of factor multigroups are presented.

Keywords: Comultisets, Multisets, Multigroups, Normal submultigroups, Factor multigroups.
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1. Introduction

Many fields of modern mathematics emerged by violating a basic principle of a known theory or concept. For example, fuzzy set theory emerged by violating the notion of definite collection of objects in cantorian set theory. Similarly, the theory of multisets (see (Knuth, 1981), (Singh *et al.*, 2007), (Syropoulos, 2001), (Wildberger, 2003) for details) has been defined by assuming that, for a given set X, an element x occurs a finite number of times. This violates the idea of distinct collection of objects.

The concept of (classical) groups is built on the foundation of cantorian (or crisp) set theory. Since group is defined over a nonempty set hence, an algebraic study of multisets is an extension of group theory. The notion of multigroup was proposed in (Nazmul *et al.*, 2013) as an algebraic structure of multiset that generalized the concept of group. The notion is consistent with other non-classical groups in (Biswas, 1989), (Rosenfeld, 1971), (Shinoj *et al.*, 2015), (Shinoj & Sunil, 2015). The term *multigroups* has been earlier mentioned in (Barlotti & Strambach, 1991), (Dresher & Ore, 1938), (Mao, 2009), (Prenowitz, 1943), (Schein, 1987), (Tella & Daniel, 2013) as an extension of group theory (with each of the authors having a divergent view). Nonetheless, the

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idea of multigroups captured in (Nazmul *et al.*, 2013) is quite acceptable because it is in consonant with other non-classical groups and defined in the light of multiset.

A complete survey on the concept of multigroups from various authors were reviewed in (Ibrahim & Ejegwa, 2016). Further studies on the concept of multigroups in the light of multisets have been carried out. See (Awolola & Ejegwa, 2017), (Awolola & Ibrahim, 2016), (Ejegwa, 2017), (Ejegwa & Ibrahim, 2017*b*), (Ejegwa & Ibrahim, 2017*c*), (Ejegwa & Ibrahim, 2017*a*), (Ibrahim & Ejegwa, 2017*a*), (Ibrahim & Ejegwa, 2017*b*) for details.

In this paper, we explore more on the concept of comultisets of a multigroup studied in (Ejegwa & Ibrahim, 2017*b*), (Nazmul *et al.*, 2013), deduce some results, and propose Lagrange theorem in multigroups context. Also, we extend the idea of factor or quotient groups to multigroups and obtain some related results. Finally, the notion of homomorphism of factor multigroups is explored.

The paper is organized as follows. Section 2 gives some basic definitions and existing results on multisets and multigroups, respectively, for the sake of reference. Section 3 presents more results on comultisets of a multigroups and proposes the analogous of Lagrange theorem in multigroup setting. Details on the notion of factor multigroups as an extension of factor groups and its homomorphic properties are explicated in Section 4. Section 5 concludes the paper.

2. Preliminaries

Definition 2.1. (Singh *et al.*, 2007) Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function,

$$C_A : X \to N = \{0, 1, ...\}$$

such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A, that is, a counting function of A (where $C_A(x) = 0$, implies $x \notin Dom(A)$).

The set X is called the ground or generic set of the class of all multisets containing objects from X. The set of all multisets over X is depicted by $MS(X)$.

Definition 2.2. (Wildberger, 2003) Let A and B be two multisets over X. Then A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 2.3. (Syropoulos, 2001) Let A and B be two multisets over X. Then the intersection and union of A and B, denoted by $A \cap B$ and $A \cup B$, respectively, are defined by the rules that for any object $x \in X$,

(i) $C_{A\cap B}(x) = C_A(x) \wedge C_B(x)$,

(ii) $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,

where $\wedge$ and $\vee$ represent minimum and maximum, respectively.

Definition 2.4. (Nazmul *et al.*, 2013) Let X be a group. A multiset G is called a multigroup of X if the count function of G,

$$C_G : X \to N = \{0, 1, ...\}$$

satisfies the following conditions:

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$,
(ii) $C_G(x^{-1}) = C_G(x) \forall x \in X$.

By implication, a multiset G is a multigroup of a group X if $\forall x, y \in X$,

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y).$$

It follows immediately that,

$$C_G(e) \geq C_G(x)\ \forall x \in X$$

where e is the identity element in X. A multigroup G is regular if

$$C_G(x) = C_G(y) \forall x, y \in X.$$

We denote the set of all multigroups of X by $MG(X)$.

The count of an element in G is the number of occurrence of the element in G, and denoted by C_G. The order of G is the sum of the count of each of the elements in G, and is given by

$$|G| = \sum_{i=1}^{n} C_G(x_i) \forall x_i \in X.$$

Remark. (Ejegwa, 2017) Every multigroup is a multiset but the converse is not necessarily true.

Theorem 2.1. (Nazmul *et al.*, 2013) *Let $A, B \in MG(X)$. Then $A \cap B \in MG(X)$.*

Definition 2.5. (Nazmul *et al.*, 2013) Let $A \in MG(X)$. Then A is said to be abelian or commutative if for all $x, y \in X$, $C_A(xy) = C_A(yx)$.

Definition 2.6. (Nazmul *et al.*, 2013) Let $A \in MG(X)$. Then the sets A_* and A^* are defined as

$$A_* = \{x \in X \mid C_A(x) > 0\}$$

and

$$A^* = \{x \in X \mid C_A(x) = C_A(e)\}.$$

Proposition 2.1. (Nazmul *et al.*, 2013) *Let $A \in MG(X)$. Then A_* and A^* are subgroups of X.*

Definition 2.7. (Ejegwa, 2017) Let $A \in MG(X)$. A nonempty submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.

A submultigroup B of A is complete if $B_* = A_*$ and incomplete otherwise. A submultigroup B of A is also a multigroup of X itself.

Definition 2.8. (Ejegwa & Ibrahim, 2017*b*) Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B if for all $x, y \in X$, it satisfies $C_A(xyx^{-1}) \geq C_A(y)$.

Definition 2.9. (Nazmul *et al.*, 2013) Let A and B be multigroups of X. Then A^{-1} and $A \circ B$ are defined by

$$C_{A^{-1}}(x) = C_A(x^{-1}) \forall x \in X$$

and

$$C_{A \circ B}(x) = \begin{cases} \bigvee_{x=yz}(C_A(y) \wedge C_B(z)), & \text{if } \exists\, y, z \in X \text{ such that } x = yz \\ 0, & \text{otherwise,} \end{cases}$$

respectively.

Definition 2.10. (Ejegwa & Ibrahim, 2017*b*) Let X be a group. For any submultigroup A of a multigroup G of X, the submultiset yA of G for $y \in G$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the left comultiset of A. Similarly, for right comultiset of A. It follows that, $xA = yA$, $Ay \circ Az = Ayz$ and $yA \circ zA = yzA$ $\forall x, y, z \in X$.

Definition 2.11. (Ejegwa & Ibrahim, 2017*c*) Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively. Then f induces a homomorphism from A to B which satisfies

(i) $C_A(f^{-1}(y_1 y_2)) \geq C_A(f^{-1}(y_1)) \wedge C_A(f^{-1}(y_2))$ $\forall y_1, y_2 \in Y$,
(ii) $C_B(f(x_1 x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2))$ $\forall x_1, x_2 \in X$,

where

(i) the image of A under f, denoted by $f(A)$, is a multiset of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$ and
(ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multiset of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \ \forall x \in X.$$

Definition 2.12. (Ejegwa & Ibrahim, 2017*c*) Let X and Y be groups and let $A \in MG(X)$ and $B \in MG(Y)$, respectively.

(i) A homomorphism f from X to Y is called a weak homomorphism from A to B if $f(A) \subseteq B$. If f is a weak homomorphism from A to B, then we say that, A is weakly homomorphic to B denoted by $A \sim B$.

(ii) An isomorphism f from X to Y is called a weak isomorphism from A to B if $f(A) \subseteq B$. If f is a weak isomorphism from A to B, then we say that, A is weakly isomorphic to B denoted by $A \simeq B$.

(iii) A homomorphism f from X to Y is called a homomorphism from A to B if $f(A) = B$. If f is a homomorphism from A to B, then A is homomorphic to B denoted by $A \approx B$.

(iv) An isomorphism f from X to Y is called an isomorphism from A to B if $f(A) = B$. If f is an isomorphism from A to B, then A is isomorphic to B denoted by $A \cong B$.

Lemma 2.1. (Ejegwa & Ibrahim, 2017*c*) *Let $f : X \to Y$ be a homomorphism of groups, $A \in MG(X)$ and $B \in MG(Y)$, respectively.*

(i) If f is an epimorphism, then $f(f^{-1}(B)) = B$.
(ii) If $kerf = \{e\}$, then $f^{-1}(f(A)) = A$.

The kernel of f is defined by $kerf = \{x \in X \mid C_A(x) = C_B(e'), f(e) = e'\}$, where e and e' are the identities of X and Y, respectively. The kernel of f is a normal subgroup of X, and always contains the identity element of X. It reduces to the identity element if and only if f is one to one.

Theorem 2.2. (Ejegwa & Ibrahim, 2017*c*) *Let X and Y be groups and $f : X \to Y$ be an isomorphism. Then $A \in MG(X) \Leftrightarrow f(A) \in MG(Y)$ and $B \in MG(Y) \Leftrightarrow f^{-1}(B) \in MG(X)$.*

Definition 2.13. (Ejegwa & Ibrahim, 2017*b*) Suppose $A \in MG(X)$. Then the normalizer of A is the set given by

$$N(A) = \{g \in X \mid C_A(gy) = C_A(yg) \forall y \in X\}.$$

Theorem 2.3. (Awolola & Ibrahim, 2016) *Let $A \in MG(X)$ with identity $e \in X$. Then $\forall x, y \in X$, $C_A(x) = C_A(y)$ if $C_A(xy^{-1}) = C_A(e)$.*

3. Some results on comultisets of a multigroup

We assume that if G is a multigroup of a group X, then $G_* = X$ (except otherwise stated). That is, every element of X is in G with its multiplicity or count. In this section, we present some result on comultisets of a multigroups.

Recall that, for any submultigroup A of a multigroup G of a group X, the submultiset yA of G for $y \in X$ defined by

$$C_{yA}(x) = C_A(y^{-1}x) \forall x \in A_*$$

is called the left comultiset of A. Similarly, the submultiset Ay of G for $y \in X$ defined by

$$C_{Ay}(x) = C_A(xy^{-1}) \forall x \in A_*$$

is called the right comultiset of A.

Example 3.1. Let $X = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$ be a permutation group on a set $S = \{1, 2, 3\}$ such that

$$\rho_0 = (1), \rho_1 = (123), \rho_2 = (132), \rho_3 = (23), \rho_4 = (13), \rho_5 = (12)$$

and $G = [\rho_0^7, \rho_1^5, \rho_2^5, \rho_3^3, \rho_4^3, \rho_5^3]$ be a multigroup of X. Then $H = [\rho_0^6, \rho_1^4, \rho_2^4]$ is an incomplete submultigroup of G.

Now, we find the left comultisets of H by pre-multiplying each element of G by H.

$$\begin{aligned}
\rho_0 H &= [\rho_0^6, \rho_1^4, \rho_2^4] \\
\rho_1 H &= [\rho_2^4, \rho_0^6, \rho_1^4] \\
\rho_2 H &= [\rho_1^4, \rho_2^4, \rho_0^6] \\
\rho_3 H &= [\rho_3^0, \rho_5^0, \rho_4^0] = \emptyset \\
\rho_4 H &= [\rho_4^0, \rho_3^0, \rho_5^0] = \emptyset \\
\rho_5 H &= [\rho_5^0, \rho_4^0, \rho_3^0] = \emptyset.
\end{aligned}$$

Similarly, the right comultisets of H are thus.

$$\begin{aligned}
H\rho_0 &= [\rho_0^6, \rho_1^4, \rho_2^4] \\
H\rho_1 &= [\rho_2^4, \rho_0^6, \rho_1^4] \\
H\rho_2 &= [\rho_1^4, \rho_2^4, \rho_0^6] \\
H\rho_3 &= [\rho_3^0, \rho_4^0, \rho_5^0] = \emptyset \\
H\rho_4 &= [\rho_4^0, \rho_5^0, \rho_3^0] = \emptyset \\
H\rho_5 &= [\rho_5^0, \rho_3^0, \rho_4^0] = \emptyset.
\end{aligned}$$

Suppose $H = [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2]$, that is, a complete submultigroup of G. The left comultisets of H are thus listed.

$$\begin{aligned}
\rho_0 H &= [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] \\
\rho_1 H &= [\rho_2^3, \rho_0^6, \rho_1^3, \rho_5^2, \rho_3^2, \rho_4^2] \\
\rho_2 H &= [\rho_1^3, \rho_2^3, \rho_0^6, \rho_4^2, \rho_5^2, \rho_3^2] \\
\rho_3 H &= [\rho_3^2, \rho_5^2, \rho_4^2, \rho_0^6, \rho_2^3, \rho_1^3] \\
\rho_4 H &= [\rho_4^2, \rho_3^2, \rho_5^2, \rho_1^3, \rho_0^6, \rho_2^3] \\
\rho_5 H &= [\rho_5^2, \rho_4^2, \rho_3^2, \rho_2^3, \rho_1^3, \rho_0^6].
\end{aligned}$$

The right comultisets of H are below.

$$\begin{aligned}
H\rho_0 &= [\rho_0^6, \rho_1^3, \rho_2^3, \rho_3^2, \rho_4^2, \rho_5^2] \\
H\rho_1 &= [\rho_2^3, \rho_0^6, \rho_1^3, \rho_4^2, \rho_5^2, \rho_3^2] \\
H\rho_2 &= [\rho_1^3, \rho_2^3, \rho_0^6, \rho_5^2, \rho_3^2, \rho_4^2] \\
H\rho_3 &= [\rho_3^2, \rho_4^2, \rho_5^2, \rho_0^6, \rho_1^3, \rho_2^3] \\
H\rho_4 &= [\rho_4^2, \rho_5^2, \rho_3^2, \rho_2^3, \rho_0^6, \rho_1^3] \\
H\rho_5 &= [\rho_5^2, \rho_3^2, \rho_4^2, \rho_1^3, \rho_2^3, \rho_0^6].
\end{aligned}$$

Remark. Let H be a submultigroup of $G \in MG(X)$. We notice that

(i) H and its comultisets are equal because a multiset is an unordered collection. Consequently, $xH = yH \; \forall x, y \in X$.

(ii) the number of comultisets of H equals the cardinality of H_*, and the union and intersection of the comultisets of H are comparable to H.

(iii) there is a one-to-one correspondence between the right comultisets and the left comultisets of H.

Proposition 3.1. *Let $A, B \in MG(X)$ such that $A \subseteq B$. If $xA = yA$, then $C_A(x) = C_A(y)\ \forall x, y \in X$.*

Proof. Let X be a group and $x \in X$. Suppose $xA = yA$, then we have

$$C_{xA}(x) = C_{yA}(x) \Rightarrow C_A(x^{-1}x) = C_A(y^{-1}x) \Rightarrow C_A(e) = C_A(y^{-1}x).$$

Then, it follows that $C_A(y) = C_A(x)\ \forall x, y \in X$ by Theorem 2.3. □

Proposition 3.2. *Let $B \in MG(X)$ and A be a submultigroup of B. If $(Ay \circ Az)^{-1} = Ay \circ Az$ and $Ay \circ Az = Ayz$, then $(Ay \circ Az)^{-1} = Ayz$.*

Proof. Straightforward from Definition 2.10. □

Theorem 3.1. *Let X be a finite group and A be a submultigroup of $B \in MG(X)$. Define*

$$\begin{aligned} H &= \{g \in X \mid C_A(g) = C_A(e)\}, \\ K &= \{x \in X \mid C_{Ax}(y) = C_{Ae}(y)\}, \end{aligned}$$

where e denotes the identity element of X. Then H and K are subgroups of X. Again $H = K$.

Proof. Let $g, h \in H$. Then

$$\begin{aligned} C_A(gh) &\geq C_A(g) \wedge C_A(h) \\ &= C_A(e) \wedge C_A(e) \\ &= C_A(e) \end{aligned}$$

$\Rightarrow C_A(gh) \geq C_A(e)$.

But, $C_A(gh) \leq C_A(e)$ from Definition 2.4. Thus, $C_A(gh) = C_A(e)$, implying that $gh \in H$. Since X is finite, it follows that H is a subgroup of X.

Now, we show that $H = K$. Let $k \in K$. Then for $y \in X$ we get

$$C_{Ak}(y) = C_{Ae}(y) \Rightarrow C_A(yk^{-1}) = C_A(y).$$

Choosing $y = e$, we obtain

$$C_A(k^{-1}) = C_A(e) \Rightarrow k^{-1} \in H,$$

and so $k \in H$ since H is a subgroup of X. Thus, $K \subseteq H$.

Again, let $h \in H$. Then $C_A(h) = C_A(e)$. Also,

$$C_{Ah}(y) = C_A(yh^{-1})\ \forall y \in X$$

and

$$C_{Ae}(y) = C_A(y)\ \forall y \in X.$$

Thus to show that $h \in K$, it suffices to prove that

$$C_A(yh^{-1}) = C_A(y)\ \forall y \in X.$$

Now,

$$\begin{aligned} C_A(yh^{-1}) &\geq C_A(y) \wedge C_A(h^{-1}) \\ &= C_A(y) \wedge C_A(h) \\ &= C_A(y) \wedge C_A(e) \\ &= C_A(y). \end{aligned}$$

Again,

$$\begin{aligned} C_A(y) &= C_A(yh^{-1}h) \\ &\geq C_A(yh^{-1}) \wedge C_A(h) \\ &= C_A(yh^{-1}) \wedge C_A(e) \\ &= C_A(yh^{-1}) \end{aligned}$$

$\Rightarrow C_A(yh^{-1}) = C_A(y)$, thus, $H \subseteq K$. Hence, $H = K$. □

Corollary 3.1. *With the same notation as in Theorem 3.1, H is a normal subgroup of X if A is a normal submultigroup of B.*

Proof. Let $y \in X$ and $x \in H$. Then

$$\begin{aligned} C_A(yxy^{-1}) &= C_A(yy^{-1}x) \text{ since } A \text{ is normal in } B \\ &= C_A(x) = C_A(e). \end{aligned}$$

Thus, $yxy^{-1} \in H$. Hence, H is normal in X. □

Theorem 3.2. *Let X be a finite group and A be a submultigroup of $B \in MG(X)$. Define*

$$H = \{g \in X \mid C_A(g) = C_A(e)\}.$$

Then for $x, y \in X$, we get $Hx = Hy \Leftrightarrow Ax = Ay$. Similarly, $xH = yH \Leftrightarrow xA = yA$.

Proof. This result gives a relationship between comultisets of a submultigroup of a multigroup and the cosets of a subgroup of a given group.

By Theorem 3.1, we recall that H is a subgroup of X and

$$H = \{x \in X \mid C_{Ax}(z) = C_{Ae}(z)\}.$$

Now, suppose that $Hx = Hy$. Then $xy^{-1} \in H$. So

$$C_{Axy^{-1}}(z) = C_{Ae}(z)$$

and

$$C_A(zyx^{-1}) = C_A(z)\ \forall z \in X.$$

Replacing z by zy^{-1}, which is also an arbitrary element of X, we get

$$C_A(zx^{-1}) = C_A(zy^{-1})\ \forall z \in X,$$

implying that $Ax = Ay$.

Conversely, suppose that $Ax = Ay$. This implies that

$$C_A(zx^{-1}) = C_A(zy^{-1})\ \forall z \in X.$$

Put $z = y$, we get

$$C_A(yx^{-1}) = C_A(e).$$

So $yx^{-1} \in H$ and therefore, $Hx = Hy$. □

Corollary 3.2. *Let X be a group. If A is a submultigroup of a multigroup B of X and $x, y \in X$. Then $xA = yA$ and $Ax = Ay \Leftrightarrow C_A(y^{-1}x) = C_A(yx^{-1}) = C_A(e)$.*

Proof. Let $x, y \in X$, and recall that $H = \{x \in X \mid C_A(x) = C_A(e)\}$. Suppose $xA = yA$ and $Ax = Ay$. Then, $y^{-1}x, yx^{-1} \in H$ as in Theorem 3.2. So, $C_A(y^{-1}x) = C_A(e) = C_A(yx^{-1})$.

Conversely, assume $C_A(y^{-1}x) = C_A(e) = C_A(yx^{-1})$. This implies that, $C_A(y^{-1}x) = C_A(x^{-1}x)$ and $C_A(yx^{-1}) = C_A(yy^{-1}) \Rightarrow C_{yA}(x) = C_{xA}(x)$ and $C_{Ax}(y) = C_{Ay}(y) \forall x, y \in X$. Hence, $xA = yA$ and $Ax = Ay$. □

Lemma 3.1. *Let X be a group. If B is a submultigroup of a finite multigroup $A \in MG(X)$, then $|B| = |xB|\ \forall x \in X$.*

Proof. Let $A \in MG(X)$. Since A is finite and $B \sqsubseteq A$, it follows that $|A| = n$ and $|B| = m$ such that $m \leq n$. Then the order of each comultisets xB of B for all $x \in X$ must be m. Hence $|B| = |xB|$ for all $x \in X$. □

Now, we state and prove the analogous of Lagrange theorem in multigroup setting.

Theorem 3.3. *Let G be a finite multigroup and let H be a complete submultigroup of G wherein the count of every element in H is a factor of the count of the corresponding element in G. Then the order of H divides the order of G.*

Proof. Let $|G| = n$ and $|H| = m$, then $m \leq n$ by Lemma 3.1. That is, since G is finite and H is a complete submultigroup of G, it follows that H is also finite and $G_* = H_*$. We prove that m is a factor of n. Because $H \sqsubseteq G$ wherein the count of every element in H is a factor of the count of the corresponding element in G, it then follows that m divides n. This completes the proof. □

Remark. Let X be a finite group and G be a regular multigroup of X, then the order of X divides the order of G.

4. Concept of factor multigroups and its homomorphic properties

In this section, we define factor multigroup as an extension of factor group and obtain some results. Unless otherwise stated, the multigroups in this section are complete.

Definition 4.1. Let A be a multigroup of a group X and B a normal submultigroup of A. Then the set of right/left comultisets of B with the property $C_{xB \circ yB}(z) = C_{xyB}(z)\ \forall\ x, y, z \in X$ form a multigroup called factor or quotient multigroup of A determined by B, denoted as A/B.

Remark. Let A be a multigroup of a group X, and C a normal submultigroup of A. Then

(i) if B is a submultigroup of A such that $C \subseteq B \subseteq A$, then B/C is a submultigroup of A/C.

(ii) every submultigroup of A/C is of the form B/C, for some submultigroup B of A such that $C \subseteq B \subseteq A$.

(iii) $|A/C| = n|C|$, where n is the number of comultisets of C in A. This is unlike in classical group where, suppose X is a group and Y a subgroup of X, then $|X/Y| = \dfrac{|X|}{|Y|}$.

Theorem 4.1. *Let A be a normal submultigroup of $B \in MG(X)$. Then A is commutative if and only if B/A is commutative.*

Proof. Let $x, y \in X$. Suppose A is commutative, then

$$C_A(xyx^{-1}y^{-1}) = C_A(e),$$

and hence,

$$C_A(xy) = C_A(yx).$$

Consequently, A is normal.
Now, since

$$C_A(xy(yx)^{-1}) = C_A(xyx^{-1}y^{-1}) = C_A(e),$$

we have

$$\begin{aligned} C_A(xy(yx)^{-1}) = C_A(e) \quad &\Rightarrow \quad C_A(xy(yx)^{-1}) = C_A(xy(xy)^{-1}) \\ &\Rightarrow \quad C_{Ayx}(xy) = C_{Axy}(xy). \end{aligned}$$

Thus, $Axy = Ayx$. It follows that, $Ax \circ Ay = Ay \circ Ax$ since $Ax \circ Ay = Axy$ and $Ay \circ Ax = Ayx$ by Definition 2.10. Hence, B/A is commutative.

Conversely, if B/A is commutative, then

$$Ax \circ Ay = Ay \circ Ax \Rightarrow Axy = Ayx.$$

Thus,

$$C_A(xy(yx)^{-1}) = C_A(e) \Rightarrow C_A(xy) = C_A(yx),$$

completes the proof. □

Theorem 4.2. *Let $A, B, C \in MG(X)$ such that A and B are normal submultigroups of C and $A \subseteq B$, then B/A is a normal submultigroup of C/A.*

Proof. Let $x \in X$. Then $C_{B/A}(x) \leq C_{C/A}(x) \forall x \in X$ since $A \subseteq B$ and A and B are normal submultigroups of C. So B/A is a submultigroup of C/A. For all $x, y \in X$,

$$C_{B/A}(yxy^{-1}) \geq C_{B/A}(x).$$

Hence, B/A is a normal submultigroup of C/A by Definition 2.8. □

Remark. Let C be a multigroup of a group X, and B a normal submultigroup of C. Then every normal submultigroup of C/A is of the form B/A, for some normal submultigroup A of C such that $A \subseteq B \subseteq C$.

Theorem 4.3. *Let $A, B \in MG(X)$ and A a normal submultigroup of B. Then $A \cap B/A_*$ is a normal submultigroup of A.*

Proof. By Proposition 2.1, A_* is a subgroup of X and $A \cap B \in MG(X)$ by Theorem 2.1. So, $A \cap B/A_*$ is a multigroup of X. Since A is a normal submultigroup of B, then $A \cap B$ is a submultigroup of B and $A \cap B/A_*$ is a submultigroup of A. We show that $A \cap B/A_*$ is a normal submultigroup of A. Let $x, y \in A_*$. Then $xyx^{-1} \in A_*$ since

$$C_A(xyx^{-1}) = C_A(y) > 0$$

by Definition of A_*. This proves that A_* is normal. Also $A \cap B$ is normal since

$$C_{A\cap B}(xyx^{-1}) = C_A(xyx^{-1}) \wedge C_B(xyx^{-1}) \geq C_A(y) \wedge C_B(y) = C_{A\cap B}(y).$$

Hence, $A \cap B/A_*$ is a normal submultigroup of A. □

Theorem 4.4. *Let $A \in MG(X)$ and $N(A)$ be a normalizer of $x \in X$. Then $N(A)$ is a subgroup of X and $A/N(A)$ is a normal submultigroup of A.*

Proof. Clearly, $e \in N(A)$. Let $x, y \in N(A)$. Then for any $z \in X$, we have

$$\begin{aligned} C_A((xy^{-1})z) &= C_A(x(y^{-1}z)) = C_A((y^{-1}z)x) \\ &= C_A(y^{-1}(zx)) = C_A(y(zx)^{-1}) \\ &= C_A(y(x^{-1}z^{-1})) = C_A(z(xy^{-1})). \end{aligned}$$

Hence, $xy^{-1} \in N(A)$. Therefore, $N(A)$ is a subgroup of X. By Definition 4.1, it follows that $A/N(A) \in MG(N(A))$ and clearly, $A/N(A)$ is a submultigroup of A. Since $C_{A/N(A)}(xyx^{-1}) = C_{A/N(A)}(y)$ $\forall x, y \in X$, it implies that $A/N(A)$ is a normal submultigroup of A. □

Theorem 4.5. *Let A be a commutative multigroup of X and B a normal submultigroup of A. Then there exists a natural homomorphism $f : A \to A/B$ defined by $C_{f(A)}(y) = C_B(x^{-1}y)$ $\forall$ $x, y \in X$.*

Proof. Let $f : A \to A/B$ be a mapping defined by

$$C_{f(A)}(y) = C_B(x^{-1}y) \; \forall \; x, y \in X.$$

That is, $C_{f(A)}(y) = C_{xB}(y) \Rightarrow f(A) = xB$ (consequently, $f(A_*) = xB_*$). Since $f : A \to A/B$ is derived from $f : A_* \to A_*/B_*$ such that B_* is a normal subgroup of A_*, then to prove that f is a homomorphism, we show that

$$C_{xyB}(z) = C_{xB\circ yB}(z) \forall z \in X \Rightarrow f(xy) = f(x)f(y).$$

Since B is commutative, then

$$C_B(xz) = C_B(zx) \Rightarrow C_B(z^{-1}xz) = C_B(x) \forall z \in X.$$

We know that,

$$C_{xB}(z) = C_B(x^{-1}z) \text{ and } C_{yB}(z) = C_B(y^{-1}z).$$

Then

$$C_{xyB}(z) = C_B((xy)^{-1}z).$$

Now,

$$\begin{aligned} C_{xB\circ yB}(z) &= \bigvee_{z=rs}(C_{xB}(r) \wedge C_{yB}(s)) \\ &= \bigvee_{z=rs}(C_B(x^{-1}r) \wedge C_B(y^{-1}s)). \end{aligned}$$

And

$$\begin{aligned} C_{xyB}(z) = C_B((xy)^{-1}z) &= C_B(y^{-1}x^{-1}z) \\ &\geq \bigvee_{z=rs}(C_B(x^{-1}r) \wedge C_B(y^{-1}s)). \end{aligned}$$

Suppose by hypothesis,

$$C_B(y^{-1}x^{-1}z) = \bigvee_{z=rs}(C_B(x^{-1}r) \wedge C_B(y^{-1}s)),$$

then it follows that $C_{xyB}(z) = C_{xB\circ yB}(z) \, \forall z \in X$. Consequently, we have $f(xy) = f(x)f(y) \, \forall x, y \in X$. Therefore, f is a homomorphism. □

Corollary 4.1. *Let $A, B \in MG(X)$ such that $C_A(x) = C_A(y) \forall x, y \in X$ and $C_A(e) \geq C_B(x) \; \forall x \in X$. If $f : A \to A/B$ is a natural homomorphism defined by $C_{f(A)}(y) = C_B(x^{-1}y) \; \forall \; x, y \in X$, then $f^{-1}(f(B)) = A \circ B$.*

Proof. Let $x \in X$. To proof the result, we assume that $f(x) = f(y) \forall x, y \in X$. Thus,

$$\begin{aligned} C_{f^{-1}(f(B))}(x) &= \bigvee_{x\in X}(C_{f(B)}(f(x))), f(x) = f(y) \\ &= \bigvee_{x\in X}(C_B(f^{-1}(f(y)))) \; \forall y \in X \\ &= C_B(y). \end{aligned}$$

Again,

$$\begin{aligned} C_{A\circ B}(x) &= \bigvee_{x=zy}(C_A(z) \wedge C_B(y)) \ \forall y, z \in X \\ &= \bigvee_{x\in X}(C_A(xy^{-1}) \wedge C_B(y)), z = xy^{-1} \ \forall y, z \in X \\ &= \bigvee_{x\in X}(C_A(e) \wedge C_B(y)) \ \forall y \in X \\ &= C_B(y). \end{aligned}$$

$\Rightarrow f^{-1}(f(B)) = A \circ B.$ □

Remark. We assume there is a bijective correspondence between every (normal) submultigroup of A that contains B and the (normal) submultigroups of A/B; if C is a (normal) submultigroup of A containing B, then the corresponding (normal) submultigroup of A/B is $f(C)$.

Theorem 4.6. *Let $f : X \to Y$ be an isomorphism of groups and A a normal submultigroup of $B \in MG(X)$ such that $C_B(x) = C_B(y)$ $\forall x, y \in X$ with $kerf = \{e\}$. Then $B/A \cong f(B)/f(A)$.*

Proof. By Theorem 2.2 and Definition 4.1, B/A and $f(B)/f(A)$ are multigroups. Let

$$h : B/A \to f(B)/f(A)$$

be defined as

$$h(Ax) = f(A)(f(x))\forall x \in X.$$

If $Ax = Ay$, then $C_A(xy^{-1}) = C_A(e)$. Since $kerf = \{e\}$ meaning $kerf \subseteq A^*$, then $f^{-1}(f(A)) = A$ by Lemma 2.1. Thus,

$$C_{f^{-1}(f(A))}(xy^{-1}) = C_{f^{-1}(f(A))}(e),$$

that is,

$$C_{f(A)}(f(xy^{-1})) = C_{f(A)}(f(e)),$$

then

$$C_{f(A)}(f(x)(f(y))^{-1}) = C_{f(A)}(f(e)),$$

so

$$C_{f(A)}(f(x)) = C_{f(A)}(f(y)e') \text{ (where } f(e) = e').$$

Hence,

$$C_{f(A)}(f(x)) = C_{f(A)}(f(y)) \Rightarrow f(A)(f(x)) = f(A)(f(y)).$$

Hence, h is well-defined. It is also a homomorphism because

$$\begin{aligned} h(AxAy) = h(Axy) &= f(A)(f(xy)) \\ &= f(A)(f(x)f(y)) \\ &= f(A)(f(x))f(A)(f(y)) \\ &= h(Ax)h(Ay). \end{aligned}$$

Suppose f is an epimorphism, then $\exists\, x \in X$ such that $f(x) = y$. So,

$$h(Ax) = f(A)(f(x)) = f(A)(y).$$

Moreover,

$$f(A)(f(x)) = f(A)(f(y)) \Rightarrow C_{f(A)}(f(x)(f(y))^{-1}) = C_{f(A)}(e') \Rightarrow$$
$$C_{f(A)}(f(xy^{-1})) = C_{f(A)}(f(e)) \Rightarrow C_{f^{-1}(f(A))}(xy^{-1}) = C_{f^{-1}(f(A))}(e)$$

implies $C_A(xy^{-1}) = C_A(e) \Rightarrow Ax = Ay$, which proves that h is an isomorphism. Hence, the result follows. □

Corollary 4.2. *Let $f : X \to Y$ be an isomorphism of groups and B a normal submultigroup of $A \in MG(Y)$ such that $C_A(x) = C_A(y)\ \forall x, y \in Y$. Then $f(A)/f(B) \cong A/B$.*

Proof. By Theorem 2.2, $f(A), f(B) \in MG(X)$ and $f(A)/f(B)$ and A/B are multigroups by Definition 4.1. Again, since $B \in MG(Y)$, then $f(f^{-1}(B)) = B$ by Lemma 2.1. If $x \in kerf$, then $f(x) = e' = f(e)$, and so

$$C_B(f(x)) = C_B(f(e)),$$

that is,

$$C_{f^{-1}(B)}(x) = C_{f^{-1}(B)}(e).$$

Hence, $x \in f^{-1}(B)$, that is, $kerf \subseteq f^{-1}(B^*)$. The proof is completed following the same process as in Theorem 4.6. □

Theorem 4.7. *Let $A, B \in MG(X)$ and A a normal submultigroup of B. Then $B/B_* \approx B/A$.*

Proof. Let f be a natural homomorphism from B_* onto B_*/A_* defined by $f(xA_*) = xA_*\ \forall x \in B_*$. Then we have

$$C_{f(B/B_*)}(xA_*) = \vee(C_{B/B_*}(z)), \forall z \in B_*, f(z) = xA_*.$$

Since B/B_* and B are bijective correspondence to each other and $z = f^{-1}(xA_*) = xA_*$, it follows that

$$\begin{aligned} C_{f(B/B_*)}(xA_*) &= \vee(C_{B/B_*}(z)), \forall z \in B_*, f(z) = xA_* \\ &= \vee(C_B(y)), \forall y \in xA_* \\ &= C_{B/A}(xA_*) \forall x \in B_*, \end{aligned}$$

because B/A and B are bijective correspondence to each other. Therefore, $B/B_* \approx B/A$. □

Lemma 4.1. *If $f : X \to Y$ and $A \in MG(X)$, then $(f(A))_* = f(A_*)$.*

Proof. Straightforward. □

Theorem 4.8. *Let $B \in MG(X)$. Suppose Y is a group and $C \in MG(Y)$ such that $B \approx C$. Then there exists a normal submultigroup A of B such that $B/A \cong C/C_*$.*

Proof. Since $B \approx C, \exists$ an epimorphism f of X onto Y such that $f(B) = C$. Define $A \in MG(X)$ as follows: $\forall\ x \in X$,

$$C_A(x) = \begin{cases} C_B(x) & \text{if } x \in kerf \\ 0, & \text{otherwise} \end{cases}$$

Clearly, $A \subseteq B$. If $x \in kerf$, then $yxy^{-1} \in kerf\ \forall y \in X$, and so

$$C_A(yxy^{-1}) = C_B(yxy^{-1}) \geq C_B(x) = C_A(x)\ \forall y \in X.$$

If $x \notin kerf$, then $C_A(x) = 0$ and so

$$C_A(yxy^{-1}) \geq C_A(x) = 0\ \forall y \in X.$$

Hence, A is a normal submultigroup of B. Also, $B \approx C \Rightarrow f(B) = C$ which further implies $(f(B))_* = C_*$ and $f(B_*) = C_*$ by Lemma 4.1. Let $f = g$. Then g is a homomorphism of B_* onto C_* and $kerg = A_*$. Thus, there exists an isomorphism h of B_*/A_* onto C_* such that $h(xA_*) = g(x) = f(x)\ \forall x \in B_*$. For such an h, we have

$$\begin{aligned} C_{h(B/A)}(z) &= \vee(C_{B/A}(xA_*)), \forall x \in B_*, h(xA_*) = z \\ &= \vee(\vee[C_B(y)], \forall y \in xA_*), \forall x \in B_*, g(x) = z \\ &= \vee(C_B(y)), \forall y \in B_*, g(y) = z \\ &= \vee(C_B(y)), \forall y \in X, f(y) = z \\ &= C_B(f^{-1}(z)) = C_{f(B)}(z) = C_C(z), \forall z \in C_*. \end{aligned}$$

Therefore, $B/A \cong C/C_*$. □

Theorem 4.9. *Let $B \in MG(X)$ and A be a normal submultigroup of B. Then $B/A \cap B \simeq A \circ B/A$.*

Proof. From Proposition 2.1, we infer that A_* is also a normal subgroup of X. By the Second Isomorphism Theorem for groups, we deduce

$$B_*/A_* \cap B_* \cong A_*B_*/A_*.$$

We know that

$$(A \cap B)_* = A_* \cap B_*,$$
$$(A \circ B)_* = A_*B_*.$$

Consequently, we have

$$B_*/(A \cap B)_* \cong (A \circ B)_*/A_*,$$

where f is given by

$$f(x(A \cap B)_*) = xA_* \forall x \in B_*.$$

Thus,

$$\begin{aligned} C_{f(B/A\cap B)}(yA_*) &= C_{B/A\cap B}(y(A \cap B)_*) \\ &= \vee(C_B(z)), \forall z \in y(A \cap B)_* \\ &\leq \vee(C_{A\circ B}(z)), \forall z \in y(A_* \cap B_*) \\ &\leq \vee(C_{A\circ B}(z)), \forall z \in yA_* \\ &= C_{A\circ B/A}(yA_*),\ \forall y \in B_*. \end{aligned}$$

Hence, $f(B/A \cap B) \subseteq A \circ B/A$. Therefore, $B/A \cap B \simeq A \circ B/A$. □

Theorem 4.10. *Let $A, B, C \in MG(X)$ such that $A \subseteq B$, and A and B are normal submultigroups of C. Then $(C/A)/(B/A) \cong C/B$.*

Proof. If $A, B \in MG(X)$ and A is a normal submultigroup of B, obviously, A_* is a normal subgroup of B_* and both A_* and B_* are normal submultigroups of C_*. From the principle of Third Isomorphism Theorem for groups, we have

$$(C_*/A_*)/(B_*/A_*) \cong C_*/B_*,$$

where f is given by

$$f(xA_*(B_*/A_*)) = xB_* \ \forall x \in C_*.$$

Then

$$\begin{aligned}
C_{f((C/A)/(B/A))}(xB_*) &= C_{(C/A)/(B/A)}(xA_*(B_*/A_*)) \\
&= \vee(C_{C/A}(yA_*)), \forall y \in C_*, yA_* \in xA_*(B_*/A_*) \\
&= \vee(\vee[C_C(z)], \forall z \in yA_*), \forall y \in C_*, yA_* \in xA_*(B_*/A_*) \\
&= \vee(C_C(z)), \forall z \in C_*, zA_* \in xA_*(B_*/A_*) \\
&= \vee(C_C(z)), \forall z \in xA_*(B_*/A_*) \\
&= \vee(C_C(z)), \forall z \in C_*, f(z) \in xB_* \\
&= \vee(C_C(z)), \forall z \in C_*, f(z) = z \\
&= C_{C/B}(xB_*)
\end{aligned}$$

$\forall x \in C_*$, where the equalities hold since f is one-to-one. Hence, the result follows. □

5. Conclusion

An indepth work on comultisets had been carried out and some results were deduced. We have extended the notion of factor groups to multigroups and explicated some properties of factor multigroups. Finally, we explored some homomorphic properties of factor multigroups. Nonetheless, more results on comultisets and factor multigroups could be exploited.

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