



Statistical Convergence and C^* -operator Algebras

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Abstract

In this article, we define in terms of Berezin symbols, reproducing kernels and statistical radial convergence the notion of generalized Engliš algebra, which is a C^* -operator algebra on the Hardy space $H^2(\mathbb{D})$, and study its some properties.

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1. Introduction

This article is mainly motivated with the papers by Engliš ([Engliš, 1995](#)), Karaev ([Karaev, 2002, 2004, 2010](#)) and Pehlivan and Karaev ([Pehlivan & Karaev, 2004](#)), where the authors systematically applied the Berezin symbols method in summability theory; and conversely, the summability methods are used in investigation of some important problems for C^* -operator algebras on the Hardy space and also on the Bergman space. In particular, Pehlivan and Karaev investigated in ([Pehlivan & Karaev, 2004](#)) compactness of the weak limit of the sequence of compact operators on a Hilbert space by using the notion of so-called statistical convergence.

In this article, we use statistical radial limits for the study of some special C^* -operator algebras on the classical Hardy space $H^2(\mathbb{D})$ over the unit disc \mathbb{D} of the complex plane \mathbb{C} . These results generalize some results in the paper ([Engliš, 1995](#)).

The Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$\|f\|_2^2 := \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < +\infty.$$

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Alternately, H^2 consists of all functions in $L^2(\mathbb{T})$ whose negative Fourier coefficients vanish; here $\mathbb{T} = \partial\mathbb{D}$ is the unit circle in \mathbb{C} . The orthogonal projection from $L^2(\mathbb{T})$ onto H^2 will be denoted P_+ , and $P_- : I - P_+$. For $\varphi \in L^\infty(\mathbb{T})$ the Toeplitz operator T_φ and Hankel operator H_φ with symbol φ are defined by $T_\varphi f = P_+ \varphi f$ and $H_\varphi f = P_- \varphi f$ and are bounded linear operators from H^2 into H^2 (i.e., $T_\varphi \in \mathcal{B}(H^2)$) and $L^2(\mathbb{T}) \ominus H^2$, respectively. For $\lambda \in \mathbb{D}$, the reproducing kernel (Szegő kernel) of H^2 is the function $k_\lambda \in H^2$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle$$

for every $f \in H^2$. The normalized reproducing kernel \widehat{k}_λ is the function $\frac{k_\lambda}{\|k_\lambda\|}$. It is well-known (and can be easily shown) that $k_\lambda(z) := \frac{1}{1-\bar{\lambda}z}$. The algebra of all bounded linear operators on the Hilbert space H is denoted by $\mathcal{B}(H)$. For $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space over some set Ω with the reproducing kernel $k_{\mathcal{H},\lambda} \in \mathcal{H}$, its Berezin symbol \widetilde{T} is the complex-valued function on Ω defined by

$$\widetilde{T}(\lambda) = \langle T \widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \rangle \quad (\lambda \in \Omega).$$

It is well-known that $\widetilde{T}_\varphi(\lambda) = \widetilde{\varphi}(\lambda)$, where $\widetilde{\varphi}$ denotes the harmonic extension of φ into the unit disc \mathbb{D} (see Engliš (Engliš, 1995), Zhu (Zhu, 1990) and Karaev (Karaev, 2002)).

Engliš determined in (Engliš, 1995) in terms of nontangential and radial convergences some C^* -operator algebras on the Hardy space $H^2(\mathbb{D})$, which is defined by the boundary behavior of $\|\widehat{T k_\lambda}\|$, $\|T^* \widehat{k}_\lambda\|$ and $|\widetilde{T}(\lambda)|$. Let \mathcal{T} be the C^* -algebra generated by $\{T_\varphi : \varphi \in L^\infty(\mathbb{T})\}$. The following celebrated result due to Douglas (Douglas, 1972) (see also in (Nikolski, 1986)).

Theorem D (Douglas). *There is a C^* -homomorphism*

$$\sigma : \mathcal{T} \rightarrow L^\infty(\mathbb{T})$$

of \mathcal{T} onto $L^\infty(\mathbb{T})$ which satisfies $\sigma(T_\varphi) = \varphi$ ($\forall \varphi \in L^\infty(\mathbb{T})$). The kernel of σ coincides with the commutator ideal of \mathcal{T} , i.e., the ideal in \mathcal{T} generated by all commutators

$$[R, S] := RS - SR \quad (R, S \in \mathcal{T}).$$

σ is sometimes called the symbol map.

Note that the major goal of the Engliš's paper (Engliš, 1995) is to develop an alternative approach for proving results akin to the Douglas theorem. The symbol of an operator $T \in \mathcal{T}$ is then obtained in (Engliš, 1995) as the nontangential boundary value of a certain function on \mathbb{D} associated with T (called the Berezin symbol (transform), \widetilde{T} , of T to be defined above). Following by (Engliš, 1995), remark that Engliš's method also works for some operator algebras larger than the Toeplitz algebra, thereby yielding a number of interesting generalizations of the classical Toeplitz symbol calculus. This method is also applicable to the Bergman space, where the resulting symbol calculus is related to the one obtained by Berger and Coburn in (Berger & Coburn, 1986, 1987), Gürdal and Şöhret in (Gürdal & Sohret, 2011) and Zhu in (Zhu, 1987).

In the present article, we replace nontangential and radial limits by so-called statistical nontangential and statistical radial limits (which are weaker than the usual one) and define generalizations

of some Engliš's algebras. It turns out that the same results are true for these generalized Engliš's algebras. Before giving our results, let us recall the definition of the statistical convergence of real or complex numbers sequence.

If K is a subset of the positive integers \mathbb{N} , the K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n . The natural density of K (see ((Niven & Zuckerman, 1980), Chapter 11) is given by $\delta(K) = \lim_{n \rightarrow \infty} \frac{|K_n|}{n}$. A sequence $(x_k : k = 1, 2, \dots)$ of (real or complex) numbers is said to be statistically convergent to some number L if for each $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero; in this case we write $st - \lim_k x_k = L$. In what follows statistical convergence studied in many further papers (see, for instance, (Braha et al., 2014), (Mursaleen et al., 2014)).

The following notion is due to Fridy (Fridy, 1985). A sequence (x_k) is said to be statistically Cauchy if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : |x_k - x_N| \geq \varepsilon\}| = 0.$$

We recall that (see (Fridy, 1985)) for two sequences $x = (x_k)$ and $y = (y_k)$ the notion " $x_k = y_k$ for almost all k " means that $\delta(\{k : x_k \neq y_k\}) = 0$. Fridy proved the following main result of this theory (Fridy, 1985).

Theorem F (Fridy). *The following statements are equivalent:*

- (1) (x_k) is a statistically convergent sequence;
- (2) (x_k) is a statistically Cauchy sequence;
- (3) (x_k) is a sequence for which there is a convergent sequence (y_k) such that $x_k = y_k$ for almost all k .

The following result is immediate from Theorem F.

Corollary 1.1. *If (x_k) is a sequence such that $st - \lim_k x_k = L$, then (x_k) has a subsequence (y_k) such that $\lim_k y_k = L$ (in the usual sense).*

2. On some properties of Berezin Symbols

In this section, we prove some results concerning to Berezin symbols.

2.1. Approach Regions

For $0 < \alpha < 1$, following by Rudin (Rudin, 1974), pp. 240-241, let us define Ω_α to be the union of the disc $D(0; \alpha) := \{z \in \mathbb{C} : |z| < \alpha\}$ and the line segments from $z = 1$ to points of $D(0; \alpha)$. In other words, Ω_α is the smallest convex open set that contains $D(0; \alpha)$ and has the point 1 in its boundary. Near $z = 1$, Ω_α is an angle, bisected by the radius of \mathbb{D} that terminates at 1, of opening 2θ , where $\alpha = \sin \theta$. Curves that approach 1 within Ω_α cannot be tangent to \mathbb{T} . Therefore Ω_α is called a nontangential approach region, with vertex 1. The regions Ω_α expand when α increases. The union is \mathbb{D} , their intersection is the radius $[0, 1)$. Rotated copies of Ω_α , with vertex at e^{it} , will be denoted by $e^{it}\Omega_\alpha$.

2.2. Statistical Nontangential and Radial Limits.

A function F , defined in \mathbb{D} , is said to have statistical nontangential limit λ at $e^{i\theta} \in \mathbb{T}$ if, for each $\alpha < 1$,

$$st - \lim_{j \rightarrow \infty} F(z_j) = \lambda$$

for every sequence $\{z_j\}$ that statistically converges to $e^{i\theta}$ and that lies in $e^{i\theta}\Omega_\alpha$. If F is a function on \mathbb{D} , and f a function on \mathbb{T} , we say that F tends to f statistically radially, written

$$st - \lim F(z_n) = f(e^{it})$$

whenever $(z_n) \rightarrow e^{i\theta}$ statistically radially (i.e., if $st - \lim_{n \rightarrow \infty} |z_n| e^{i\theta} = e^{i\theta}$) for almost all $\theta \in [0, 2\pi)$.

2.3. Some Properties of Berezin Symbols of Operators.

For any two functions $\varphi, \psi \in L^\infty(\mathbb{T})$, let us denote $[T_\varphi, T_\psi] := T_{\varphi\psi} - T_\varphi T_\psi$, which is called their semicommutator.

Theorem 2.1. *For $\varphi \in L^\infty(\mathbb{T})$, $\|H_\varphi \widehat{k}_\lambda\| \rightarrow 0$ statistically radially. Consequently, for any $\varphi, \psi \in L^\infty(\mathbb{T})$, $[T_\varphi, T_\psi]^\sim(\lambda) \rightarrow 0$ statistically radially.*

Proof. Following the method of the paper (Engliš, 1995), let $y := P_- \varphi$. By considering that $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{H^2}} \in H^\infty$, we may write

$$H_\varphi \widehat{k}_\lambda = P_- (\varphi \widehat{k}_\lambda) = P_- ((P_+ \varphi) \widehat{k}_\lambda) + P_- (y \widehat{k}_\lambda) = P_- (y \widehat{k}_\lambda) = H_y \widehat{k}_\lambda,$$

and hence

$$\|H_\varphi \widehat{k}_\lambda\|^2 = \|y \widehat{k}_\lambda\|^2 - \|T_y \widehat{k}_\lambda\|^2 = \langle |y|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \rangle - \|T_y \widehat{k}_\lambda\|^2 = \widetilde{|y|^2}(\lambda) - \|T_y \widehat{k}_\lambda\|^2.$$

Then, for any $\beta \in \mathbb{D}$, we have

$$\langle T_y k_\lambda, k_\beta \rangle = \langle k_\lambda, \bar{y} k_\beta \rangle = \bar{y}(\lambda) \langle k_\lambda, k_\beta \rangle,$$

since \bar{y} is the boundary value of an analytic function. It follows that $T_\varphi \widehat{k}_\lambda = \bar{y}(\lambda) \widehat{k}_\lambda$ and

$$\|H_\varphi \widehat{k}_\lambda\|^2 = \widetilde{|y|^2}(\lambda) - |\bar{y}(\lambda)|^2.$$

By Fatou's theorem, both $\widetilde{|y|^2}$ and $|\bar{y}|^2$ tend radially to $|y|^2$, which implies that they tend statistically radially to $|y|^2$, and so their difference statistically tends to zero and we are done.

Observe that, $[T_\varphi, T_\psi] = H_\varphi^* H_\psi$, and therefore we obtain that

$$\begin{aligned} |[T_\varphi, T_\psi]^\sim(\lambda)| &= |\langle H_\varphi^* H_\psi \widehat{k}_\lambda, \widehat{k}_\lambda \rangle| = |\langle H_\psi \widehat{k}_\lambda, H_\varphi \widehat{k}_\lambda \rangle| \leq \\ &\leq \|H_\psi \widehat{k}_\lambda\| \|H_\varphi \widehat{k}_\lambda\| \rightarrow 0 \end{aligned}$$

statistically radially, and the second assertion follows. The theorem is proved. \square

Let us call the closed ideal in \mathcal{T} , generated by all semicommutators $[T_\varphi, T_\psi]$, $\varphi, \psi \in L^\infty(\mathbb{T})$, the semicommutator ideal. The following strengthens Theorem 2.1.

Theorem 2.2. *If $T \in \mathcal{B}(H^2)$ belongs to the semicommutator ideal of \mathcal{T} , then $\widetilde{T} \rightarrow 0$ statistically radially.*

Proof. Since the linear combinations of Toeplitz operators of the form

$$T_{\varphi_1} T_{\varphi_2} \dots T_{\varphi_n} [T_a, T_b] T_{\psi_1} T_{\psi_2} \dots T_{\psi_m} \quad (1)$$

form a dense subset of the semicommutator ideal (and therefore it is a statistically dense subset of the ideal in question), it suffices to prove the assertion when T is of the form (1). So, we obtain (see also (Engliš, 1995))

$$\begin{aligned} T_c [T_a, T_b] &= T_c T_{ab} - T_c T_a T_b \\ &= (T_c T_{ab} - T_{cab}) + (T_{cab} - T_{ca} T_b) + (T_{ca} T_b - T_c T_a T_b) \\ &= -[T_c, T_{ab}] + [T_{ca}, T_b] + [T_c, T_a] T_b. \end{aligned}$$

It follows that we may even assume T to be of the form

$$T = [T_\varphi, T_\psi] A = H_\varphi^* H_\psi A, \quad A \in \mathcal{T}, \quad \varphi, \psi \in L^\infty(\mathbb{T}).$$

Now by using that $[T_\varphi, T_\psi] = H_\varphi^* H_\psi$, we have

$$\begin{aligned} |\widetilde{T}(\lambda)| &= |\langle T \widehat{k}_\lambda, \widehat{k}_\lambda \rangle| = |\langle H_\varphi^* H_\psi A \widehat{k}_\lambda, \widehat{k}_\lambda \rangle| \\ &= |\langle H_\psi A \widehat{k}_\lambda, H_\varphi \widehat{k}_\lambda \rangle| \\ &\leq \|H_\psi A\| \|H_\varphi \widehat{k}_\lambda\| \quad (\text{Cauchy-Schwarz inequality}) \end{aligned}$$

and by considering that $\|H_\varphi \widehat{k}_\lambda\| \rightarrow 0$ statistically radially (see Theorem 2.1), we obtain that $\widetilde{T} \rightarrow 0$ statistically radially, as desired. \square

Now we prove more general theorem which improves Engliš's result (see [4, Theorem 2]) and implies the theorem of Douglas (Theorem D) as an easy corollary.

Theorem 2.3. *For any T in \mathcal{T} , $\widetilde{T} \rightarrow \varphi$ statistically radially for some function $\varphi \in L^\infty(\mathbb{T})$. The mapping*

$$\sigma : \mathcal{T} \rightarrow L^\infty(\mathbb{T}), \quad T \mapsto \varphi$$

is a C^ -algebra morphism, its kernel is precisely the commutator ideal of \mathcal{T} , and $\sigma(T_\psi) = \psi$ for any Toeplitz operator T_ψ . Thus, σ coincides with the symbol map from Theorem D.*

Proof. Let \mathcal{J} be the semicommutator ideal in \mathcal{T} . As in (Berger & Coburn, 1987), by repeated applications of the identity $AT_a T_b B - AT_{ab} B = -A[T_a, T_b]B$, we have that $T_{\varphi_1} T_{\varphi_2} \dots T_{\varphi_n} - T_{\varphi_1} T_{\varphi_2} \dots \varphi_n \in \mathcal{J}$ for any $\varphi_1, \varphi_2, \dots, \varphi_n \in L^\infty(\mathbb{T})$. By considering that linear combinations of operators of the form

$$T = T_\varphi + S, \quad \varphi \in L^\infty(\mathbb{T}), \quad S \in \mathcal{J}, \quad (2)$$

form a statistically dense subset of \mathcal{T} . According to the fact that $\widetilde{T}_\varphi = \widetilde{\varphi}$ and Theorem 2.2, $\widetilde{T} \rightarrow \varphi$ statistically radially, and thus we have $\|\varphi\|_\infty \leq \|\widetilde{T}\|_\infty \leq \|T\|$. It follows that the mapping

$$\sigma : T_\varphi + S \mapsto \varphi$$

is well defined and extends continuously to the whole of \mathcal{T} , so that, in particular, every operator in \mathcal{T} is of the form (2), because a statistical limit of Toeplitz operators is again a Toeplitz operator. Indeed, if

$$st - \lim_n \|T_{\varphi_n} - X\| = 0$$

for some $X \in \mathcal{B}(H^2)$, then by Corollary 1, the sequence $(\|T_{\varphi_n} - X\|)_{n \geq 1}$ has a subsequence $(\|T_{\varphi_{n_k}} - X\|)_k$ such that $\lim_k \|T_{\varphi_{n_k}} - X\| = 0$ in the usual sense, which easily implies that X is a Toeplitz operator, and so \mathcal{T} is statistically closed.

This mapping is clearly linear, preserves conjugation and, owing to Theorem 2.2, is also multiplicative. It is clear that its kernel is precisely the semicommutator ideal \mathcal{J} of \mathcal{T} . Now it remains only to show that \mathcal{J} coincides with the commutator ideal \mathcal{G} . Since $[T_\varphi, T_\psi] = [T_\psi, T_\varphi] - [T_\varphi, T_\psi]$, the inclusion $\mathcal{G} \subset \mathcal{J}$ is trivial. The reverse inclusion $\mathcal{J} \subset \mathcal{G}$ is proved in (Engliš, 1995), and therefore we omit it. The theorem is proved. \square

Corollary 2.1. *If $A \in \mathcal{T}$, then $[A^*, A]^\sim \rightarrow 0$ statistically radially.*

3. Generalized Engliš algebras and extending the Toeplitz calculus

In the present section, we introduce the concept of generalized Engliš algebra of operators, study some properties and exhibit a family of generalized Engliš C^* -algebras containing \mathcal{T} for which analogs of Theorem 2.3 still hold.

For this reason, let us define the following generalized Engliš algebra:

$$\mathcal{E} := \left\{ H \in \mathcal{B}(H^2) : \|\widehat{Hk}_\lambda\| \text{ and } \|\widehat{H^*k}_\lambda\| \rightarrow 0 \text{ statistically radially} \right\}.$$

In other words, we demand that

$$st - \lim_{r \nearrow 1} \|\widehat{Hk}_{re^{i\theta}}\| = 0$$

for all $\theta \in [0, 2\pi) \setminus E$, where E is a set (depending on H) of zero Lebesgue measure; similarly for H^* . Note that in case of usual radial limits, this algebra is the usual Engliš algebra (Engliš, 1995).

In the following theorem we give some important properties of algebra \mathcal{E} .

Theorem 3.1. *We have:*

- (a) \mathcal{E} is a C^* -algebra;
- (b) If $T_\varphi \in \mathcal{E}$, then $\varphi = 0$;
- (c) For $\varphi, \psi \in L^\infty(\mathbb{T})$, $[T_\varphi, T_\psi] \in \mathcal{E}$;
- (d) \mathcal{E} is an "ideal with respect to Toeplitz operators", that is, $H \in \mathcal{E}$ and $\varphi \in L^\infty(\mathbb{T})$ implies that $HT_\varphi, T_\varphi H \in \mathcal{E}$.

Proof. (a) Everything is trivial, except may be for the implication $A, B \in \mathcal{E} \Rightarrow AB \in \mathcal{E}$. But, $0 \leq \|AB\widehat{k}_\lambda\| \leq \|A\| \|\widehat{Bk}_\lambda\| \rightarrow 0$ statistically radially, and similarly for B^*A^* .

(b) Indeed, $\|T_\varphi\widehat{k}_\lambda\| \rightarrow 0$ statistically radially implies that

$$|\widetilde{T}_\varphi(\lambda)| = \left| \langle T_\varphi\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \leq \|T_\varphi\widehat{k}_\lambda\| \rightarrow 0$$

statistically radially, and hence $\widetilde{T}_\varphi(\lambda) \rightarrow 0$ statistically radially. But, we know that $\widetilde{T}_\varphi \rightarrow \varphi$ statistically radially, so $\varphi = 0$.

(c) The proof is immediate from Theorem 2.1 and the equality $[T_\varphi, T_\psi] = H_\varphi^* H_\psi$.

(d) Indeed, we have that

$$0 \leq \|T_\varphi H\widehat{k}_\lambda\| \leq \|T_\varphi\| \|H\widehat{k}_\lambda\| \rightarrow 0$$

statistically radially, and similarly for $T_{\overline{\varphi}} H^* \widehat{k}_\lambda$. So, the corresponding assertions for $HT_\varphi\widehat{k}_\lambda$ are immediate from the following fact. \square

Proposition 1. Let $\varphi \in L^\infty(\mathbb{T})$, and denote, as before, by $\widetilde{\varphi}$ its harmonic extension (by the Poisson formula) into \mathbb{D} . Then $T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda \rightarrow 0$ statistically radially, i.e.,

$$\lim_{r \nearrow 1} \left\| T_\varphi \widehat{k}_{re^{i\theta}} - \widetilde{\varphi}(re^{i\theta}) \widehat{k}_{re^{i\theta}} \right\| = 0$$

for almost all $t \in [0, 2\pi)$.

The proof of this proposition is immediate from Theorem 6 in [4] and Corollary 1.1 in Section 1.

Denote

$$\mathcal{A}_1 := \{T_\varphi + H : \varphi \in L^\infty(\mathbb{T}), H \in \mathcal{E}\}.$$

The following theorem, which generalizes the Douglas theorem, can be proved by using Corollary 1.1 and the method of the proof of Theorem 7 in (Engliš, 1995) and therefore its proof is omitted.

Theorem 3.2. We have;

(i) \mathcal{A}_1 is a C^* -algebra.

(ii) For any $T \in \mathcal{A}_1$, there exists a statistical radial limit, denoted $\sigma_{st}(T)$, of $\widetilde{T}(\lambda)$:

$$\widetilde{T} \rightarrow \sigma_{st}(T) \in L^\infty(\mathbb{T}) \text{ statistically radially.}$$

(iii) $\sigma_{st} : \mathcal{A}_1 \rightarrow L^\infty(\mathbb{T})$ induces a C^* -isomorphism of $\mathcal{A}_1/\mathcal{E}$ onto $L^\infty(\mathbb{T})$ which maps T_φ into φ , for any $\varphi \in L^\infty(\mathbb{T})$.

The following result shows that Theorem 3.2 can also be used for the characterization of the class \mathcal{T} .

Proposition 2. Let $A \in \mathcal{B}(H^2)$ be an operator. If $A \in \mathcal{T}$, then

- (i) \tilde{A} has statistical radial limit : $\tilde{A} \rightarrow \varphi$ statistically radially for some $\varphi \in L^\infty(\mathbb{T})$; and
- (ii) $\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\|$ and $\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\| \rightarrow 0$ statistically radially.

Proof. Let $A \in \mathcal{T} \subset \mathcal{A}_1$. Then clearly $A = T_\varphi + H$ for some $\varphi \in L^\infty(\mathbb{T})$ and $H \in \mathcal{E}$. Hence $\tilde{A} \rightarrow \varphi$ statistically radially, which proves (i), and $\|A\widehat{k}_\lambda - T_\varphi\widehat{k}_\lambda\| = \|H\widehat{k}_\lambda\| \rightarrow 0$ statistically radially. By virtue of Proposition 1, this is equivalent to $\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| \rightarrow 0$ statistically radially, because

$$\begin{aligned} \|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| &\leq \|A\widehat{k}_\lambda - T_\varphi\widehat{k}_\lambda\| + \|T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| \\ &\leq \|H\widehat{k}_\lambda\| + \|T_\varphi\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| \rightarrow 0 \end{aligned}$$

statistically radially. Similarly, it can be proved that $\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\| \rightarrow 0$ statistically radially, which completes the proof of (ii). So, the proposition is proved. \square

Below we give some results further extending the result of Engliš from (Engliš, 1995) by means of statistical Banach limits. The proofs of them are slight modification of analogous results of the paper (Engliš, 1995), and therefore omitted.

Following by (Engliš, 1995), recall that if $BC[0, 1)$ is the C^* -algebra of all bounded continuous functions on the half-open interval $[0, 1)$. It is known by Gelfand theory that $BC[0, 1) \simeq C(M)$, where M is the maximal ideal space of $BC[0, 1)$. M consist of a homeomorphic copy of $[0, 1)$ plus a certain fiber, denoted M_1 , over the point 1. Each multiplicative linear functional $Lim \in M_1$ will be called a Banach limit. For $f \in L^\infty(\mathbb{D})$ and $\varphi \in L^\infty(\mathbb{T})$, we say that f tends to φ statistically radially with respect to Lim , written $f \xrightarrow{Lim} \varphi$ statistically radially, when

$$Lim(r \xrightarrow{st} f(re^{i\theta})) = \varphi(e^{i\theta})$$

for all $\theta \in [0, 2\pi)$ except for a set of measure zero. Define

$\mathcal{E}_{st-Lim} := \{H \in \mathcal{B}(H^2) : \|H\widehat{k}_\lambda\| \text{ and } \|H^*\widehat{k}_\lambda\| \xrightarrow{Lim} 0 \text{ statistically radially}\}$. It can be easily shown that all of assertions in Theorem 3.1 remain in force when \mathcal{E} is replaced by \mathcal{E}_{st-Lim} , and thus we obtain the following analogs of Theorem 3.2 and Proposition 2.

Theorem 3.3. Let $\mathcal{A}_{st-Lim} := \{T_\varphi + H : \varphi \in L^\infty(\mathbb{T}), H \in \mathcal{E}_{st-Lim}\}$. Then

- (i) \mathcal{A}_{st-Lim} is a C^* -algebra;
- (ii) $\forall T \in \mathcal{A}_{st-Lim} \exists \varphi \in L^\infty(\mathbb{T})$ such that $\tilde{T} \xrightarrow{Lim} \varphi$ statistically radially;
- (iii) The mapping $T \mapsto \varphi$ is a C^* morphism of $\mathcal{A}_{st-Lim}/\mathcal{E}_{st-Lim}$ onto $L^\infty(\mathbb{T})$ which maps T_φ onto φ .

Proposition 3. Let A be an operator on H^2 . A necessary and sufficient condition for $A \in \mathcal{A}_{st-Lim}$ is that

- (i) there is $\varphi \in L^\infty(\mathbb{T})$ such that $\tilde{A} \xrightarrow{Lim} \varphi$ statistically radially;
- (ii) $\|A\widehat{k}_\lambda - \widetilde{\varphi}(\lambda)\widehat{k}_\lambda\| \xrightarrow{Lim} 0$ statistically radially and $\|A^*\widehat{k}_\lambda - \overline{\widetilde{\varphi}(\lambda)}\widehat{k}_\lambda\| \xrightarrow{Lim} 0$ statistically radially.

In conclusion we define one more generalized Engliš algebra (which in case of usual radial limits is defined by Engliš (Engliš, 1995)) and study its some properties.

Define

$$\mathcal{A}_{st} := \left\{ T \in \mathcal{B}(H^2) : \left\| T\widehat{k}_\lambda \right\|^2 - \left| \langle T\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \text{ and } \left\| T^*\widehat{k}_\lambda \right\|^2 - \left| \langle T^*\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2 \rightarrow 0 \text{ statistically radially} \right\} \quad (3)$$

or, in other words, $\left\| T\widehat{k}_\lambda \right\|^2 - |\widetilde{T}(\lambda)|^2 \rightarrow 0$ statistically radially and similarly for T^* . Following by the arguments of the paper (Engliš, 1995), if we decompose $T\widehat{k}_\lambda$ as

$$T\widehat{k}_\lambda = c_\lambda \widehat{k}_\lambda + d_\lambda, \quad c_\lambda \in \mathbb{C}, \quad d_\lambda \perp \widehat{k}_\lambda \quad (4)$$

then $\widetilde{T}(\lambda) = \langle T\widehat{k}_\lambda, \widehat{k}_\lambda \rangle = c_\lambda$ and $\left\| T\widehat{k}_\lambda \right\|^2 = |c_\lambda|^2 + \|d_\lambda\|^2$, so the first condition in (3) reads just

$$d_\lambda \rightarrow 0 \text{ statistically radially} \quad (5)$$

Proposition 4. \mathcal{A}_{st} is a C^* -algebra.

Proof. In fact, it follows from (4) and (5) that \mathcal{A}_{st} is linear, and from (3) that (see (Engliš, 1995)) it is self-adjoint and statistically norm closed. If $T, T' \in \mathcal{A}_{st}$ and $T\widehat{k}_\lambda = c_\lambda \widehat{k}_\lambda + d_\lambda$, $T'\widehat{k}_\lambda = c'_\lambda \widehat{k}_\lambda + d'_\lambda$ are the decomposition (4), then

$$TT'\widehat{k}_\lambda = Td'_\lambda + c'_\lambda c_\lambda \widehat{k}_\lambda + c'_\lambda d_\lambda.$$

Let $TT'\widehat{k}_\lambda = c''_\lambda \widehat{k}_\lambda + d''_\lambda$ be the decomposition (4) for TT' . Then

$$(c'_\lambda c_\lambda - c''_\lambda) \widehat{k}_\lambda = (d''_\lambda - c'_\lambda d_\lambda) - Td'_\lambda. \quad (6)$$

Since $d_\lambda \perp \widehat{k}_\lambda$ and $d''_\lambda \perp \widehat{k}_\lambda$, taking the inner product with \widehat{k}_λ on both sides gives

$$\begin{aligned} |c'_\lambda c_\lambda - c''_\lambda| &= \left| \langle Td'_\lambda, \widehat{k}_\lambda \rangle \right| \\ &\leq \|T\| \|d'_\lambda\| \rightarrow 0 \text{ statistically radially (by (5)).} \end{aligned} \quad (7)$$

Now putting this back into (6) shows that

$$d''_\lambda - (c'_\lambda d_\lambda + Td'_\lambda) \rightarrow 0 \text{ statistically radially.}$$

Now $|c'_\lambda| \leq \|T'\|$, so by (5) and the boundedness of T and T' it follows that

$$c'_\lambda d_\lambda + Td'_\lambda \rightarrow 0 \text{ statistically radially.}$$

Consequently, $d''_\lambda \rightarrow 0$ statistically radially. By a similar arguments for T'^*T^* it can be easily proved that $TT' \in \mathcal{A}_{st}$, that is \mathcal{A}_{st} is also closed under multiplication. So, \mathcal{A}_{st} is a C^* -algebra, as desired. \square

Proposition 5. For $\varphi \in L^\infty(\mathbb{T})$, $T_\varphi \in \mathcal{A}_{st}$.

Proof. Immediate from Proposition 1 and the fact that $\widetilde{T_\varphi} = \widetilde{\varphi}$. □

Proposition 6. If $H \in \mathcal{E}$, then $H \in \mathcal{A}_{st}$.

Proof. The assertion that $\|H\widehat{k}_\lambda\| \rightarrow 0$ statistically radially implies that $\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \rightarrow 0$ statistically radially (because $|\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \rangle| \leq \|H\widehat{k}_\lambda\|$), and hence $\|H\widehat{k}_\lambda\|^2 - |\langle H\widehat{k}_\lambda, \widehat{k}_\lambda \rangle| \rightarrow 0$ statistically radially as well; similarly for H^* , and thus $\mathcal{E} \subset \mathcal{A}_{st}$, as desired. □

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