Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 7 (1) (2017) 72 – 77

Fuzzy M-Open Sets

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Abstract

In this paper, we introduce the relatively new notion of fuzzy $M-$open subset which is strictly weaker than fuzzy open. We prove that the collection of all fuzzy $M-$open subsets of a fuzzy space forms a fuzzy topology that is finer than the original one. Several characterizations and properties of this class are also given as well as connections to other well-known "fuzzy generalized open" subsets.

Keywords: Fuzzy M-open, Fuzzy countable set, Fuzzy anti locally countable space.
2010 MSC: 54C08, 54H40.

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1. Introduction

Fuzzy topological spaces were first introduced by (Chakraborty & Ahsanullah, 1992; Chang, 1968). Let $(X, \mathfrak{T})$ be a fuzzy topological space (simply, Fts). If λ is a fuzzy set (simply, F-set), then the closure of λ, the interior of λ and the derived set of λ will be denoted by $Cl_{\mathfrak{T}}(\lambda)$, $Int_{\mathfrak{T}}(\lambda)$ and $d_{\mathfrak{T}}(\lambda)$, respectively. If no ambiguity appears, we use $\overline{\lambda}$, $\overset{o}{\lambda}$ and $\acute{\lambda}'$ instead, respectively. A F-set λ is called *F-semi-open* (simply, *FSO*) (Mahmoud *et al.*, 2004) if there exists a fuzzy open (simply, F-open) set μ such that $\mu \leq \lambda \leq Cl_{\mathfrak{T}}(\mu)$. Clearly λ is a FSO-set if and only if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda))$. A complement of a FSO-set is called *F-semi-closed* (simply, *FSC*). The fuzzy semi-interior of λ is the union of all fuzzy semi-open subsets contained in λ and is denoted by $sInt(\lambda)$. λ is called *fuzzy preopen* (simply, *FPO*) if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda)).\lambda$ is called fuzzy $\alpha-open$ if $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ and fuzzy $\beta-open$ if $\lambda \leq Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda)))$. Finally, λ is called *fuzzy regular-open* (simply, *FRO*) if $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$. Complements of FRO-sets are called *fuzzy regular-closed* (simply, *FRC*). The collection of all FSO (resp., FPO, FRO, FRC, F$\alpha - open$ and F$\beta - open$) subsets of

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[1] This work has been done during the author's sabbatical leave at Jordan University of Science and Technology—Jordan.

X is denoted by $SO(X, \mathfrak{T})$ (resp., F$PO(X, \mathfrak{T})$, F$RO(X, \mathfrak{T})$, F$RC(X, \mathfrak{T})$, F$\alpha(X, \mathfrak{T})$ and F$\beta(X, \mathfrak{T})$). We remark that F$\alpha(X, \mathfrak{T})$ is a topological space and F$\alpha(X, \mathfrak{T}) = FSO(X, \mathfrak{T}) \wedge FPO(X, \mathfrak{T})$. A fuzzy space $(X, \mathfrak{T})$ is called *locally countable (P-space, anti locally countable,* respectively*)* if each $\lambda \in X$ has a countable neighborhood (countable intersections of fuzzy open subsets are fuzzy open, non-empty fuzzy open subsets are uncountable, respectively). For more on the preceding notions, the reader is referred to (Al-Hawary, 2017, 2008; Chakraborty & Ahsanullah, 1992; Chang, 1968; Chaudhuri & Das, 1993; Mahmoud *et al.*, 2004; Wong, 1974).

In this paper, we introduce the relatively new notions of FMO, which is weaker than the class of fuzzy open subsets. In section 2, we also show that the collection of all FMO subsets of a space $(X, \mathfrak{T})$ forms a fuzzy topology that is finer than $\mathfrak{T}$ and we investigate the connection of FMO notion to other classes of "fuzzy generalized open" subsets as well as several characterizations of FMO and fuzzy $M-$closed notions via the operations of interior and closure. In section 3, several interesting properties and constructions of FMO subsets are discussed in the case of anti locally countable spaces.

2. Fuzzy M-open set

We begin this section by introducing the notion of FMO and fuzzy $M-$closed subsets.

Definition 2.1. A fuzzy subset λ of a space $(X, \mathfrak{T})$ is called FMO (simply, FMO) if for every $\upsilon \leq \lambda$, there exists an open fuzzy subset $\omega \epsilon X$ such that $\upsilon \leq \omega$ and such that $\omega \backslash sInt(\lambda)$ is countable. The complement of a FMO subset is called fuzzy *$M-$closed* (simply, FMC).

Clearly every FO-set is FMO, but the converse needs not be true.

Example 2.1. Let $X = \{a, b\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a\}}\}$. Set $\lambda = \chi_{\{b\}}$. Then λ is FMO but not FO.

Next, we show that the collection of all FMO subsets of a space $(X, \mathfrak{T})$ forms a fuzzy topology $\mathfrak{T}_M$ that contains $\mathfrak{T}$.

Theorem 2.1. *If $(X, \mathfrak{T})$ is a fuzzy space, then $(X, \mathfrak{T}_M)$ is a space such that $\mathfrak{T} \leq \mathfrak{T}_M$.*

Proof. We only need to show $(X, \mathfrak{T}_M)$ is a space. Clearly since 0 and 1 are FO-sets, they are FMO. If $\lambda, \psi \in \mathfrak{T}_M$ and $\upsilon \leq \lambda \wedge \psi$, then there exist FO-sets ω, υ in X both containing λ such that $\omega \backslash sInt(\lambda)$ and $\upsilon \backslash sInt(\psi)$ are countable. Now $\lambda \leq \omega \wedge \upsilon$ and for every $\delta \leq (\omega \wedge \upsilon) \backslash sInt(\lambda \wedge \psi) = (\omega \wedge \upsilon) \backslash (sInt(\lambda) \wedge sInt(\psi))$ either $\delta \leq \omega \backslash sInt(\lambda)$ or $\delta \leq \upsilon \backslash sInt(\psi)$. Thus $(\omega \wedge \upsilon) \backslash sInt(\lambda \wedge \psi) \leq \omega \backslash sInt(\lambda)$ or $(\omega \wedge \upsilon) \backslash sInt(\lambda \wedge \psi) \leq \omega \backslash sInt(\psi)$ and thus $(\omega \wedge \upsilon) \backslash sInt(\lambda \wedge \psi)$ is countable. Therefore, $\lambda \wedge \psi \in \mathfrak{T}_M$. If $\{\lambda_\alpha : \alpha \in \Delta\}$ is a collection of FMO subsets of X, then for every $\lambda \leq \vee\{\lambda_\alpha : \alpha \in \nabla\}$, $\upsilon \leq \lambda_\beta$ for some $\beta \in \Delta$. Hence there exists a FO-set ω of X containing λ such that

$\omega \backslash sInt(\lambda)$ is countable. Now as $\omega \backslash sInt(\vee\{\lambda_\alpha : \alpha \in \nabla\}) \leq \omega \backslash \vee \{sInt(\lambda_\alpha) : \alpha \in \nabla\} \leq \omega \backslash sInt(\lambda)$, $\omega \backslash sInt(\vee\{\lambda_\alpha : \alpha \in \nabla\})$ is countable and hence $\vee\{\lambda_\alpha : \alpha \in \nabla\} \in \mathfrak{T}_M$. □

Corollary 2.1. *If $(X, \mathfrak{T})$ is a p-space, then $\mathfrak{T} = \mathfrak{T}_M$.*

Next we show that FMO notion is independent of both FPO and FSO notions.

Example 2.2. Consider $\mathbb{R}$ with the standard fuzzy topology. Then $\chi_{\mathbb{Q}}$ is FPO but not FMO. Also $\chi_{[0,1]}$ is FSO but not FMO.

Example 2.3. In Example 2.1, $\chi_{\{b\}}$ is FMO but neither FPO nor FO.

Next we characterize $\mathfrak{T}_M$ when X is a locally countable fuzzy space.

Theorem 2.2. *If $(X, \mathfrak{T})$ is a locally countable fuzzy space, then $\mathfrak{T}_M$ is the discrete fuzzy topology.*

Proof. Let λ be a fF-set in X and $\upsilon \leq \lambda$. Then there exists a countable neighborhood ω of λ and hence there exists a FO-set η containing λ such that $\eta \leq \omega$. Clearly $\eta \backslash sInt(\lambda) \leq \omega \backslash sInt(\lambda) \leq \omega$ and thus $\eta \backslash sInt(\lambda)$ is countable. Therefore λ is FMO and so $\mathfrak{T}_M$ is the discrete fuzzy topology. □

Corollary 2.2. *If $(X, \mathfrak{T})$ is a countable fuzzy space, then $\mathfrak{T}_M$ is the discrete fuzzy topology.*

The following result, in which a new characterization of FMO subsets is given, will be a basic tool throughout the rest of the paper.

Lemma 2.1. *A subset λ of a fuzzy space X is FMO if and only if for every $\upsilon \leq \lambda$, there exists a FO- subset ω containing λ and a countable subset π such that $\omega - \pi \leq sInt(\lambda)$.*

Proof. Let $\lambda \in \mathfrak{T}_M$ and $\upsilon \leq \lambda$, then there exists a FO- subset ω containing λ such that $\omega \backslash sInt(\lambda)$ is countable. Let $\pi = \omega \backslash sInt(\lambda) = \omega \wedge (X \backslash sInt(\lambda))$. Then $\omega - \pi \leq sInt(\lambda)$. □

Conversely, let $\upsilon \leq \lambda$. Then there exists a FO- subset ω containing λ and a countable subset π such that $\omega - \pi \leq sInt(\lambda)$. Thus $\omega \backslash sInt(\lambda) = \pi$ is countable.

The next result follows easily from the definition and the fact that the intersection of fuzzy M-closed sets is again fuzzy M-closed.

Lemma 2.2. *A subset λ of a fuzzy space X is fuzzy M-closed if and only if $Cl_M(\lambda) = \lambda$.*

We next study restriction and deletion operations.

Theorem 2.3. *If λ is FMO subset of X, then $\mathfrak{T}_M|_\lambda \subseteq (\mathfrak{T}|_\lambda)_M$.*

Proof. Let $\rho \in \mathfrak{T}_M|_\lambda$. Then $\rho = \upsilon \wedge \lambda$ for some FMO subset υ. For every $\lambda \leq \rho$, there exist $\delta_\upsilon, \delta_\lambda \in \mathfrak{T}$ containing λ and countable sets γ_υ and γ_λ such that $\delta_\upsilon - \gamma_\upsilon \leq sInt(\upsilon)$ and $\delta_\lambda - \gamma_\lambda \leq sInt(\lambda)$. Therefore, $\upsilon \leq \lambda \wedge (\delta_\upsilon \wedge \delta_\lambda) \in \mathfrak{T}_\lambda$, $\gamma_\upsilon \vee \gamma_\lambda$ is countable and

$$\begin{aligned} \lambda \wedge (\delta_\upsilon \wedge \delta_\lambda) - (\gamma_\upsilon \vee \gamma_\lambda) &\leq (\delta_\upsilon \wedge \delta_\lambda) \wedge (1 - \gamma_\upsilon) \wedge (1 - \gamma_\lambda) \\ &= (\delta_\upsilon - \gamma_\upsilon) \wedge (\delta_\lambda - \gamma_\lambda) \\ &\leq sInt(\upsilon) \wedge sInt(\lambda) \wedge \lambda \\ &= sInt(\upsilon \wedge \lambda) \wedge \lambda \\ &= sInt(\rho) \wedge \lambda \\ &\leq sInt_\lambda(\rho). \end{aligned}$$

Therefore, $\rho \in (\mathfrak{T}|_\lambda)_M$. □

Corollary 2.3. *If λ is a FO subset of X, then $\mathfrak{T}_M|_\lambda \leq (\mathfrak{T}|_\lambda)_M$.*

In the next example, we show that if λ in the preceding Theorem is not FMO, then the result needs not be true.

Example 2.4. Consider $\mathbb{R}$ with the standard fuzzy topology and let $\lambda = \chi_{\mathbb{R}\setminus\mathbb{Q}}$. Then λ is not FMO and so not FO. As $\chi_{(0,1)}$ is FMO, then $\theta = \chi_{(0,1)} \wedge \lambda \in \mathfrak{T}_M|_\lambda$ while if $\theta \in (\mathfrak{T}|_\lambda)_M$ then for every $\lambda \leq \theta$, there exists $\omega \in \mathfrak{T}|_\lambda$ and a countable $\delta \leq \lambda$ such that $\omega - \delta \leq sInt(\theta) = 0$. Thus $\omega \leq \delta$ and hence ω is countable which is a contradiction.

In the next example, we show that $(\mathfrak{T}|_\lambda)_M$ needs not be a subset of $\mathfrak{T}_M|_\lambda$.

Example 2.5. Consider $\mathbb{R}$ with the standard fuzzy topology, $\lambda = \chi_{\mathbb{Q}}$ and $\mu = \chi_{(0,2)}$. If $\mu \in \mathfrak{T}_M|_\lambda$, then $\mu = \delta \wedge \lambda$ for some $\delta \in \mathfrak{T}_M$ which is impossible as $\chi_{\sqrt{2}} \leq \delta - \lambda$. On the other hand to show $\mu \in (\mathfrak{T}|_\lambda)_M$, let $\upsilon \leq \mu$. If $\upsilon \leq \lambda$, pick $q_1, q_2 \leq \lambda$ such that $0 < q_1 < \upsilon < q_2 < 2$ and let $\omega = \chi_{(q_1,q_2)} \wedge \lambda$. Then $\lambda \leq \omega - 0 \leq \mu = sInt(\mu)$. Thus in both cases $\mu \in (\mathfrak{T}|_\lambda)_M$.

Theorem 2.4. *Let $(X, \mathfrak{T})$ be a fuzzy space and λ is FMC-set. Then $Cl_{\mathfrak{T}}(\lambda) \leq \gamma \vee \vartheta$ for some closed subset γ and a countable subset ϑ.*

Proof. Let λ be FMC-set. Then $1 - \lambda$ is FMO and hence for every $\lambda \leq 1 - \lambda$, there exists a FO-set ω containing λ and a countable set ϑ such that $\omega - \vartheta \leq sInt(1 - \lambda) \leq 1 - Cl_{\mathfrak{T}}(\lambda)$. Thus

$$Cl_{\mathfrak{T}}(\lambda) \leq 1 - (\omega - \vartheta) \leq 1 - (\omega \wedge (X - \vartheta)) \leq 1 \wedge ((1 - \omega) \vee \vartheta) \leq (1 - \omega) \vee \vartheta.$$

Letting $\gamma = 1 - \omega$. Then γ is closed such that $Cl_{\mathfrak{T}}(\lambda) \leq \gamma \vee \vartheta$.‘ □

3. Anti-locally countable fuzzy spaces

In this section, several interesting properties and constructions of FMO subsets are discussed in case of anti locally countable fuzzy spaces.

Theorem 3.1. *A fuzzy space* $(X, \mathfrak{T})$ *is anti locally countable if and only if* $(X, \mathfrak{T}_M)$ *is anti locally countable.*

Proof. Let $\lambda \in \mathfrak{T}_M$ and $\upsilon \leq \lambda$. Then by Lemma 2.1, there exists a FO- subset ω containing λ and a countable μ such that $\omega - \mu \leq sInt(\lambda)$. Hence $sInt(\lambda)$ is uncountable and so is λ. The converse follows from the fact that every FO-set is FMO. □

Corollary 3.1. *If* $(X, \mathfrak{T})$ *is anti locally countable fuzzy space and* λ *is FMO, then* $Cl_{\mathfrak{T}}(\lambda) = Cl_{\mathfrak{T}_M}(\lambda)$.

Proof. Clearly $Cl_{\mathfrak{T}_M}(\lambda) \leq Cl_{\mathfrak{T}}(\lambda)$. On the other hand, let $\lambda \leq Cl_{\mathfrak{T}}(\lambda)$ and μ be an FMO subset containing λ. Then by Lemma 2.1, there exists a FO- subset υ containing λ and a countable set η such that $\upsilon - \eta \leq sInt(\mu)$. Thus $(\upsilon - \eta) \wedge \lambda \leq sInt(\mu) \wedge \lambda$ and so $(\upsilon \wedge \lambda) - \eta \leq sInt(\mu) \wedge \lambda$. As $\lambda \in \upsilon$ and $\lambda \in Cl_{\mathfrak{T}}(\lambda)$, $\upsilon \wedge \lambda \neq 0$ and then as υ and λ are FMO, $\upsilon \wedge \lambda$ is FMO and as X is ani locally countable, $\upsilon \wedge \lambda$ is uncountable and so is $(\upsilon \wedge \lambda) - \eta$. Thus $\upsilon \wedge \lambda$ is uncountable as it contains the uncountable set $sInt(\mu) \wedge \lambda$. Therefore, $\mu \wedge \lambda \neq 0$ which means that $\lambda \in Cl_{\mathfrak{T}_M}(\lambda)$. □

By a similar argument, we can easily prove the following result:

Corollary 3.2. *If* $(X, \mathfrak{T})$ *is anti locally countable and* λ *is FMC, then* $Int_{\mathfrak{T}}(\lambda) = Int_{\mathfrak{T}_M}(\lambda)$.

Theorem 3.2. *Let* $(X, \mathfrak{T})$ *be an anti locally countable fuzzy space. Then* $F\alpha(X, \mathfrak{T}) \subseteq F\alpha(X, \mathfrak{T}_M)$.

Proof. If $\lambda \in F\alpha(X, \mathfrak{T})$, then $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(Int_{\mathfrak{T}}(\lambda)))$ and by Corollary 3.1, $\lambda \leq Int_{\mathfrak{T}}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda)))$. Now by Corollary 3.2 and as $Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda))$ is FMC, $\lambda \leq Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}}(\lambda)))$ and by Corollary 3.2 again, $\lambda \leq Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(Int_{\mathfrak{T}_M}(\lambda)))$ which means $\lambda \in F\alpha(X, \mathfrak{T}_M)$. □

The converse of the preceding result needs not be true as shown next.

Example 3.1. Consider $\mathbb{R}$ with the standard fuzzy topology and let $\lambda = \chi_{\mathbb{R}\setminus\mathbb{Q}}$. Then $\lambda \in F\alpha(\mathbb{R}, \mathfrak{T}_M)$ but $\lambda \notin F\alpha(\mathbb{R}, \mathfrak{T})$.

Similarly, one can show that in an anti locally countable fuzzy space, $F\beta(X, \mathfrak{T}_M) \leq F\beta(X, \mathfrak{T})$.

Theorem 3.3. *Let $(X, \mathfrak{T})$ be an anti locally countable fuzzy space. Then $d_{\mathfrak{T}}(\mu) = d_{\mathfrak{T}_M}(\mu)$ for every subset F-set μ.*

Proof. If $\lambda \leq d_{\mathfrak{T}}(\mu)$ and υ is any FMO subset containing λ, then there exists a FO- subset ω containing λ and a countable γ such that $\omega - \gamma \leq sInt(\upsilon) \leq \upsilon$. Thus $(\omega - \gamma) \wedge (\lambda - \{\lambda\}) \leq sInt(\upsilon) \wedge (\mu - \lambda) \leq \upsilon \wedge (\mu - \lambda)$ and as $\lambda \in d_{\mathfrak{T}}(\mu)$ and υ^o is open containing λ, we have $\upsilon^o \wedge (\mu - \lambda) \neq 0$ and so $\upsilon \wedge (\mu - \lambda) \neq 0$. Therefore $\lambda \in d_{\mathfrak{T}_M}(\mu)$. □

The converse is obvious as every FO subset is FMO.

Theorem 3.4. *Let $(X, \mathfrak{T})$ be an anti locally countable fuzzy space. Then $FRO(X, \mathfrak{T}) = FRO(X, \mathfrak{T}_M)$.*

Proof. If $\lambda \in FRO(X, \mathfrak{T})$, then $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$ and by Corollary 3.1, $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}_M}(\lambda))$. Now by Corollary 3.2 and as $Cl_{\mathfrak{T}_M}(\lambda)$ is FMC, $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(\lambda))$ which means $\lambda \in FRO(X, \mathfrak{T}_M)$. Conversely, if $\lambda \in FRO(X, \mathfrak{T}_M)$, then $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}_M}(\lambda))$. Then as λ is FMO, by Corollary 3.1, $\lambda = Int_{\mathfrak{T}_M}(Cl_{\mathfrak{T}}(\lambda))$ and as $Cl_{\mathfrak{T}}(\lambda)$ is FMC being a FC-set, then by $\lambda = Int_{\mathfrak{T}}(Cl_{\mathfrak{T}}(\lambda))$ which means $\lambda \in FRO(X, \mathfrak{T})$. □

The converse of the preceding result need not be true as shown next.

Example 3.2. Let $X = \{a, b, c, d, e\}$ and $\mathfrak{T} = \{0, 1, \chi_{\{a\}}, \chi_{\{a,b\}}, \chi_{\{a,b,c\}}, \chi_{\{a,b,c,d\}}\}$. Then $(X, \mathfrak{T})$ is an anti locally countable fuzzy space such that $FRO(X, \mathfrak{T}) = \{0, 1\}$ while $FRO(X, \mathfrak{T}_M) = \mathfrak{T}$.

References

Al-Hawary, T. (2008). Fuzzy ω_0-open sets. *Bull. Korian Math. Soc.* **45**(4), 749–755.

Al-Hawary, T. (2017). Fuzzy l-closed sets, to appear in *MATEMATIKA*.

Chakraborty, M.K. and T.M.G. Ahsanullah (1992). Fuzzy topology on fuzzy sets and tolerance topology. *Fuzzy Sets and Systems* **45**(1), 103 – 108.

Chang, C.L (1968). Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications* **24**(1), 182 – 190.

Chaudhuri, A.K. and P. Das (1993). Some results on fuzzy topology on fuzzy sets. *Fuzzy Sets and Systems* **56**(3), 331 – 336.

Mahmoud, F.S., M.A. Fath Alla and S.M. Abd Ellah (2004). Fuzzy topology on fuzzy sets: fuzzy semicontinuity and fuzzy semiseparation axioms. *Applied Mathematics and Computation* **153**(1), 127 – 140.

Wong, C.K (1974). Fuzzy points and local properties of fuzzy topology. *Journal of Mathematical Analysis and Applications* **46**(2), 316 – 328.