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Fuzzy Differential Superordination

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Abstract

S.S. Miller and P.T. Mocanu in (Miller & Mocanu, 2003) the notion of differential superordination as a dual concept of differential subordination (Miller & Mocanu, 2000) . In (Oros & Oros, 2011) The authors define the notion of fuzzy subordination, in (Oros & Oros, 2012*b*) they define the notion of fuzzy differential subordination and in (Oros & Oros, Jun2012*a*) they determine conditions for a function to be a dominant of the fuzzy differential subordination and they also gave the best dominant. In this paper, we introduced the concept of fuzzy differential superordination and we set conditions for a function to be subordinant of fuzzy differential superordination and we also give the best subordinant.

Keywords: fuzzy set, fuzzy differential subordination, fuzzy differential superordination , fuzzy subordinant, fuzzy best subordinant, sandwich theorem.
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1. Introduction and Preliminaries

The general form of differential superordination method can be presented as follows: Let Ω and Δ be any set in C, let p be analytic in the unit disk U and let $\varphi(r, s, t; z) : C^3 \times U \longrightarrow C$. The problem is to study the following implication:

$$\Omega \subset \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow \Delta \subset p(z). \tag{1.1}$$

If Δ is a simply connected domain containing the point a and $\Delta \neq C$, then there is a conformal mapping q of U onto Δ such that $q(0) = a$. In this case, relation (1.1) can be rewritten as

$$\Omega \subset \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.2}$$

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If Ω is also a simply connected domain and $\Omega \neq C$,then there is conformal mapping h of U onto Ω such that $h(0) = \varphi(a, 0, 0; 0)$.If in addition, the function $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ is univalent in U, then (1.2) can be rewritten as

$$h(z) \prec \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z) \Rightarrow q(z) \prec p(z). \quad (1.3)$$

For further details on the differential superordination method, the valuable monograph (Miller & Mocanu, 2003) can be seen.
Let U denote the unit disc of the complex plane

$$U = \{z \in C : |z| < 1\}, \overline{U} = \{z \in C : |z| \leq 1\}$$

and $H(U)$ denote the class of analytic function in U.For $a \in C$ and $n \in N$, we denote by

$$H[a, n] = \left\{f \in H(U) : f(z) = a + a_{(n+1)}z^{(n+1)} + \ldots, z \in U\right\},$$
$$A_n = \left\{f \in H(U) : f(z) = z + a_{(n+1)}z^{(n+1)} + \ldots, z \in U\right\}$$

with $A_1 = A$. Let $S = \{f \in A : f \text{ univalent in } U\}$ be the class of analytic and univalent functions in the open unit disk U, with condition $f(0) = 0$, $f^{'}(0) = 1$, that is the analytic and univalent functions with the following power series development

$$f(z) = z + a_2z^2 + \ldots, z \in U.$$

Denote by
$S^* = \left\{f \in A : Re\left(\frac{zf^{'}(z)}{f(z)}\right) > 0, z \in U\right\}$, the class of normalized starlike functions in U, and
$C = \left\{f \in A : Re\left(\frac{zf^{''}(z)}{f^{'}(z)}\right) > 0, z \in U\right\}$, the class of normalized convex functions in U, and
$K = \left\{f \in A : Re\left(\frac{f^{'}(z)}{g^{'}(z)}\right) > 0, g(z) \in C, z \in U\right\}$, the class of normalized close to convex functions in U.

In order to introduce the notation of fuzzy differential superordination, we use the following definitions and lemmas:

Definition 1.1 (Miller & Mocanu, 2000) We denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $E(q) = \left\{\zeta \in \partial U : \lim_{\zeta \to \infty} q(z) = \infty\right\}$, and are such that $q^{'}(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.The set $E(q)$ is called exemption set.

Definition 1.2 (Zadeh, 1965) Let X be a non-empty set. An application $F : X \to [0, 1]$ is called fuzzy subset. An alternate definition, more precise, would be the following:
A pair(A, F_A), where

$$F_A : X \to [0, 1] \quad \text{and} \quad A = \{x \in X : 0 < F_A \leq 1\} = supp(A, F_A),$$

is called fuzzy subset. The function F_A is called membership function of the fuzzy set (A, F_A).

Definition 1.3 (Zadeh, 1965) Let two fuzzy subsets of X,(M, F_M) and (N, F_N). We say that fuzzy subsets M and N are equal if and only if $F_M(x) = F_N(x)$,$x \in X$ and we denote this by $(M, F_M) = (N, F_N)$. The fuzzy subset (M, F_M) is contained in the fussy subset (N, F_N) if and only if $F_M(x) \leq$

$F_N(x)$, $x \in X$ and we denote the inclusion relation by $(M, F_M) \subseteq (N, F_N)$.

Definition 1.4 (Oros & Oros, 2011) Let $D \subseteq C, z_0 \in D$ be a fixed point, and let the functions $f, g \in H(D)$. The function f is said to be fuzzy subordinate to g, written $f \prec_F g$ or $f(z) \prec_F g(z)$ if the following conditions are satisfied:

1. $f(z_0) = g(z_0)$,
2. $F_{f(D)}(f(z)) \leq F_{g(D)}(g(z)), z \in U$.

Definition 1.5 (Oros & Oros, Jun2012*a*) A function $L(z,t)$,$z \in U$,$t \geq 0$, is a fuzzy subordination chain if $L(.,t)$ is analytic and univalent in U. For all $t \geq 0$,$L(z,t)$ is continuously differentiable on $[0,\infty)$ for all $z \in U$, and

$$F_{L[U\times[0,\infty)]}(L(z,t_1)) \leq F_{L[U\times[0,\infty)]}(L(z,t_2)), \qquad t_1 \leq t_2.$$

Remark (Oros & Oros, 2011) Let the functions $f, g \in H(U)$ and g is an univalent functionthen $f \prec g$ if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. From here if g is an univalent function, then $f \prec_F g$ if and only if $f \prec g$.

Lemma 1.6 (Miller & Mocanu, 2000) Let $q \in Q(a)$ and let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...,$ be analytic in U,$q(z) \neq a$ and $n \geq 1$, if q is not subordinate to p, then there exist points $z_0 = r_0 e^i \in U$ and $\zeta_0 \in \partial U \setminus E(p)$ and $m \geq n \geq 1$ for which $q(U_{r_0}) \subset p(U)$,

1. $q(z_0) = p(\zeta_0)$,
2. $z_0 q^{'}(z_0) = m\zeta_0 p^{'}(\zeta_0)$, and
3. $Re\left(\frac{z_0 q^{''}(z_0)}{q^{'}(z_0)} + 1\right) \geq mRe\left(\frac{\zeta_0 p^{''}(\zeta_0)}{p^{'}(\zeta_0)} + 1\right)$.

Lemma 1.7 (Oros & Oros, 2012*b*) Let h be a convex function with $h(0) = a$, and let $\gamma \in C^*$ be a complex number with $Re(\gamma) \geq 0$. If $p \in H[a,n]$ with $p(0) = a$ and $\psi : C^2 \times U \to C$, $\psi(p(z), zp^{'}(z)) = p(z) + \frac{1}{\gamma} zp^{'}(z)$, is analytic in U, then

$$F_{\psi(C^2\times U)}[p(z) + \frac{1}{\gamma} zp^{'}(z)] \leq F_{h(U)}h(z), implies, F_{p(U)}p(z) \leq F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n-1}} dt.$$

The function q is convex and is the fuzzy best (a,n) dominant.

Lemma 1.8 (Oros & Oros, 2012*b*) Let h be starlike in U, with $h(0) = 0$. If $p \in H[0,1] \cap Q$ is univalent in U, then $zp^{'}(z) \prec_F h(z)$, implies $p(z) \prec_F q(z), z \in U$ where q is given by

$$q(z) = \int_0^z h(t) t^{-1} dt.$$

The function q is convex and is the fuzzy best dominant.

Lemma 1.9 (Pascu, 2006) If $L_\gamma : A \to A$ is the integral operator defined by $L_\gamma[f] = F$, given by $L_\gamma[f](z) = F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z h(t) t^{\gamma-1} dt$ and $Re(\gamma) > 0$ then

1. $L_\gamma[S^*] \subset S^*$
2. $L_\gamma[K] \subset K$
3. $L_\gamma[C] \subset C$.

Lemma 1.10 (Oros & Oros, 2012b) The function $L(z,t) = a_1(t)z + a_2(t)z^2 + ...$,with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is fuzzy subordination chain if and only if

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0, z \in U. \tag{1.4}$$

2. Main Results

Let

$$\Omega = supp(\Omega, F_\Omega) = \{z \in C : 0 < F_\Omega(z) \leq 1\},$$

$$\Delta = supp(\Delta, F_\Delta) = \{z \in C : 0 < F_\Delta(z) \leq 1\},$$

$$p(U) = supp(p(U), F_{p(U)}) = \{z \in C : 0 < F_{p(U)}(z) \leq 1\}$$

and

$$\varphi(C^3 \times U) = supp(\varphi(C^3 \times U), F_{\varphi(C^3\times U)}) = \{\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z\}$$

.

Definition 2.1 Let Ω be a set in C and $q \in H[a,n]$ with$q^{'}(z) \neq 0$. The class of admissible functions $\Phi_n[\Omega, q]$, consist of those functions $\varphi : C^3 \times U \to C$ that satisfy the admissibility condition:

$$F_{\varphi(C^3\times U)}(\varphi(r,s,t;\zeta) \leq F_\Omega(z).i.e.F_\Omega(\varphi(r,s,t;\zeta)) > 0, \tag{2.1}$$

Whenever

$$r = q(z), s = \frac{zq^{'}(z)}{m}, Re\left(\frac{t}{s}+1\right) \geq \frac{1}{m}Re\left(\frac{zq^{''}(z)}{q^{'}(z)}+1\right),$$

where $\zeta \in \partial U$, $z \in U$ and $m \geq n \geq 1$. When $n = 1$ we write $\Phi_1[\Omega, q]$ as $\Phi[\Omega, q]$.
In the special case when h is an analytic mapping of U onto $\Omega \neq C$, we denote this class $\Phi_n[h(U), q]$ by $\Phi_n[h, q]$.
If $\varphi : C^2 \times U \to C$ and $q \in H[a,n]$, then the admissibility condition (2.1) reduces to

$$F_\Omega(\varphi(q(z), (zq^{'}(z))/m; \zeta)) > 0, \quad \text{when} \quad z \in U, \zeta \in \partial U \quad \text{and} \quad m \geq n \geq 1$$

.
If $\varphi : C \times U \to C$, then the admissibility condition (2.1) reduces to $F_\Omega(\varphi(q(z);\zeta)) > 0$, when $z \in U, \zeta \in \partial U$.
Let (Ω, F_Ω) and(Δ, F_Δ) be any fuzzy sets in C, let p be an analytic function in the unit disc U and let $\varphi(r,s,t;z) : C^3 \times U \to C$. To study the following implication:

$$F_\Omega(z) \leq F_{\varphi(C^3\times U)}(\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)), \Rightarrow F_\Delta(z) \leq F_{p(U)}(p(z)). \tag{2.2}$$

There are there distinct cases to consider in analyzing this implication, which we list as the following Problems.

Problem 2.2 Given (Ω, F_{Ω}) and (Δ, F_{Δ}) any fuzzy sets in C, find condonations on the function φ so that (2.2) holds. We call such a φ an admissible function.
Problem 2.3 Given φ and (Ω, F_{Ω}), find (Δ, F_{Δ}) so that (2.2) holds. Furthermore, find the "largest" such Δ.
Problem 2.4 Given φ and (Δ, F_{Δ}), find (Ω, F_{Ω}) so that (2.2) holds. Furthermore find the "smallest" such Ω.
If either (Ω, F_{Ω}) or (Δ, F_{Δ}) in (2.2) is a simply connected domain. Then it may be possible to rephrase (2.2) in terms of fuzzy differential superordination. If p is univalent in U, and if (Δ, F_{Δ}) is simply connected domain with$\Delta \neq C$, then there is a conformal mapping q of U onto Δ such that $q(0) = p(0)$. In this case (2.2) can be rewritten as $F_{\Omega}(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp^{'}(z), z^2 p^{''}(z); z))$ implies

$$F_{q(U)}(q(z)) \leq F_{p(U)}p(z), z \in U, i.e.q(z) \prec_F p(z). \tag{2.3}$$

If (Ω, F_{Ω}) is also a simply connected domain and $\Omega \neq C$. Then there is a conformal mapping h of U onto Ω such that $h(0) = (p(0), 0, 0; 0)$, if in addition, the function $\varphi(p(z), zp^{'}(z), z^2 p^{''}(z); z)$ is univalent in U, then (2.3) can be rewritten as

$$h(z) \prec_F \varphi(p(z), zp^{'}(z), z^2 p^{''}(z); z), \quad \Rightarrow q(z) \prec_F p(z). \tag{2.4}$$

This implication also has meaning if h and q are analytic and not necessarily univalent.
Definition 2.5 Let $\varphi : C^3 \times U \to C$ and let h be analytic in U. If p and $\varphi(p(z), zp^{'}(z), z^2 p^{''}(z); z)$ are univalent in U and satisfy the (second-order) fuzzy differential superordination

$$F_{h(U)}h(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp^{'}(z), z^2 p^{''}(z); z))$$

.i.e.

$$h(z) \prec_F \varphi(p(z), zp^{'}(z), z^2 p^{''}(z); z),$$

then p is called a fuzzy solution of the fuzzy differential superordination. An analytic function q is called fuzzy subordinant of the fuzzy differential superordination, or more simply a fuzzy subordination if $q(z) \prec_F p(z)$,$z \in U$, for all p satisfying (2.4).
A univalent fuzzy subordination $\tilde{q}$ that satisfies $q \prec_F \tilde{q}$ for all fuzzy subordinate q of (2.4) is said to be the fuzzy best subordinate of (2.4).
Note that the fuzzy best subordinant is unique uo to a relation of U. In the special case when the set inclusions of (2.2) can be replaced by the fuzzy superordination of (2.4) we can reinterpret the three problem referred to above as follows:
Problem 2.6 Given analytic functions h and q , find a class of admissible functions $\Phi[h, q]$ such that (2.4) holds.
Problem 2.7 Given the fuzzy differential superordination in (2.4), find a fuzzy subordination q, moreover, find the fuzzy best subordinant.
Problem 2.8 Given φ and fuzzy subordinant q, find the largest class of analytic function h such that holds. The next theorem is key results.

Theorem 2.9 Let $\varphi \in \Phi_n[\Omega, q]$ and let $q \in H[a, n]$ .If $p \in Q(a)$ and $\varphi(p(z), zp^{'}(z), z^2 p^{''}(z); z)$ is univalent in U, then

$$F_{\Omega}(z) \leq F_{\varphi(C^3 \times U)}(\varphi(p(z), zp^{'}(z), z^2 p^{''}(z); z)), \quad z \in U \quad \Rightarrow q(z) \prec_F p(z). \tag{2.5}$$

Proof. Form (2.5) and Definition (1.3). We have

$$\Omega \subset (\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)). \tag{2.6}$$

Assume $q(z) \not\prec p(z)$. By Lemma (1.6), there exist points $z_0 = r_0e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(p)$ and $m \geq n \geq 1$.
That satisfy

$$q(z_0) = p(\zeta_0), z_0q^{'}(z_0) = m\zeta_0p^{'}(\zeta_0) \quad \text{and} \quad Re\left(\frac{zq^{''}(z)}{q^{'}(z)} + 1\right) \geq mRe\left(\frac{\zeta_0p^{''}(\zeta_0)}{p^{'}(\zeta_0)} + 1\right).$$

Using these condition with $r = p(\zeta_0), s = \zeta_0p^{'}(\zeta_0), t = \zeta_0^2p^{''}(\zeta_0)$ and $\zeta = \zeta_0$ in definition (2.1) we obtain

$$F_{\varphi(C^3\times U)}(\varphi(p(\zeta_0), \zeta_0p^{'}(\zeta_0), \zeta_0^2p^{''}(\zeta_0); \zeta_0)) \leq F(\zeta_0) \tag{2.7}$$

Since this contradict (2.6) we must have $q(z) \prec_F p(z)$.
We next consider the special situation when h is analytic on U and $h(U) = \Omega \neq C$. In this case, the class $\Phi_n[h(U), q]$ is written as $\Phi_n[h, q]$ and the following result an immediate consequence of Theorem (2.9).
Theorem 2.10 Let $q \in H[a, n]$, let h be analytic in U and let $\varphi \in \Phi_n[h, q]$, if $p \in Q(a)$ and $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ is univalent in U, then

$$h(z) \prec_F \varphi(p(z), zp^{'}(z), z^2p^{P''}(z); z) \Rightarrow q(z) \prec_F p(z). \tag{2.8}$$

Theorem (2.9) and Theorem (2.10) can only used to obtain fuzzy subordinates of a fuzzy differential superordination of the form (2.6) or (2.8) the following theorem proves the existence of the fuzzy best subordinate of q for certain φ and also provides a method for finding the fuzzy best subordinant.
Theorem 2.11 Let h be analytic in U and let $\varphi : C^3 \times U \to C$ suppose that the differential equation

$$\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z) = h(z), \tag{2.9}$$

has a solution $q \in Q(a)$.If $\varphi \in \Phi_n[h, q], p \in Q(a)$ and $\varphi(p(z), zp^{'}(z), z^2p^{''}(z); z)$ is univalent in U, then

$$h(z) \prec_F \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z) \Rightarrow q(z) \prec_F p(z), \tag{2.10}$$

and q is the best subordinate.
Proof. Since $\varphi \in \Phi[h, q]$, by applying Theorem (2.10) we deduce that q is a fuzzy subordinant, of (2.10), since q also satisfies (2.9), it is also a solution of the fuzzy differential superordination (2.10) and therefore all subordinates of (2.10) will be fuzzy subordinant to q. Hence q will be the fuzzy best subordinant of (2.10). From this theorem we see that the problem of finding the fuzzy best subordinant of (2.10) essentially reduces to showing that differential equation 2.9 has a univalent solution and checking that $\varphi \in \Phi_n[h, q]$. The conclusion of the theorem can written in the symmetric form.

$$\varphi(q(z), zq^{'}(z), z^2q^{''}(z); z) \prec_F \varphi(p(z), zp^{'}(z), z^2p^{''}(z); z), \Rightarrow q(z) \prec_F p(z).$$

We can simplify Theorems (2.9), (2.10) and (2.11) for the case of first-order fuzzy differential subordination. The following results are immediately obtained by using these theorems and admissibility condition (2.1).

Theorem 2.12 Let $\Omega \subset C, q \in H[a,n], \varphi : C^2 \times U \to C$ and suppose that $F_{\varphi(C^2\times U)}\varphi(q(z), zq^{'}(z); \zeta) \leq F_{\Omega}(z)$, for $z \in U, \zeta \in \partial U$ and $0 < t \leq \frac{1}{n} < 1$, if $p \in Q(a)$ and $\varphi(p(z), zp^{'}(z); z)$ is univalent in U, then

$$F_{\Omega}(z) \leq F_{\varphi(C^2\times U)}\varphi(p(z), zp^{'}(z); z) \Rightarrow F_{q(U)}q(z) \leq F_{p(U)}p(z) \quad i.e. \quad q(z) \prec_F p(z).$$

Theorem 2.13 Let h be univalent in U, $q \in H[a,n], \varphi : C^2 \times U \to C$ and suppose that $F_{\varphi(C^2\times U)}(\varphi(q(z), zq^{'}(z); z)) \leq F_{h(U)}h(z)$, for $z \in U, \zeta \in \partial U$ and $0 < t \leq \frac{1}{n} < 1$, if $p \in Q(a)$ and $\varphi(p(z), zp^{'}(z); z)$ is univalent in U, then

$$h(z) \prec_F \varphi(p(z), zp^{'}(z); z) \Rightarrow q(z) \prec_F p(z). \tag{2.11}$$

Furthermore if $\varphi(p(z), zp^{'}(z); z) = h(z)$ has a univalent solution $q \in Q(a)$ then q is fuzzy best subordinant.

Georgia and Gheorghe (Oros & Oros, 2012*b*) considered the fuzzy subordination

$$F_{\psi(C^2\times U)}[p(z) + \frac{1}{\gamma}zp^{'}(z)] \leq F_{h_2(U)}h_2(z), \tag{2.12}$$

where h_2 is convex function in U,$h_2(0) = a$,$\gamma \neq 0$ and $Re(\gamma) \geq 0$. They showed if $p \in H[a,1]$ satisfies(2.12), then

$$F_{p(U)}p(z) \leq F_{q_2(U)}q_2(z) \leq F_{h_2(U)}h_2(z), z \in U, \tag{2.13}$$

where

$$q_2(z) = \frac{1}{nz^{\gamma}} \int_0^z h_2(t)t^{\gamma-1}dt.$$

The function q is convex and is the fuzzy best dominant of (2.12).

We next prove an analogous result for the corresponding fuzzy differential subordination.

Theorem 2.14 Let $h_1$be convex in U, with $h_1(0) = a$,$\gamma \neq 0$, with $Re(\gamma) \geq 0$, and $p \in H[a,1] \cap Q$ if $p(z) + \frac{1}{\gamma}zp^{'}(z)$ is univalent in U,

$$h_1(z) \prec_F \frac{1}{\gamma}zp^{'}(z), \tag{2.14}$$

and

$$q_1(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h_1(t)t^{\gamma-1}dt, \tag{2.15}$$

then $q_1(z) \prec_F p(z)$, and the function q_1 is convex and is the fuzzy best subordinate.

Proof. If we let ,$\varphi : C^2 \times U \to C, \varphi(r,s) = r + \frac{1}{\gamma}s$, for $r = p(z), s = zp^{'}(z), z \in U$, then relation (2.14) becomes

$$h_1(z) \prec_F \varphi(p(z), zp^{'}(z); z).$$

The integral given by (2.15), with the exception of a different normalization $q(0) = a$ has the form

$$q_1(z) = \frac{\gamma}{nz^\gamma}\int_0^z h(t)t^{\gamma-1}dt = \frac{\gamma}{nz^\gamma}\int_0^z (a + a_n z^n + \dots)t^{\gamma-1}dt$$
$$= a + \frac{a_1}{\gamma+1}z + \dots, z \in U,$$

which gives $q \in [a, 1]$, since h is convex and $Re(\gamma) \geq 0$, we deduce form (2) of Lemma (1.10) that q is convex and univalent. A simple colocation shows that q_1 also satisfies the differential equation.

$$q_1(z) + \frac{1}{\gamma}zq_1^{'}(z) = \varphi(p(z), zp^{'}(z)) = h_1(z). \tag{2.16}$$

Since q_1 is the univalent solution of the differential equation (2.16) associated with fuzzy differential subordination (2.14), we can prove that it is the fuzzy best subordinate of (2.14) by applying Theorem (2.13). Without loss of generality, we can assume that h_1 and q_1 are analytic and univalent on $\overline{U}$ and $q_1^{'}(\zeta)$ for $|\zeta| = 1$. If not, then we could replace h_1 with $h_1(\rho z)$ and q_1 with $q_1(\rho z)$, where $0 < \rho < 1$. These new function would then have the desired properties and we would prove the Theorem by using Theorem (2.14) and then letting $\varphi \to 1$ with our assumptions, to apply Theorem (2.13) only need to show that $\varphi \in \Phi[h_1, q_1]$. This is equivalent to showing that

$$\varphi_0 = \varphi(q_1(z) + tzq_1^{'}(z)) = q_1(z) + \frac{1}{\gamma}zq_1^{'}(z) \in h_1(U),$$

for $z \in U$ and $t \in (0, 1]$. Form (2.16) we see that (2.12) is satisfied with p, h replaced by q_1, h_1. Hence, from (2.11) we obtain

$$q_1(z) \prec_F p(z).$$

Since $h_1(U)$ is convex domain and $t \in (0, 1]$. We conclude that $\varphi_0 \in h_1(U)$ which proves that q_1 is the fuzzy best subordinat.

Theorem 2.15 Let $q \in H[a, 1]$,$\varphi : C^2 \times U \to C$ and set $\varphi(q(z), zq^{'}(z)) = h(z)$. If $L(z, t) = \varphi(q(z), tzq^{'}(z))$ is fuzzy subordination chain and $p \in H[a, 1] \cap Q$, then

$$h(z) \prec_F \varphi(p(z), zp^{'}(z)) \Rightarrow (z) \prec_F p(z).$$

Furthermore, if $\varphi(q(z), zq^{'}(z)) = h(z)$, has a univalent solution $q \in Q$,then q is the fuzzy best subordinant.

Proof. Since L is a fuzzy subordination chain $L(z, t) \prec_F L(z, 1)$,or equivalently, $(p(z), zp^{'}(z)) \prec_F h(z)$, for all $z \in U$ and $t \in (0, 1]$. Since this implies that (2.11) is satisfied ,we obtain the desired conclusion by applying of this result by again considering the fuzzy differential superordination

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2\times U)}(p(z) + \frac{1}{\gamma}zp^{'}(z)), \tag{2.17}$$

with corresponding differential equation

$$q(z) + \frac{1}{\gamma}zq^{'}(z) = h(z). \tag{2.18}$$

In Theorem (2.14), we assumed that h in (2.18) was convex, which implied that solution q was convex. On the other hand if we assuming that q is convex and h is defined by (2.18) and by simple calculation we have

$$Re\left\{\frac{h'(z)}{q'(z)}\right\} = Re\left\{\frac{(\gamma+1)}{\gamma} + \frac{1}{\gamma}\frac{zq''(z)}{q'(z)}\right\} > 0,$$

then h is close to convex, therefore h is univalent function.By using the fuzzy subordination chain as given in Theorem (2.15) to obtain fuzzy best subordinant for (2.17).

In next theorem we introduce an example of a solution of problem (2.4),(2.8) referred to the introduction.

Theorem 2.16 Let q be convex in U and let h be defined by $h(z) = q(z) + \frac{1}{\gamma}zq'(z)$,with $Re(\gamma) > 0$. If $p \in [a, 1] \cap Q$,$p(z) + \frac{1}{\gamma}zp'(z)$ is univalent in U and

$$F_{h(U)}(h(z)) \leq F_{(C^2\times U)}(p(z) + \frac{1}{\gamma}zp'(z))$$

then

$$q(z) \prec_F p(z),$$

where

$$q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t)t^{\gamma-1}dt.$$

The function q is the fuzzy best subordinant.

Proof . Let $L(z, t) = \varphi(q(z), tzq'(z)) = q(z) + \frac{t}{\gamma}zq'(z)$. By simple calculation, we get

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = Re\left\{\gamma + t\frac{zq''(z)}{q'(z)}\right\},$$

since q convex function ,$Re(\gamma) > 0$ and $t \in (0, 1]$,we obtain

$$Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\}$$

Using lemma (1.9), we deduce that L is fuzzy subordination chain. By Theorem (2.15), we conclude that q is fuzzy subordinant of fuzzy differential superordination

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2\times U)}(p(z) + \frac{1}{\gamma}zp'(z)),$$

Furthermore, since q is a univalent solution of (2.18), it also is the fuzzy best subordinant of

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2\times U)}(p(z) + \frac{1}{\gamma}zp'(z)).$$

Example 2.17 . Let $h(z) = \frac{1-z}{1+z}$ and $p(z) = 1+z, z \in U$, it is clear to show that $h(0) = 1, h'(z) = \frac{-2}{(1+z)^2}$, $h''(z) = \frac{4}{(1+z)^3}$ and $p(z) + zp'(z) = 1 + 2z$. Since

$$Re\left\{\frac{zh''(z)}{h'(z)} + 1\right\} = Re\left\{\frac{1-z}{1+z}\right\} = Re\left\{\frac{(1 - r(\cos\theta + i\sin\theta)}{(1 + r(\cos\theta + i\sin\theta)}\right\}$$

$$= \frac{1-r^2}{1 + 2r\cos\theta + r^2} > 0,$$

where $r = |z| < 1, \theta \in R$. Then the function h is convex in U.
We have

$$q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = \frac{2\ln(1+z)}{z} - 1.$$

Using Theorem (2.14),we obtain $\frac{1-z}{1+z} \prec_F 1 + 2z$,induce $\frac{2\ln(1+z)}{z} - 1 \prec_F 1 + z, \quad z \in U.$
In next result, we introduce fuzzy differential superordination for which the fuzzy subordinant function h is a starlike function.
Theorem 2.18 Let h be starlike in U , with $h(0) = 0$,If $p \in [0, 1] \cap Q$ and $zp'(z)$ is univalent in U, then

$$F_{h(U)}(h(z)) \leq F_{\varphi(C^2\times U)}(zp'(z)) \Rightarrow F_{q(U)}q(z) \leq F_{p(U)}p(z), z \in U. \tag{2.19}$$

Where

$$q(z) = \int_0^z h(t)t^{-1}dt, \tag{2.20}$$

The function q is convex and is the fuzzy best subordinant.
Proof. Differentiating (2.20), we have $zq'(z) = h(z)$, if we let $\varphi : C^2 \times U \to C, \varphi(s) = s$, for $s = zp'(z), z \in U$,relation (2.19) becomes

$$F_{h(U)}h(z) \leq F_{\varphi(C^2\times U)}(\varphi(zp'(z))),$$

the function q is the solution of $\varphi(zq'(z)) = zq'(z) = h(z)$. Since h is starlike, we deduce from Alexander's theorem that q is convex and univalent. As in the previous theorem we can assume that h and q are analytic and univalent on$\overline{U}$ and $q'(\zeta) \neq 0$ for$|\zeta| = 1$, the conclusion of this theorem follows form Theorem (2.13), if we show that $\varphi \in \Phi[h, q]$, we get this immediately since $h(U)$ is starlike domain and

$$\varphi(tzq'(z)) = tzq'(z) = th(z) \in h(U), \quad z \in U \quad \text{and} \quad 0 < t \leq 1 \tag{2.21}$$

Form (2.21), we have

$$F_{\varphi(C^2\times U)}(tzq'(z)) \leq F_{h(U)}(h(z))$$

Using Definition (2.1), we obtain $\varphi \in \Phi[h, q]$,and applying Theorem (2.13), we conclude that q is the fuzzy best subordinant.
Example 2.19 Let $h(z) = z + z^2, z \in U$ It is clear to show that

$$Re\left\{\frac{zh'(z)}{h(z)}\right\} = Re\left\{1 + \frac{z}{1+z}\right\} = Re\left\{1 + \frac{r(\cos\theta + i\sin\theta)}{1 + r(\cos\theta + i\sin\theta)}\right\}$$
$$= 1 + \frac{1 + r\cos\theta}{1 + 2r\cos\theta + r^2} > 0.$$

If $p \in H[0, 1] \cap Q$ and $zp'(z)$ is univalent in U, then

$$z + z^2 \prec_F zp'(z) \Rightarrow z + z^2 \prec_F p(z).$$

In the next section, we will combine some theorems for the Sandwich theorem.

3. Sandwich theorem

We can combine Theorem (2.14) to gather with Lemma (1.7) to obtain the following fuzzy differential "sandwich theorem".

Theorem 3.1 Let h_1 and h_2 be convex in U, with $h_1(z) = h_2(z) = a$. Let $\gamma \neq 0$, with $Re(\gamma) > 0$ and let the function q_i be defined by

$$q_i = \frac{\gamma}{z^\gamma} \int_0^z h_i(t) t^{\gamma-1} dt,$$

for $i = 1, 2$. If $p \in H[a, 1] \cap Q$ and $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent , then

$$h_1(z) \prec_F p(z) + \frac{1}{\gamma} z p'(z) \prec_F h_2(z) \Rightarrow q_1(z) \prec_F p(z) \prec_F q_2(z), \quad z \in U. \tag{3.1}$$

The function, q_1 and q_2 are convex and they are respectively the fuzzy best subordinant and fuzzy best dominant.

If we set $f(z) = p(z) + \frac{1}{\gamma} z p'(z)$, then (3.1), can be expressed as the following "sandwich theorem"involving fuzzy subordination preserving integral operator.

Corollary 3.2 Let h_1 and h_2 be convex in U and f be univalent function in U, with $h_1(0) = h_2(0) = f(0)$, Let $\gamma \neq 0$, with $Re(\gamma) > 0$. If

$$h_1(z) \prec_F f(z) \prec_F h_2(z),$$

then

$$\frac{\gamma}{z^\gamma} \int_0^z h_1(t) t^{\gamma-1} dt \prec_F \frac{\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \prec_F \frac{\gamma}{z^\gamma} \int_0^z h_2(t) t^{\gamma-1} dt$$

When the middle integral is univalent. If we combine Theorem (2.18), with Lemma (1.8), we obtain the following "sandwich result".

Theorem 3.3 Let h_1 and h_2 be starlike functions in U, with $h_1(0) = h_1(0) = 0$ and let the function $q_(i)$ be defined by

$$q_i = \int_0^z h_i(t) t^{-1} dt,$$

for $i = 1, 2$. If $p \in H[0, 1] \cap Q$ and $z p'(z)$is univalent in U, then

$$h_1(z) \prec_F z p'(z) \prec_F h_2(z) \Rightarrow q_1(z) \prec_F p(z) \prec_F q_2(z), \quad z \in U.$$

The functions q_1 and q_2 are convex and they are respectively the fuzzy best subordinant and fuzzy best dominant. If we set $f(z) = z p'(z)$, then this last theorem can be expressed as the following "sandwich theorem"involving fuzzy subordination preserving integral operator.

Corollary 3.3 Let h_1 and h_2 be starlike functions in U and f be univalent in U, with $h_1(0) = h_2(0) = 0$. If

$$h_1(z) \prec_F f(z) \prec_F h_2(z), \quad z \in U,$$

then

$$\int_0^z h_1(t) t^{-1} dt \prec_F \int_0^z f(t) t^{-1} dt \prec_F \int_0^z h_2(t) t^{-1} dt \prec_F$$

when the middle integral is univalent.

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