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On the Growth of Solutions of Higher Order Complex Differential Equations with finite $[p, q]$-Order

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Abstract

In this paper, we study the growth of entire solutions of higher order linear complex differential equations with entire coefficients of finite $[p, q]$-order. We give another conditions that generalize some results due to (Belaïdi, 2015), (Liu *et al.*, 2010) and (Li & Cao, 2012).

Keywords: Complex differential equation, Meromorphic solution, Entire solution, $[p, q]$-Order.
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1. Introduction

In this article, we use the standard notation and fundamental results of the Nevanlinna value distribution theory of meromorphic functions, see (Hayman, 1964; Laine, 1993; Yang & Yi, 2003). We define, for $r \in [0, +\infty)$, $\exp_0 r := r$, $\exp_1 r := e^r$ and $\exp_{n+1} r := \exp(\exp_n r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log_0 r := r$, $\log_1 r := \log r$ and $\log_{n+1} r := \log(\log_n r)$, $n \in \mathbb{N}$. Moreover, we denote by $\exp_{-1} r := \log r$ and $\log_{-1} r := \exp_1 r$.

For a meromorphic function f in complex plane $\mathbb{C}$, the order of growth is defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. The exponents of convergence of sequence of the zeros and distinct zeros of f are respectively defined by

$$\lambda(f) = \limsup_{r\to+\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}, \quad \overline{\lambda}(f) = \limsup_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r},$$

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where $N\left(r,\frac{1}{f}\right)$ $\left(\text{resp. } \overline{N}\left(r,\frac{1}{f}\right)\right)$ is the integrated counting function of zeros (resp. distinct zeros) of $f(z)$ in the disc $\{z : |z| \le r\}$.

(Juneja *et al.*, 1976, 1977) have investigated some properties of entire functions of $[p,q]$-order and obtained some results about their growth. In order to maintain accordance with general definitions of the entire function f of iterated p-order[1], (Liu *et al.*, 2010) gave a minor modification of the original definition of the $[p,q]$-order given by (Juneja *et al.*, 1976, 1977).

We recall the following definition,

Definition 1.1. (Kinnunen, 1998) Let $p \ge 1$ be an integer. The iterated p-order $\sigma_p(f)$ of a meromorphic function f is defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r}.$$

Now, we shall introduce the definition of meromorphic functions of $[p,q]$-order, where p,q are positive integers satisfying $p \ge q \ge 1$ or $2 \le q = p+1$. In order to keep accordance with Definition 1.1, (Li & Cao, 2012; Belaïdi, 2015) have gave a minor modification to the original definition of $[p,q]$-order (e.g. see, (Juneja *et al.*, 1976, 1977)). We recall the following definitions

Definition 1.2. (Belaïdi, 2015; Li & Cao, 2012; Liu *et al.*, 2010) Let $p \ge q \ge 1$ or $2 \le q = p+1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the $[p,q]$-order is defined by

$$\sigma_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

It is easy to see that $0 \le \sigma_{[p,q]}(f) \le +\infty$. If $f(z)$ is rational, then $\sigma_{[p,q]}(f) = 0$ for any $p \ge q \ge 1$. By Definition 1.2, we note that $\sigma_{[1,1]}(f) = \sigma(f)$ (order of growth), $\sigma_{[2,1]}(f) = \sigma_2(f)$ (hyper-order), $\sigma_{[1,2]}(f) = \sigma_{\log}(f)$ (logarithmic order) and $\sigma_{[p,1]}(f) = \sigma_p(f)$ (iterated p-order).

Definition 1.3. (Belaïdi, 2015; Li & Cao, 2012) Let $p \ge q \ge 1$ or $2 \le q = p+1$ be integers. The $[p,q]$ convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q r}.$$

Similarly, the $[p,q]$ convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q r}.$$

[1] see (Kinnunen, 1998), for the definition of the iterated p-order.

We recall also the following definitions. The linear measure of a set $E \subset (0, +\infty)$ is defined as

$$m(E) = \int_0^{+\infty} \chi_E(t)dt$$

and the logarithmic measure of a set $F \subset (1, +\infty)$ is defined as

$$\ell m(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt,$$

where $\chi_H(t)$ is the characteristic function of the set H. The upper density of a set $E \subset (0, +\infty)$ is defined by

$$\overline{\text{dens}}\,(E) = \limsup_{r\to+\infty} \frac{m\,(E \cap [0, r])}{r}.$$

The upper logarithmic density of a set $F \subset (1, +\infty)$ is defined by

$$\overline{\log\text{dens}}\,(F) = \limsup_{r\to+\infty} \frac{\ell m\,(F \cap [1, r])}{\log r}.$$

Proposition 1.1. (Belaïdi, 2015) *For all $H \subset [1, +\infty)$ the following statements hold :*
(i) If $\ell m\,(H) = \infty$, then $m\,(H) = \infty$,
(ii) if $\overline{\text{dens}}\,(H) > 0$, then $m\,(H) = \infty$,
(iii) if $\overline{\log\text{dens}}\,(H) > 0$, then $\ell m\,(H) = \infty$.

For $a \in \overline{\mathbb{C}}$, the deficiency of a with respect to a meromorphic function f is defined by

$$\delta\,(a, f) = \liminf_{r\to+\infty} \frac{m\left(r, \frac{1}{f-a}\right)}{T\,(r, f)} = 1 - \limsup_{r\to+\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{T\,(r, f)}.$$

Consider the differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = 0. \tag{1.1}$$

(Liu *et al.*, 2010) studied the growth of solutions of the homogeneous differential equation (1.1) with coefficients that are entire functions of finite $[p, q]$-order and obtained following result

Theorem 1.1. (Liu *et al.*, 2010) *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying* $\max\left\{\sigma_{[p,q]}(A_j) : j \neq s\right\} < \sigma_{[p,q]}(A_s) < \infty$*. Then every solution $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_s)$. Furthermore, at least one solution of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$.*

Theorem 1.2. (Liu *et al.*, 2010) *Let $A_0, A_1, ..., A_{k-1}$ be entire functions, and let $s \in \{0, ..., k-1\}$ be the largest index for which $\sigma_{[p,q]}(A_s) = \max_{0\leq j\leq k-1} \sigma_{[p,q]}(A_j)$. Then there are at least $k - s$ linearly independent solutions $f(z)$ of (1.1) such that $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$. Moreover, all solutions of (1.1) satisfy $\sigma_{[p+1,q]}(f) \leq \rho$ if and only if $\sigma_{[p,q]}(A_j) \leq \rho$ for all $j = 0, 1, ..., k-1$.*

Theorem 1.3. (Liu *et al.*, 2010) *Let H be a set of complex numbers satisfying $\overline{\text{dens}}\{|z| : z \in H\} > 0$ and let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \alpha.$$

Suppose that there exists a positive constant β satisfying $\beta < \alpha$ such that any given $\varepsilon\,(0 < \varepsilon < \alpha - \beta)$, we have

$$|A_0(z)| \ge \exp_{p+1}\left\{(\alpha - \varepsilon)\log_q r\right\}$$

and

$$\left|A_j(z)\right| \le \exp_{p+1}\left\{\beta \log_q r\right\} \quad (j = 1, \ldots, k-1)$$

for $z \in H$. Then, every solution $f \not\equiv 0$ of the equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \alpha$.

Recently, (Belaïdi, 2015) has obtained the following results which generalize and improve Theorem 1.3 and also improve some results due to (Li & Cao, 2012).

Theorem 1.4. (Belaïdi, 2015) *Let H be a set of complex numbers satisfying $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$ and let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be meromorphic functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α and β satisfying $0 \le \beta < \alpha$ such that

$$|A_0(z)| \ge \exp_p\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right) \tag{1.2}$$

and

$$\left|A_j(z)\right| \le \exp_p\left(\beta\left[\log_{q-1} r\right]^{\rho}\right), \quad (j = 1, \cdots, k-1) \tag{1.3}$$

as $|z| = r \to +\infty$ for $z \in H$. Then the following statements hold :
(i) If $p \ge q \ge 2$ or $3 \le q = p + 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
(ii) If $p = 1, q = 2$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[2,2]}(f) \ge \rho$.

Theorem 1.5. (Belaïdi, 2015) *Let H be a set of complex numbers satisfying $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$ and let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be meromorphic functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two positive constants α and β such that, we have

$$m(r, A_0) \ge \exp_{p-1}\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right) \tag{1.4}$$

and

$$m(r, A_j) \le \exp_{p-1}\left(\beta\left[\log_{q-1} r\right]^{\rho}\right), \quad (j = 1, \cdots, k-1) \tag{1.5}$$

as $|z| = r \to +\infty$ for $z \in H$. Then the following statements hold :
(i) If $p \ge q \ge 2$ and $0 \le \beta < \alpha$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
(ii) If $3 \le q = p + 1$, $0 \le \beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \not\equiv 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,p+1]}(f) = \rho$.
(iii) If $p = 1, q = 2$, $0 \le (k-1)\beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \not\equiv 0$ of equation (1.1) satisfies $\sigma_{[2,2]}(f) \ge \rho$.

2. Main results

Now, a natural question is whether somewhat similar results to Theorem 1.4 and Theorem 1.5 could be obtained for the differential equation (1.1), where $A_j(z)$ $(j = 0, 1, \cdots, k)$ are entire functions and the dominant coefficient is some $A_s(z)$ $(0 \le s \le k-1)$ instead of $A_0(z)$? The main purpose of this article is to answer the above question and improving and generalizing the previous results.

Theorem 2.1. *Let H be a set of complex numbers satisfying $\overline{\log\operatorname{dens}}\{|z| : z \in H\} > 0$. Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α and β satisfying $0 \le \beta < \alpha$ and let $s \in \{0, ..., k-1\}$ be an integer for which

$$|A_s(z)| \ge \exp_p\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right), \ 0 \le s \le k-1 \tag{2.1}$$

and

$$\left|A_j(z)\right| \le \exp_p\left(\beta\left[\log_{q-1} r\right]^{\rho}\right), \quad j \ne s, \tag{2.2}$$

as $|z| = r \to +\infty$, $z \in H$. Then,
(i) If $p \ge q \ge 1$, then every polynomial solution $f \not\equiv 0$ of equation (1.1) is of $\deg f \le s-1$ $(s \ge 1)$ and every transcendental solution f of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
(ii) If $2 \le q = p+1, \rho > 1$, then every polynomial solution $f \not\equiv 0$ of equation (1.1) is of $\deg f \le s-1$ $(s \ge 1)$ and every transcendental solution f of equation (1.1) satisfies $\rho \le \sigma_{[p+1,p+1]}(f) \le \rho + 1$.

Corollary 2.1. *Let H be a set of complex numbers satisfying $\overline{\log\operatorname{dens}}\{|z| : z \in H\} > 0$. Let $F(z) \not\equiv 0, A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions. Suppose that $H, A_j(z)$ $(j = 0, 1, \ldots, k-1)$ satisfy the hypotheses in Theorem 2.1. Consider the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F. \tag{2.3}$$

(i) Let $p \ge q \ge 1$, if $\sigma_{[p+1,q]}(F) \le \rho$, then every transcendental solution f of equation (2.3) satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$; if $\rho_{[p+1,q]}(F) > \rho$, then every transcendental solution f of equation (2.3) satisfies $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) Let $2 \le q = p+1$ and $\rho > 1$, if $\sigma_{[p+1,p+1]}(F) \le \rho$, then every transcendental solution f of equation (2.3) satisfies $\overline{\lambda}_{[p+1,p+1]}(f) = \lambda_{[p+1,p+1]}(f) = \sigma_{[p+1,p+1]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$; if $\rho_{[p+1,p+1]}(F) > \rho$, then every transcendental solution f of equation (2.3) satisfies $\rho_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(F)$.

Theorem 2.2. *Let H be a set of complex numbers satisfying $\overline{\log\operatorname{dens}}\{|z| : z \in H\} > 0$. Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 0, 1, \ldots, k-1\right\} \le \rho, \quad 0 < \rho < +\infty.$$

Suppose that there exist two real numbers α and β satisfying $0 \leq \beta < \alpha$ and let $s \in \{0, ..., k-1\}$ be an integer for which

$$m(r, A_s) \geq \exp_{p-1}\left(\alpha \left[\log_{q-1} r\right]^{\rho}\right), \ 0 \leq s \leq k-1 \tag{2.4}$$

and

$$m\left(r, A_j\right) \leq \exp_{p-1}\left(\beta \left[\log_{q-1} r\right]^{\rho}\right), \ \ j \neq s, \tag{2.5}$$

as $|z| = r \to +\infty$, $z \in H$. Then the following statements hold :
(i) If $p \geq q \geq 1$ and $0 \leq \beta < \alpha$, then every polynomial solution $f \not\equiv 0$ of (1.1) is of $\deg f \leq s-1$ $(s \geq 1)$, and every transcendental solution f satisfies $\sigma_{[p,q]}(f) \geq \rho \geq \sigma_{[p+1,q]}(f)$.
(ii) If $2 \leq q = p+1$ and $0 \leq (k-1)\beta < \alpha$, then every polynomial solution $f \not\equiv 0$ of (1.1) is of $\deg f \leq s-1$ $(s \geq 1)$, and every transcendental solution f satisfies $\rho \leq \sigma_{[p,p+1]}(f)$ and $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$.

3. Some preliminary lemmas

Lemma 3.1. (Gundersen, 1988) *Let f be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exists a set $E_1 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and $s, j (0 \leq s < j)$, such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$*

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right]^{j-s}.$$

Lemma 3.2. (Gundersen, 1988) *Let f be a meromorphic function, and let j be a given positive integer, and let $\alpha > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r \geq R$ we have*

$$T\left(r, f^{(j)}\right) \leq (j+2)\, T(\alpha r, f).$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, $\mu_f(r)$ be the maximum term, i.e., $\mu_f(r) = \max\{|a_n| r^n; n = 0, 1, \cdots\}$, and let $\nu_f(r)$ be the central index of f, i.e., $\nu_f(r) = \max\{m; \mu_f(r) = |a_m| r^m\}$.

Lemma 3.3. (Hayman, 1974) *Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r, f)$. Then for all $|z| = r$ outside a set E_2 of r of finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \ \ j \in \mathbb{N},$$

where $\nu_f(r)$ is the central index of $f(z)$.

Lemma 3.4. (Juneja *et al.*, 1976) *Let $f(z)$ be an entire function of $[p, q]$-order, and let $\nu_f(r)$ be the central index of $f(z)$. Then*

$$\sigma_{[p,q]}(f) = \limsup_{r \to +\infty} \frac{\log_p \nu_f(r)}{\log_q r}.$$

Lemma 3.5. *Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions of finite $[p,q]$-order. Then,*
(i) If $p \geq q \geq 1$, then every solution $f \not\equiv 0$ of equation (1.1) satisfies

$$\sigma_{[p+1,q]}(f) \leq \max\left\{\sigma_{[p,q]}(A_j) : j = 0, 1, \ldots, k-1\right\}.$$

(ii) If $2 \leq q = p+1$, then every solution $f \not\equiv 0$ of equation (1.1) satisfies

$$\sigma_{[p+1,p+1]}(f) \leq \max\left\{\sigma_{[p,p+1]}(A_j) : j = 0, 1, \ldots, k-1\right\} + 1.$$

Proof. We prove only (ii) . For the proof of (i) see (Liu *et al.*, 2010). Let $f \not\equiv 0$ be a solution of equation (1.1) . By (1.1) , we have

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}|\left|\frac{f^{(k-1)}}{f}\right| + |A_{k-2}|\left|\frac{f^{(k-2)}}{f}\right| + \cdots + |A_1|\left|\frac{f'}{f}\right| + |A_0|. \tag{3.1}$$

Set $\max\left\{\sigma_{[p,p+1]}(A_j) : j = 0, 1, \ldots, k-1\right\} = \rho$. For any given $\varepsilon > 0$, when r is sufficiently large, we have

$$\left|A_j(z)\right| \leq \exp_{p+1}\left((\rho+\varepsilon)\left[\log_{p+1} r\right]\right), \; j = 0, 1, ..., k-1. \tag{3.2}$$

By Lemma 3.3, there exists a set $E_2 \subset [1, +\infty)$ with logarithmic measure $\ell m E_2 < \infty$, we can choose z satisfying $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r, f)$, such that

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \; j = 1, ..., k \tag{3.3}$$

holds. Substituting (3.2) and (3.3) into (3.1) , we obtain

$$\left(\frac{\nu_f(r)}{|z|}\right)^k |1 + o(1)| \leq k \exp_{p+1}\left((\rho+\varepsilon)\left[\log_{p+1} r\right]\right)\left(\frac{\nu_f(r)}{|z|}\right)^{k-1} |1 + o(1)|, \tag{3.4}$$

where z satisfies $|z| = r \notin [0,1] \cup E_2$ and $|f(z)| = M(r, f)$. By (3.4), we get

$$\nu_f(r)|1 + o(1)| \leq kr|1 + o(1)| \exp_{p+1}\left((\rho+\varepsilon)\left[\log_{p+1} r\right]\right). \tag{3.5}$$

So, from (3.5), we obtain

$$\limsup_{r\to+\infty} \frac{\log_{p+1} \nu_f(r)}{\log_{p+1} r} \leq \rho + 1 + \varepsilon. \tag{3.6}$$

Since $\varepsilon > 0$ is arbitrary, by (3.6) and Lemma 3.4 we have $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$. □

Remark. Lemma 3.5 (ii) has been proved for $p = 1$ and $q = 2$ by (Cao *et al.*, 2013).

Lemma 3.6. (Chen & Shon, 2004) *Let $f(z)$ be a transcendental entire function. Then there is a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point z satisfying $|z| = r \notin [0,1] \cup E_3$ and $|f(z)| = M(r, f)$, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq 2r^s, \quad s \in \mathbb{N}.$$

Lemma 3.7. *Let f be a transcendental meromorphic function of finite $[p,q]$-order. Then the following statements hold :*
(i) If $p \geq q \geq 1$, then $\rho_{[p,q]}(f') = \rho_{[p,q]}(f)$.
(ii) If $2 \leq q = p+1$, then $\rho_{[p,p+1]}(f') = \rho_{[p,p+1]}(f)$.

Proof. We prove only (ii). For the proof of (i) see (Belaïdi, 2015). Let f be a transcendental meromorphic function of finite $[p,q]$-order. By lemma of logarithmic derivative [2], we have

$$T(r,f') = m(r,f') + N(r,f') \leq m(r,f) + m\left(r,\frac{f'}{f}\right) + 2N(r,f)$$

$$\leq 2T(r,f) + m\left(r,\frac{f'}{f}\right) \leq 2T(r,f) + O(\log T(r,f) + \log r) \tag{3.7}$$

holds outside of an exceptional set $E_4 \subset (0,+\infty)$ with finite linear measure. By (3.7), it is easy to see that $\rho_{[p,p+1]}(f') \leq \rho_{[p,p+1]}(f)$ if $2 \leq q = p+1$. On the other hand, by (Chuang, 1951), ((Yang & Yi, 2003), p. 35), we have for $r \to +\infty$

$$T(r,f) < O(T(2r,f') + \log r). \tag{3.8}$$

Hence, by using (3.8) we obtain $\rho_{[p,p+1]}(f) \leq \rho_{[p,p+1]}(f')$ if $2 \leq q = p+1$. Thus, $\rho_{[p,p+1]}(f') = \rho_{[p,p+1]}(f)$ if $2 \leq q = p+1$. □

Remark. Lemma 3.7 (ii) has been proved for $p = 1$ and $q = 2$ by (Chern, 2006).

Lemma 3.8. (Belaïdi, 2015) *Let A_j $(j = 0,1,\ldots,k-1)$, $F \not\equiv 0$ be meromorphic functions. Then the following statements hold :*
(i) If $p \geq q \geq 1$, then every every meromorphic solution f of equation (2.3) such that

$$\max\left\{\sigma_{[p,q]}\left(A_j\right); \sigma_{[p,q]}(F) : j = 0,1,\ldots,k-1\right\} < \sigma_{[p,q]}(f)$$

satisfies $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.
(ii) If $2 \leq q = p+1$, then every meromorphic solution f of equation (2.3) such that

$$\max\left\{1; \sigma_{[p,q]}\left(A_j\right); \sigma_{[p,q]}(F) : j = 0,1,\ldots,k-1\right\} < \sigma_{[p,q]}(f)$$

satisfies $\overline{\lambda}_{[p,p+1]}(f) = \lambda_{[p,p+1]}(f) = \rho_{[p,p+1]}(f)$.

4. Proofs of main results

Proof of Theorem 2.1 It's should be noticed that the case $s = 0$ returns to Theorem 1.4. So, we will prove Theorem 2.1 in case $s > 0$.

[2] see, (Hayman, 1964; Yang & Yi, 2003).

(i) Case : $p \geq q \geq 1$. Suppose that $f \not\equiv 0$ is a polynomial solution of the equation (1.1), let $f(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$ and suppose that $n \geq s$, i.e., $f^{(s)}(z) \not\equiv 0$. Then from (1.1), we have

$$|A_s| A_n^s |a_n| r^{n-s} (1 + o(1)) \leq |A_s| \left| f^{(s)}(z) \right| \leq \sum_{\substack{j=0 \\ j \neq s}}^{k} \left| A_j \right| \left| f^{(j)}(z) \right| \leq \sum_{\substack{j=0 \\ j \neq s}}^{k} \left| A_j \right| A_n^j |a_n| r^{n-j} (1 + o(1)), \quad (4.1)$$

where $A_k \equiv 1$ and $A_n^j = n(n-1)\cdots(n-j+1)$. It follows from (4.1), (2.1) and (2.2) that

$$\exp_p\left(\alpha \left[\log_{q-1} r\right]^{\rho}\right) r^{-s} \leq O\left(\exp_p\left(\beta \left[\log_{q-1} r\right]^{\rho}\right)\right). \quad (4.2)$$

Since $\alpha > \beta$, we see that (4.2) is a contradiction as $r \to +\infty$. Then $\deg f \leq s - 1$.
Now, suppose that f is a transcendental solution of the equation (1.1). From the conditions of Theorem 2.1, there is a set H of complex numbers satisfying $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$, and there exists A_s $(0 \leq s \leq k-1, k \geq 2)$ such that for all $z \in H$ we have (2.1) and (2.2) as $|z| \to +\infty$. Set $H_1 = \{|z| : z \in H\}$, since $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$, then H_1 is a set with $\ell m(H_1) = \infty$.

From (1.1), we have

$$-A_s = \frac{f}{f^{(s)}} \left(\frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \cdots + A_{s+1} \frac{f^{(s+1)}}{f} + A_{s-1} \frac{f^{(s-1)}}{f} + \cdots + A_1 \frac{f'}{f} + A_0 \right). \quad (4.3)$$

By Lemma 3.1, there exists a set $E_1 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B \left[T(2r, f) \right]^{j+1}, \quad j = 1, 2, \ldots, k-1. \quad (4.4)$$

By Lemma 3.6, there is a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point z satisfying $|z| = r \notin [0, 1] \cup E_3$ and $|f(z)| = M(r, f)$, we have

$$\left| \frac{f(z)}{f^{(s)}(z)} \right| \leq 2r^s. \quad (4.5)$$

It follows from (4.3) – (4.5), (2.1) and (2.2) that

$$\exp_p\left(\alpha \left[\log_{q-1} r\right]^{\rho}\right) \leq 2kB \left[T(2r, f)\right]^{k+1} r^s \exp_p\left(\beta \left[\log_{q-1} r\right]^{\rho}\right). \quad (4.6)$$

for all $|z| = r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_3)$ and $|f(z)| = M(r, f)$. Then by (4.6), we obtain $\rho \leq \sigma_{[p+1,q]}(f)$. On the other hand, by Lemma 3.5 (i), we have $\sigma_{[p+1,q]}(f) \leq \rho$. Hence, every transcendental solution f of the equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.
(ii) Case : $2 \leq q = p + 1, \rho > 1$. Suppose that $f \not\equiv 0$ is a polynomial solution of the equation (1.1), let $f(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$ and suppose that $n \geq s$, i.e. $f^{(s)}(z) \not\equiv 0$. From (4.2), we have

$$\exp_p\left(\alpha \left[\log_p r\right]^{\rho}\right) r^{-s} \leq O\left(\exp_p\left(\beta \left[\log_p r\right]^{\rho}\right)\right). \quad (4.7)$$

Since $\alpha > \beta$, we see that (4.7) is a contradiction as $r \to +\infty$. Then $\deg f \leq s - 1$.
Now, suppose that f is a transcendental. Then from (4.6) we have

$$\exp_p\left(\alpha\left[\log_p r\right]^\rho\right) \leq 2kBr^s\left[T\left(2r, f\right)\right]^{k+1}\exp_p\left(\beta\left[\log_p r\right]^\rho\right) \tag{4.8}$$

holds for all z satisfying $|z| = r \in H_1 \setminus ([0,1] \cup E_1 \cup E_3)$, as $r \to +\infty$. By (4.8), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p+1,p+1]}(f) \geq \rho$, and by Lemma 3.5 (ii), we have $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$, thus $\rho \leq \sigma_{[p+1,p+1]}(f) \leq \rho + 1$.

Proof of Corollary 2.1 (i) (a) Let $p \geq q \geq 1$. Let f be a transcendental solution of the equation (2.3) and $\{f_1, f_2, \ldots, f_k\}$ is a solution base of the corresponding homogeneous equation (1.1) of (2.3). By Theorem 2.1, we know that for $j = 1, 2, \ldots, k$

$$\sigma_{[p+1,q]}\left(f_j\right) = \rho.$$

Then f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + \cdots + B_k(z) f_k(z), \tag{4.9}$$

where $B_1, B_2, \ldots, B_k$ are suitable meromorphic functions satisfying

$$B_j' = F \cdot G_j(f_1, f_2, \ldots, f_k) \cdot (W(f_1, f_2, \ldots, f_k))^{-1}, \quad j = 1, 2, \ldots, k, \tag{4.10}$$

where $G_j(f_1, f_2, \ldots, f_k)$ are differential polynomials in $f_1, f_2, \ldots, f_k$ and their derivatives with constant coefficients, thus

$$\sigma_{[p+1,q]}\left(G_j\right) \leq \max_{j=1,2,\ldots,k} \sigma_{[p+1,q]}\left(f_j\right) = \rho, \quad j = 1, 2, \ldots, k. \tag{4.11}$$

Since the Wronskian $W(f_1, f_2, \ldots, f_k)$ is a differential polynomial in $f_1, f_2, \ldots, f_k$, it is easy to deduce also that

$$\sigma_{[p+1,q]}(W) \leq \max_{j=1,2,\ldots,k} \sigma_{[p+1,q]}\left(f_j\right) = \rho. \tag{4.12}$$

Since $\sigma_{[p+1,q]}(F) \leq \rho$, then by using Lemma 3.7 (i) and (4.10) – (4.12) we get for $j = 1, 2, \ldots, k$

$$\sigma_{[p+1,q]}\left(B_j\right) = \sigma_{[p+1,q]}\left(B_j'\right) \leq \max\left\{\sigma_{[p+1,q]}(F); \rho\right\} = \rho. \tag{4.13}$$

Then by (4.9) and (4.13), we obtain

$$\sigma_{[p+1,q]}(f) \leq \max_{j=1,2,\ldots,k}\left\{\sigma_{[p+1,q]}\left(f_j\right); \sigma_{[p+1,q]}\left(B_j\right)\right\} = \rho. \tag{4.14}$$

Now, we assert that every transcendental solution f of (2.3) satisfies $\sigma_{[p+1,q]}(f) = \rho$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$. In fact, if f^* is another transcendental solution with $\sigma_{[p+1,q]}(f^*) < \rho$ of (2.3), then $\sigma_{[p+1,q]}(f_0 - f^*) < \rho$, but $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.1), and this is a contradiction with the results of Theorem 2.1. Then, $\sigma_{[p+1,q]}(f) = \rho$ holds for every transcendental solution f of (2.3) with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \rho$. By Lemma 3.8, every transcendental solution f of (2.3) with $\sigma_{[p+1,q]}(f) = \rho$ satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \rho$.

(b) If $\rho < \rho_{[p+1,q]}(F)$, then by using Lemma 3.7 (i), (4.11) and (4.12), we have from (4.10) for $j = 1, 2, \cdots, k$

$$\rho_{[p+1,q]}\left(B_j\right) = \rho_{[p+1,q]}\left(B'_j\right)$$

$$\leq \max\left\{\rho_{[p+1,q]}(F), \rho_{[p+1,q]}\left(f_j\right) : j = 1, 2, \cdots, k\right\} = \rho_{[p+1,q]}(F). \tag{4.15}$$

Then from (4.15) and (4.9), we get

$$\rho_{[p+1,q]}(f) \leq \max\left\{\rho_{[p+1,q]}\left(f_j\right), \rho_{[p+1,q]}\left(B_j\right) : j = 1, 2, \cdots, k\right\} \leq \rho_{[p+1,q]}(F). \tag{4.16}$$

On the other hand, if $\rho < \rho_{[p+1,q]}(F)$, it follows from equation (2.3) that a simple consideration of $[p, q]$−order implies $\rho_{[p+1,q]}(f) \geq \rho_{[p+1,q]}(F)$. By this inequality and (4.16) we obtain $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.
(ii) For $2 \leq q = p+1, \rho > 1$, by the similar proof in case (i), we can also obtain that the conclusions of case (ii) hold.

Proof of Theorem 2.2 Suppose that $f \not\equiv 0$ is a solution of the equation (1.1). From the conditions the Theorem 2.2, there is a set H of complex numbers satisfying $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$, and there exists A_s $(0 \leq s \leq k-1, k \geq 2)$ such that for all $z \in H$ we have (2.4) and (2.5) as $|z| \to +\infty$. Set $H_1 = \{|z| : z \in H\}$, since $\overline{\log\text{dens}}\{|z| : z \in H\} > 0$ then H_1 is a set with $\ell m(H_1) = \infty$.
(i) Let $p \geq q \geq 1$ and $0 \leq \beta < \alpha$. Suppose that $f \not\equiv 0$ is a polynomial with $\deg f = n \geq s$, then $f^{(s)} \not\equiv 0$, implies that $\frac{f^{(j)}}{f^{(s)}}$ $(j = 0, 1, \ldots, k)$ is a rational, hence $T\left(r, \frac{f^{(j)}}{f^{(s)}}\right) = O(\log r)$ for r sufficiently large. From (4.3) we have

$$T(r, A_s) \leq \sum_{\substack{j=0 \\ j \neq s}}^{k-1} T\left(r, A_j\right) + O(\log r). \tag{4.17}$$

It follows by (4.17), (2.4) and (2.5) that

$$\exp_{p-1}\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right) \leq O\left(\exp_{p-1}\left(\beta\left[\log_{q-1} r\right]^{\rho}\right)\right) \tag{4.18}$$

which is a contradiction since $\alpha > \beta$ and $r \to +\infty$. Then, every polynomial solution $f \not\equiv 0$ of (1.1) is of $\deg f \leq s-1$.
Now, suppose that f is a transcendental solution of (1.1). By using the first main theorem of Nevanlinna and properties of the characteristic function, we obtain from (4.3)

$$\begin{aligned} T(r, A_s) \leq\ & T\left(r, f^{(k)}\right) + kT\left(r, f^{(s)}\right) + \sum_{j=0,\ j\neq s}^{k-1} T\left(r, f^{(j)}\right) \\ & + \sum_{j=0,\ j\neq s}^{k-1} T\left(r, A_j\right) + O(1). \end{aligned} \tag{4.19}$$

By Lemma 3.2, there exists a constant $R > 0$ such that for all z satisfying $|z| = r > R$, we rewrite (4.19) as follows

$$\begin{aligned} m(r, A_s) &= T(r, A_s) \leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) + \sum_{j=0,\ j\neq s}^{k-1} T\left(r, A_j\right) + O(1) \\ &= \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) + \sum_{j=0,\ j\neq s}^{k-1} m\left(r, A_j\right) + O(1). \end{aligned} \tag{4.20}$$

It follows by (4.20), (2.4) and (2.5) that

$$\begin{aligned} \exp_{p-1}\left(\alpha\left[\log_{q-1} r\right]^{\rho}\right) &\leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) \\ &\quad + (k-1)\exp_{p-1}\left(\beta\left[\log_{q-1} r\right]^{\rho}\right) + O(1) \end{aligned} \tag{4.21}$$

holds for all z satisfying $|z| = r \in H_1$ as $r \to +\infty$. Then, by (4.21), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p,q]}(f) \geq \rho$, and by Lemma 3.5 (i), we have $\sigma_{[p+1,q]}(f) \leq \rho$. Thus, $\sigma_{[p,q]}(f) \geq \rho \geq \sigma_{[p+1,q]}(f)$.
(ii) Let $2 \leq q = p+1$ and $0 \leq (k-1)\beta < \alpha$. Suppose that $f \not\equiv 0$ is a polynomial with $\deg f = n \geq s$, then $f^{(s)} \not\equiv 0$. By the same reasoning as in the proof in case (i), it is clear that $f(z)$ is a polynomial with $\deg f \leq s-1$.
Now, suppose that f is a transcendental solution of (1.1). Then by (4.21)

$$\begin{aligned} \exp_{p-1}\left(\alpha\left[\log_{p} r\right]^{\rho}\right) &\leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) \\ &\quad + (k-1)\exp_{p-1}\left(\beta\left[\log_{p} r\right]^{\rho}\right) + O(1) \end{aligned} \tag{4.22}$$

holds for all z satisfying $|z| = r \in H_1$ as $r \to +\infty$. Then, by (4.22), every transcendental solution f of equation (1.1) satisfies $\sigma_{[p,p+1]}(f) \geq \rho$, and by Lemma 3.5 (ii), we have $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$. Hence, $\rho \leq \sigma_{[p,p+1]}(f)$ and $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$.

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