Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 6 (2) (2016) 150 – 169

Co-Universal Algebraic Extension with Hidden Parameters

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Abstract

In the research of underlying algebraic structures of real world phenomena, we can find some behavior anomalies that depend on external parameters that are not ruled by their axiom systems. These are not visible straightaway and we have to deduce their existence from the effects they cause. To add them in mathematical constructions, we introduce co-universal extensions of algebras and co-algebras based upon the dual construction of the Kleisli category associated to a monad.

To illustrate this topic we introduce two applications. The first one is an artificial example. In the second application we analyze language algebraic structures with a method that states a bridge between language and logic blindly, that is to say, handling statements through their expressions in those languages satisfying some adequate conditions, and disregarding their meanings.

Keywords: Algebraic extensions, hidden parameters, algebraic language structures, co-monad, Kleisli categories, blind logic.
2010 MSC: 18C99, 18C20, 68T50, 03B65.

1. Introduction

When we investigate the mathematical structures of real world phenomena, we can observe some anomalies that depend on parameters that are not ruled by those axioms that define their algebraic structures. For instance, the states of a Turing machine, contexts when we interpret sentences in any language, environments, positions, etc. Recall that only tape symbols are the visible part of Turing machines. By contrast, moves and states are not displayed in their tapes. They work in the background as hidden parameters, but we can deduce their existence from the behavior changes they cause.

In positional notations, the meaning of each word or symbol depends on their position. For instance, consider the following sentences: 1) "*Programmers know how to write code fast;*" and 2) "*Programmers know how to write fast code.*" Both consist of the same words, but their meanings

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are order-dependent. We can consider both orders as hidden parameters, and the meanings of the former sentences depend on them. Accordingly, to define a map μ sending each sentence in the English language E into its meaning in M, we have to add a parameter set $\mathcal{H}$ the members of which are associated to orders, contexts and styles. Thus, the domain of μ must be the Cartesian product $E \times \mathcal{H}$; where E denotes the set of all English sentences.

We can also find hidden parameters in psychology, physics, and random phenomena. For instance, the probability of remembering a name increases with the occurrence frequency, or when some noticeable fact is associated to it. Thus, frequency and remarkable facts can work as hidden parameters that can modify probabilities. In section 4, we analyze an artificial example of this kind.

To learn and interpret any language, we have to handle abstractions and inferences between the definitions of sentence meanings (Tudor-Răzvan & Manolescu, 2011). The topic goes as follows. If two words, say W_1 and W_2, have the same meaning, when we swap them in any sentence, we obtain an equivalent one. We introduce language structure conditions to build the inverse method. Thus, we can find logical relations and abstractions between the meanings of W_1 and W_2 when we observe that some set of proper sentences T_1 becomes T_2 when we swap W_1 and W_2 and each member of T_2 is a proper sentence too. To know that T_2 consists of right sentences, we need not know their meanings. It is sufficient to find them in any scholar paper. The method works as a blind logic and can give rise to many ambiguities, that we can avoid deducing the existence of hidden parameters. This topic is an enlargement of what Newell stated in (Newell & Simon, 1976). We do not dive in this topic deeply, because we only expose these ideas to illustrate applications of co-universal algebraic extensions that we introduce.

The main aim of this article consists of introducing an algebraic device to enrich categories with sets of external (hidden) parameters that are not ruled by the axioms defining them. We term these constructions co-universal because are based upon co-monads together with the associated dual constructions of Kleisli categories. Well-known universal extensions of **Set**, associated to monads, are categories of sets with fuzzy subsets (Mawanda, 1988). These extensions of **Set** arise from an endofunctor that sends each ordinary set X into $X \times M$, where M is a monoid of truth-values. We introduce co-universal extensions by a similar endofunctor $X \mapsto X \times M$ such that M is the set of hidden parameters.

2. Preliminaries

To simplify expressions, we state some auxiliary definitions and notations. We write in bold face font those symbols denoting categories. In particular, **Set** denotes the category of ordinary sets and maps. We use the symbol ◃ as an end-of-definition marker.

Notation. For each couple of sets X and Y, we denote by $X^Y_{\geq n}$ the subset of X^Y defined as follows.

$$X^Y_{\geq n} = \{f \in X^Y \mid \#(\mathrm{img}(f)) \geq n\}.$$

For instance, $X^Y_{\geq 2}$ consists of every non-constant map in X^Y.

For each subcategory $\mathbf{C}$ of **Set** and every non-empty set $\mathcal{H}$, we denote the members of the set $(\mathrm{Hom}_{\mathbf{C}}(X, Y))^{\mathcal{H}}$ by symbols with the accent ˇ to indicate that are maps from an arbitrary set $\mathcal{H}$ into a homset. For each member $\breve{f}$ of $(\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}$ we write the values of the independent variable $\mathcal{H}$ as subscripts. Thus, the expression $\breve{f}_\alpha \in \mathrm{Hom}_{\mathbf{C}}(X,Y)$ denotes the image of $\alpha \in \mathcal{H}$ under $\breve{f}$.

Definition 2.1. Let **C** be a subcategory of **Set**. For every set $\mathcal{H}$ with cardinality greater than 1, we term $\mathcal{H}$-extension of **C** the category $\mathbf{C}[\mathcal{H}]$, with the same object class as **C**, such that, for every couple of sets X and Y,

$$\mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y) = \mathrm{Hom}_{\mathbf{C}}(X,Y) \bigcup \left\{ \coprod_{\alpha\in\mathcal{H}} \{\breve{h}_\alpha\} \middle| \breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2} \right\}. \tag{2.1}$$

Since **C** is a subcategory of $\mathbf{C}[\mathcal{H}]$, we only have to define those compositions involving morphisms in $\left\{\coprod_{\alpha\in\mathcal{H}}\{\breve{h}_\alpha\} \middle| \breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2}\right\}$. We denote this composition by the infix symbol $\diamond$. For every couple of morphisms $f : X \to Y \in \mathrm{Hom}_{\mathbf{C}}(X,Y)$ and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(Y,Z)$ we define their composition as follows.

$$\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\right) \diamond f = \coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha \circ f\} \tag{2.2}$$

Likewise, the composition of f and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(T,X)$ is

$$f \diamond \left(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\right) = \coprod_{\alpha\in\mathcal{H}}\{f \circ \breve{g}_\alpha\} \tag{2.3}$$

Finally, we define the composition of two morphisms $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y)$ and $\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\} \in \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(Y,Z)$ by

$$\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\right) \diamond \left(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}\right) = \coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha \circ \breve{f}_\alpha\}. \tag{2.4}$$

Since **C** is a subcategory of $\mathbf{C}[\mathcal{H}]$ with the same object class, identities are the same in both categories. ◁

Theorem 2.1. *Let* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *be two subcategories of* **Set**. *For every set* $\mathcal{H}$ *with cardinality greater than* 1, *and each functor* $T : \mathbf{C}_1 \to \mathbf{C}_2$, *the following statements hold.*

1) *There is an extension* $T^* : \mathbf{C}_1[\mathcal{H}] \to \mathbf{C}_2[\mathcal{H}]$ *of* T *with the same object-map.*

2) *If* $X_1 \xrightarrow{\sigma} T(X_2)$ *is a* T*-universal arrow, then* $X_1 \xrightarrow{\sigma} T^*(X_2)$ *is a* T^**-universal one.*

3) *If for every* $\alpha \in \mathcal{H}$, $X_1 \xrightarrow{\breve{\sigma}_\alpha} T(X_2)$ *is a* T*-universal arrow, then*

$$X_1 \xrightarrow{\coprod_{\alpha\in\mathcal{H}}\{\breve{\sigma}_\alpha\}} T^*(X_2)$$

is a T^**-universal one.*

Proof.

1) We define the extension T^* of T in the following terms. The object-maps of both T and T^* are the same. Recall that, by definition, $\mathrm{Obj}(\mathbf{C}_1) = \mathrm{Obj}(\mathbf{C}_1[\mathcal{H}])$. The images $T(f)$ and $T^*(f)$ of every morphism $f \in \mathrm{Mor}(\mathbf{C}_1)$ are the same. The image of each morphism $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\} \in \mathrm{Mor}(\mathbf{C}_1[\mathcal{H}]) \setminus \mathrm{Mor}(\mathbf{C}_1)$ is given by

$$T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}\Big) = \coprod_{\alpha\in\mathcal{H}}\{T(\breve{f}_\alpha)\} \tag{2.5}$$

The former definition is possible because, by equation (2.1), $\breve{f}_\alpha$ belongs to $\mathrm{Mor}(\mathbf{C}_1)$, for every $\alpha \in \mathcal{H}$.

It remains to be shown that T^* preserves morphism composition and identities. Since the restriction of T^* to $\mathrm{Mor}(\mathbf{C}_1)$ coincides with T, the extension T^* preserves identities and compositions between members of $\mathbf{C}_1$. We only have to show that T^* preserves morphism compositions involving some members of $\mathrm{Mor}(\mathbf{C}_1[\mathcal{H}]) \setminus \mathrm{Mor}(\mathbf{C}_1)$. For compositions like (2.2), taking into account (2.5),

$$\begin{aligned} T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{(\breve{g}_\alpha \circ f)\}\Big) &= \coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha \circ f)\} = \coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha) \circ T(f)\} = \\ &\Big(\coprod_{\alpha\in\mathcal{H}}\{T(\breve{g}_\alpha)\}\Big) \diamond T(f) = T^*\Big(\coprod_{\alpha\in\mathcal{H}}\{\breve{g}_\alpha\}\Big) \diamond T^*(f) \end{aligned} \tag{2.6}$$

The proofs for compositions of the form (2.3) and (2.4) go as in the preceding case.

2) We have to show that, for every object Y and every morphism $f : X_1 \to T^*(Y)$ there is a unique $f^* : X_2 \to Y$ such that the following triangle commutes.

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & T^*(X_2) \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle T^*(f^*)} \\ & & T^*(Y) \end{array} \tag{2.7}$$

If $f \in \mathrm{Mor}(\mathbf{C}_1)$, by hypothesis, this condition must be satisfied. Now, suppose that $f = \coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}$. Since for every α, the morphism $\breve{f}_\alpha : X \to T^*(Y)$ belongs to $\mathrm{Mor}(\mathbf{C}_1)$, there is a unique $\breve{f}^*_\alpha : X_2 \to T^*(Y) = T(Y)$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \sigma\ } & T^*(X_2) = T(X_2) \\ & {\scriptstyle \breve{f}_\alpha}\searrow & \downarrow {\scriptstyle T^*(\breve{f}^*)=T(\breve{f}_\alpha)} \\ & & T^*(Y) = T(Y) \end{array} \tag{2.8}$$

By virtue of (2.2) the following triangle is also commutative

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \sigma\ } & T^*(X_2) \\ & {\scriptstyle \coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}}\searrow & \downarrow {\scriptstyle T^*\left(\coprod_{\alpha\in\mathcal{H}}\{\breve{f}^*\}\right)} \\ & & T^*(Y)) \end{array} \tag{2.9}$$

3) Let $X_1 \xrightarrow{\coprod_{\alpha\in\mathcal{H}}\{\check{f}_\alpha\}} T^*(Y)$ be a $\mathbf{C}_1$-morphism. By assumption, for every $\alpha \in \mathcal{H}$, there is a $\mathbf{C}_1$-morphism $\check{f}^*_\alpha$ such that the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{\check{\sigma}_\alpha} & T(X_2) \\ & \searrow^{\check{f}_\alpha} & \big\downarrow {\scriptstyle T(\check{f}_\alpha)} \\ & & T(Y) \end{array} \tag{2.10}$$

hence, the following triangle is also commutative.

$$\begin{array}{ccc} X_1 & \xrightarrow{\check{\sigma}_\alpha} & T(X_2) \\ & \searrow^{\coprod_{\alpha\in\mathcal{H}}\{\check{f}_\alpha\}} & \big\downarrow {\scriptstyle T\left(\coprod_{\alpha\in\mathcal{H}}\{\check{f}_\alpha\}\right)} \\ & & T(Y) \end{array} \tag{2.11}$$

The uniqueness of $\coprod_{\alpha\in\mathcal{H}}\{\check{f}^*_\alpha\}$ is a consequence of being unique each $\check{f}^*_\alpha$ that satisfies the commutativity of (2.10), for every $\alpha \in \mathcal{H}$.

□

3. Co-universal algebraic extensions with hidden parameters

For every subcategory $\mathbf{C}$ of **Set**, being stable under Cartesian products, and each non-empty set $\mathcal{H}$ in Obj($\mathbf{C}$), we denote by $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$ the functor sending each set $X \in$ Obj($\mathbf{C}$) into $X \times \mathcal{H}$, and every map $f : X \to Y$ into

$$\mathcal{H}^\dagger(f) = f \times \mathrm{id}_{\mathcal{H}} : X \times \mathcal{H} \to Y \times \mathcal{H}. \tag{3.1}$$

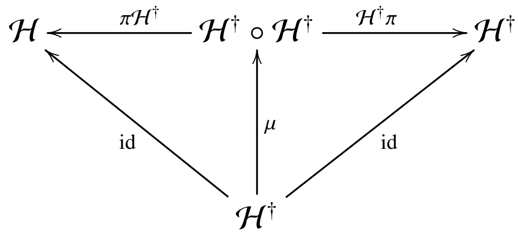
Notation. For every endofunctor $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$, we denote by π the natural transformation $\mathcal{H}^\dagger \xrightarrow{\pi} \mathrm{Id}$ such that, for each set X, the map $\pi_X : X \times \mathcal{H} \to X$ is the canonical projection; where $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ denotes the identity endofunctor. Likewise, $\mathcal{H}^\dagger \xrightarrow{\mu} \mathcal{H}^\dagger \circ \mathcal{H}^\dagger$ is the natural transformation

$$\mu_X =: X \times \mathcal{H} \to X \times \mathcal{H} \times \mathcal{H} \tag{3.2}$$

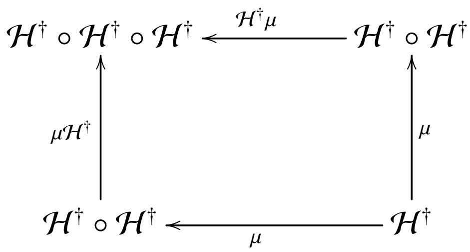
that sends each $(x, v) \in X \times \mathcal{H}$ into $(x, v, v) \in X \times \mathcal{H} \times \mathcal{H}$.

Proposition 3.1. *Let* $\mathbf{C}$ *be a subcategory of* **Set** *being stable under Cartesian products. For every nonempty set* $\mathcal{H} \in$ Obj($\mathbf{C}$), *the endofunctor* $\mathcal{H}^\dagger : \mathbf{C} \to \mathbf{C}$ *together with both natural transformations* π *and* μ *form a comonad* $(\mathcal{H}^\dagger, \pi, \mu)$.

Proof. We show that the following diagrams commute.



(3.3)



(3.4)

Let X be a set and (x, v) any member of $\mathcal{H}^\dagger(X) = X \times \mathcal{H}$. By straightforward computations we obtain

$$\pi_{X\times\mathcal{H}}(\mu_X(x, v)) = \pi_{X\times\mathcal{H}}(x, v, v) = (x, v);$$

accordingly, $\pi\mathcal{H}^\dagger \circ \mu = \text{id}$. The proofs for the right triangle and quadrangle (3.4) are similar. □

Definition 3.1. Let $\mathbf{C}$ be a subcategory of **Set** being stable under Cartesian products. For each set $\mathcal{H} \in \text{Obj}(\mathbf{C})$ with cardinality greater than 1, a co-universal $\mathcal{H}$-extension of $\mathbf{C}$ with *hidden parameters* is the category $\mathbf{C}_\mathcal{H}$ defined as follows.

1. The object-classes of both $\mathbf{C}_\mathcal{H}$ and $\mathbf{C}$ are the same.
2. For each couple of objects X and Y, the set $\text{Hom}_{\mathbf{C}_\mathcal{H}}(X, Y)$ consists of all maps from $\mathcal{H}^\dagger(X) = X \times \mathcal{H}$ into Y such that there is $\breve{f} \in (\text{Hom}_\mathbf{C}(X, Y))^\mathcal{H}$ that satisfies the relation
$$\forall\alpha \in \mathcal{H} : \ f(x, \alpha) = \breve{f}_\alpha(x).$$
3. The composition $f \star g$ of two $\mathbf{C}_\mathcal{H}$-morphisms $g \in \text{Hom}_{\mathbf{C}_\mathcal{H}}(X, Y)$ and $f \in \text{Hom}_{\mathbf{C}_\mathcal{H}}(Y, Z)$ is given by
$$f \star g = f \circ \mathcal{H}^\dagger(g) \circ \mu_X \tag{3.5}$$
4. The identity associated to each $\mathbf{C}_\mathcal{H}$-object X is the projection
$$\pi_X : \mathcal{H}^\dagger(X) = X \times \mathcal{H} \to X.$$

◁

Notation. As in the preceding definition, for every co-universal $\mathcal{H}$-extension $\mathbf{C}_\mathcal{H}$ of a subcategory $\mathbf{C}$ of **Set**, we denote the morphism composition by the infix symbol $\star$.

Definition 3.2. Let $\mathbf{C}$ be a subcategory of **Set** such that there is the co-universal $\mathcal{H}$-extension $\mathbf{C}_\mathcal{H}$. We say a $\mathbf{C}_\mathcal{H}$-morphism $f : X \times \mathcal{H} \to Y$ to be π-factorizable whenever there is $f^* \in \text{Hom}_\mathbf{C}(X, Y)$ that satisfies the equation $f = f^* \circ \pi_X$. ◁

Lemma 3.1. *Let* **C** *be a subcategory of* **Set**, *being stable under Cartesian products. For every set* $\mathcal{H} \in \text{Obj}(\mathbf{C})$ *and each* $\mathbf{C}_{\mathcal{H}}$*-object* X, *the associated* $\mathbf{C}_{\mathcal{H}}$*-identity* π_X *is* π*-factorizable. In addition,* $\pi_X^* = \text{id}_X$.

Proof. Setting $\pi_X^* = \text{id}_X$, the relation $\pi_X = \text{id}_X \circ \pi_X$ leads to $\pi_X = \pi_X^* \circ \pi_X$. □

Lemma 3.2. *A* $\mathbf{C}_{\mathcal{H}}$*-morphism* $f(x, \alpha) = \check{f}_\alpha(x)$ *is* π*-factorizable if and only if* $\check{f} \in \text{Hom}_{\mathbf{C}}(X, Y)^{\mathcal{H}}$ *is constant.*

Proof. Assume $\check{f}$ to be a constant map, therefore the value of $f(x, \alpha)$ depends only on x. Thus, setting $f^*(x) = \check{f}_\alpha(x)$, for every $(x, \alpha) \in X \times \mathcal{H}$, the relation $f = f^* \circ \pi_X$ holds. The proof for the converse implication is similar. □

Lemma 3.3. *The composition of* π*-factorizable morphisms is again* π*-factorizable.*

Proof. Let $f \circ \pi_X : X \times \mathcal{H} \to Y$ and $g \circ \pi_Y : Y \times \mathcal{H} \to Z$ be two π-factorizable morphisms. According to (3.5)

$$(g \circ \pi_Y) \star (f \circ \pi_X) = (g \circ \pi_Y) \circ \mathcal{H}^\dagger(f \circ \pi_X) \circ \mu_X = (g \circ \pi_Y) \circ ((f \circ \pi_X) \times \text{id}_{\mathcal{H}}) \circ \mu_X = g \circ f \circ \pi_X \quad (3.6)$$

therefore $(g \circ \pi_Y) \star (f \circ \pi_X) = (g \circ f) \circ \pi_X$ is π-factorizable. □

Definition 3.3. Let **C** be any subcategory of **Set**, being stable under Cartesian products. For each set $\mathcal{H}$ with cardinality greater than 1, and each $\alpha \in \mathcal{H}$, we define the map $\Gamma_{\alpha,A} : \text{Mor}(\mathbf{C}_{\mathcal{H}}) \to \text{Mor}(\mathbf{C})$ as follows. For every couple $\mathbf{C}_{\mathcal{H}}$-objects X and Y, and each $f \in \text{Hom}_{\mathbf{C}_{\mathcal{H}}}(X, Y)$:

$$\Gamma_{\alpha,A}(f) = \begin{cases} f^* \text{ if } f = f^* \circ \pi_X \text{ is } \pi\text{-factorizable} \\ \check{f}_\alpha \text{ where } \check{f} \in (\text{Hom}_{\mathbf{C}}(X, Y))^{\mathcal{H}}_{\geq 2} \text{ otherwise.} \end{cases} \quad (3.7)$$

being $\check{f}$ the map such that $\forall (x, \alpha) \in X \times \mathcal{H}$: $\check{f}_\alpha(x) = f(x, \alpha)$. ◁

To agree with Lemma 3.2, in the former definition, when $f = f^* \circ \pi_X$ is π-factorizable, its image $\Gamma_{\alpha,A}(f)$ does not depend on the parameter α.

Proposition 3.2. *Let* $\mathbf{C}_{\mathcal{H}}$ *be a co-universal* $\mathcal{H}$*-extension of a subcategory* **C** *of* **Set** *with hidden parameters. For every* $\alpha \in \mathcal{H}$, *the map* $\Gamma_{\alpha,A} : \text{Mor}(\mathbf{C}_{\mathcal{H}}) \to \text{Mor}(\mathbf{C})$ *preserves identities and morphism compositions.*

Proof. We have to show that, for every couple of morphisms $f : X \times \mathcal{H} \to Y$ and $g : Y \times \mathcal{H} \to Z$, and each $\alpha \in \mathcal{H}$, the map $\Gamma_{\alpha,A}$ satisfies the following relation.

$$\Gamma_{\alpha,A}(g \star f) = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f) \quad (3.8)$$

If both $f = f^* \circ \pi_X$ and $g = g^* \circ \pi_Y$ are π-factorizable, then

$$\begin{aligned}\Gamma_{\alpha,A}(g \star f) = \Gamma_{\alpha,A}((g^* \circ \pi_Y) \star (f^* \circ \pi_X)) = \\ \Gamma_{\alpha,A}((g^* \circ \pi_Y) \circ \mathcal{H}^\dagger(f^* \circ \pi_X) \circ \mu_X) = \\ \Gamma_{\alpha,A}((g^* \circ \pi_Y) \circ ((f^* \circ \pi_X) \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X) = \\ \Gamma_{\alpha,A}(g^* \circ f^* \circ \pi_X)) = g^* \circ f^* = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f)\end{aligned} \tag{3.9}$$

Thus, $\Gamma_{\alpha,A}$ preserves the composition of π-factorizable morphisms.

For non-π-factorizable morphisms, the expression $(g \star f)(x, \alpha)$ can be written explicitly as follows.

$$\begin{aligned}\forall (x, \alpha) \in X \times \mathcal{H} : \quad (g \star f)(x, \alpha) = \\ (g \circ \mathcal{H}^\dagger(f) \circ \mu_X)(x, \alpha) = (g \circ (f \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X)(x, \alpha) = \\ (g(f(x, \alpha), \alpha) = \breve{g}_\alpha(\breve{f}_\alpha(x)) = (\breve{g}_\alpha \circ \breve{f}_\alpha)(x);\end{aligned} \tag{3.10}$$

and by definition, $\Gamma_{\alpha,A}(f) = \breve{f}_\alpha$ and $\Gamma_{\alpha,A}(g) = \breve{g}_\alpha$; therefore

$$\Gamma_{\alpha,A}(g \star f) = (g \star f)\breve{}_\alpha = \breve{g}_\alpha \circ \breve{f}_\alpha = \Gamma_{\alpha,A}(g) \circ \Gamma_{\alpha,A}(f); \tag{3.11}$$

hence $\Gamma_{\alpha,A}$ also preserves the composition of non-π-factorizable morphisms.

If $g = g^* \circ \pi_Y$ is π-factorizable and f is not, the same procedure yields

$$\forall (x, \alpha) \in X \times \mathcal{H} : \quad (g \star f)(x, \alpha) = (g(f(x, \alpha), \alpha) = g^*(\breve{f}_\alpha(x)); \tag{3.12}$$

and this equation leads to (3.11). The proof when f is π-factorizable and g is not, is similar.

It remains to be shown that $\Gamma_{\alpha,A}$ preserves identities. According to Lemma 3.1 and equation (3.7), $\Gamma_{\alpha,A}(\pi_X) = \mathrm{id}_X$. □

Corollary 3.1. *With the same assumptions as in Proposition 3.2, for every fixed* $\alpha \in \mathcal{H}$*, the identity* $\mathrm{Id} : \mathrm{Obj}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Obj}(\mathbf{C})$ *and the map* $\Gamma_{\alpha,A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C})$ *form a functor* $\Gamma_\alpha = (\mathrm{Id}, \Gamma_{\alpha,A})$.

Proof. By definition, the object classes of $\mathbf{C}_{\mathcal{H}}$ and $\mathbf{C}$ are the same; hence the identity can be the object map of Γ_α. By Proposition 3.2 the map Γ_α, A preserves identities and morphism composition. □

Notation. For every subcategory $\mathbf{C}$ of **Set** being stable under Cartesian products, and each set $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$ with cardinality greater than 1, the expression

$$F_{\mathcal{H},A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C}[\mathcal{H}])$$

denotes the map such that, for each pair X and Y in $\mathrm{Obj}(\mathbf{C}_{\mathcal{H}})$ and every $f \in \mathrm{Hom}_{\mathbf{C}_{\mathcal{H}}}(X, Y)$:

$$F_{\mathcal{H},A}(f) = \begin{cases} \Gamma_\alpha(f) \text{ if } f \text{ is } \pi\text{-factorizable} \\ \coprod_{\alpha \in \mathcal{H}} \{\Gamma_\alpha(f)\} = \coprod_{\alpha \in \mathcal{H}} \{\breve{f}_\alpha\} \text{ otherwise.} \end{cases} \tag{3.13}$$

where Γ_α is the functor defined in Corollary 3.1.

Theorem 3.1 (Main)**.** *For every subcategory* **C** *of* **Set** *being stable under Cartesian products, and each set* $\mathcal{H} \in \mathrm{Obj}(\mathbf{C})$ *with cardinality greater than* 1*, the following statements hold.*

1) *The identity* $\mathrm{Id} : \mathrm{Obj}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Obj}(\mathbf{C}[\mathcal{H}])$ *together with the arrow map*

$$F_{\mathcal{H},A} : \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) \to \mathrm{Mor}(\mathbf{C}[\mathcal{H}])$$

form an isomorphism $F_{\mathcal{H}} = (\mathrm{Id}, F_{\mathcal{H},A})$ *between both categories* $\mathbf{C}_{\mathcal{H}}$ *and* $\mathbf{C}[\mathcal{H}]$.

2) *If* **D** *is a subcategory of* **Set***, being stable under Cartesian products such that* $\mathcal{H}$ *belongs to* $\mathrm{Obj}(\mathbf{D})$*, then every functor* $T : \mathbf{C} \to \mathbf{D}$ *gives rise to another one* $T^{\natural}_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{D}_{\mathcal{H}}$*, having the same object map as* T*, which satisfies the following relation.*

$$\forall f \in \mathrm{Mor}(\mathbf{C}_{\mathcal{H}}) : \quad \Gamma_{\alpha} \circ T^{\natural}_{\mathcal{H}}(f) = T \circ \Gamma_{\alpha}(f) \tag{3.14}$$

3) *With the same conditions as in the preceding statement, if for every* $\alpha \in \mathcal{H}$*, the* **C***-morphism* $X_1 \xrightarrow{\breve{\sigma}_{\alpha}} T(X_2)$ *is a* T*-universal arrow, then* $X_1 \xrightarrow{\sigma} T^{\natural}_{\mathcal{H}}(X_2)$ *is a* $T^{\natural}_{\mathcal{H}}$*-universal one; where* σ *denotes the* $\mathbf{C}_{\mathcal{H}}$*-morphism* $\sigma : X_1 \times \mathcal{H} \to T^{\natural}_{\mathcal{H}}(X_2)$ *such that,* $\forall (x, \alpha) \in X_1 \times \mathcal{H}$*:* $\sigma(x, \alpha) = \breve{\sigma}_{\alpha}(x)$.

4) *With the same assumptions as in Statement 2), every* $T^{\natural}$*-algebra (co-algebra) is the extension with hidden parameters of an ordinary* $T^{\natural}_{\mathcal{H}}$*-algebra (co-algebra).*

Proof.

1) We have to show that $F_{\mathcal{H}}$ is a functor. For every object X, the $\mathbf{C}_{\mathcal{H}}$-identity is $\pi_X : X \times \mathcal{H} \to X$. According to Proposition 3.2, its image under $F_{\mathcal{H}}$ is $\Gamma_{\alpha}(\pi_X) = \mathrm{id}_X$. Thus, $F_{\mathcal{H}}$ preserves identities.

To show that $F_{\mathcal{H}}$ preserves morphism composition, let $f = f^* \circ \pi_X : X \times \mathcal{H} \to Y$ and $g = g^* \circ \pi_Y : Y \times \mathcal{H} \to Z$ be two π-factorizable morphisms. By equation (3.13) and taking into account Lemma 3.3,

$$\begin{aligned} F_{\mathcal{H}}(g \star f) = \Gamma_{\alpha}(g \star f) = &\\ \Gamma_{\alpha}(g^* \circ \pi_Y \circ \mathcal{H}^{\dagger}(f^* \circ \pi_X) \circ \mu_X) = \Gamma_{\alpha}(g^* \circ f^* \circ \pi_X) = &\\ g^* \circ f^* = \Gamma_{\alpha}(g) \circ \Gamma_{\alpha}(f) = \Gamma_{\alpha}(g) \diamond \Gamma_{\alpha}(f); & \end{aligned} \tag{3.15}$$

therefore

$$F_{\mathcal{H}}(g \star f) = \Gamma_{\alpha}(g \star f) = \Gamma_{\alpha}(g) \diamond \Gamma_{\alpha}(f) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.16}$$

If f and g are two non-π-factorizable morphisms, by definition,

$$\begin{aligned} (g \star f)(x, \alpha) = (g \circ \mathcal{H}^{\dagger}(f) \circ \mu_X)(x, \alpha) = &\\ (g \circ (f \times \mathrm{id}_{\mathcal{H}}) \circ \mu_X)(x, \alpha) = g(f(x, \alpha), \alpha) & \end{aligned} \tag{3.17}$$

Let $\breve{f} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}$ and $\breve{g} \in (\mathrm{Hom}_{\mathbf{C}}(Y,Z))^{\mathcal{H}}$ be the maps such that

$$\begin{cases} \forall (x,\alpha) \in X \times \mathcal{H}: & \breve{f}_\alpha(x) = f(x,\alpha) \\ \forall (y,\alpha) \in Y \times \mathcal{H}: & \breve{g}_\alpha(x) = g(y,\alpha). \end{cases} \tag{3.18}$$

These relations together with (3.17) lead to

$$\forall x \in X : (g \star f)(x) = g((f(x,\alpha),\alpha) = \breve{g}_\alpha(\breve{f}_\alpha(x)) = (\breve{g}_\alpha \circ \breve{f}_\alpha)(x), \tag{3.19}$$

for every fixed $\alpha \in \mathcal{H}$. Consequently, by virtue of (2.4) and (3.13),

$$F_{\mathcal{H}}(g \star f) = \coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha \circ \breve{f}_\alpha\} = \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{g}_\alpha\}\Big) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{f}_\alpha\}\Big) = \\ \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(g)\}\Big) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(f)\}\Big) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.20}$$

If $g = g^* \circ \pi_Y$ is π-factorizable and f is not, the same procedure yields,

$$F_{\mathcal{H}}(g \star f) = \coprod_{\alpha\in\mathcal{H}} \{g^* \circ \breve{f}_\alpha\} = g^* \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\breve{f}_\alpha\}\Big) = \\ \Gamma_\alpha(g) \diamond \Big(\coprod_{\alpha\in\mathcal{H}} \{\Gamma_\alpha(f)\}\Big) = F_{\mathcal{H}}(g) \diamond F_{\mathcal{H}}(f). \tag{3.21}$$

By the same method, we can build the proof when f is π-factorizable and g is not.

Since $F_{\mathcal{H}}$ preserves identities and morphism composition, it is a functor.

To be an isomorphism, $F_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{C}[\mathcal{H}]$ must be full, faithful, and bijective on objects. By definition, the object-classes of $\mathbf{C}$, $\mathbf{C}_{\mathcal{H}}$, and $\mathbf{C}[\mathcal{H}]$ are the same. Because the object map Id of $F_{\mathcal{H}}$ is the identity, $F_{\mathcal{H}}$ is bijective on objects.

It remains to be shown that $F_{\mathcal{H}}$ is full and faithful. The class $\mathrm{Mor}(\mathbf{C}[\mathcal{H}])$ consists of the ordinary maps in $\mathrm{Mor}(\mathbf{C})$ together with the coproduct class

$$\mathrm{Cprd}(\mathbf{C},\mathcal{H}) = \Big\{ \coprod_{\alpha\in\mathcal{H}} \{\breve{h}_\alpha\} \Big| \breve{h} \in (\mathrm{Hom}_{\mathbf{C}}(X,Y))^{\mathcal{H}}_{\geq 2}) \wedge (X,Y) \in \mathrm{Obj}(\mathbf{C}) \times \mathrm{Obj}(\mathbf{C}) \Big\}$$

For every map $f : X \to Y$ in $\mathrm{Mor}(\mathbf{C})$ there is the preimage $F^{-1}_{\mathcal{H}}(f) = f \circ \pi_X$, because $F_{\mathcal{H}}(f \circ \pi_X) = \Gamma_\alpha(f \circ \pi_x) = f$. Likewise, for each $\mathbf{C}[\mathcal{H}]$-morphism $\coprod_{\alpha\in\mathcal{H}}\{\breve{f}_\alpha\}$ lying in $\mathrm{Cprd}(\mathbf{C},\mathcal{H}) \subsetneq \mathrm{Hom}_{\mathbf{C}[\mathcal{H}]}(X,Y)$, the preimage is the morphism $f : X \times \mathcal{H} \to Y$ that satisfies the relation $f(x,\alpha) = \breve{f}_\alpha(x)$, for each fixed $\alpha \in \mathcal{H}$ and every $x \in X$; hence $F_{\mathcal{H}}$ is full.

To see that $F_{\mathcal{H}}$ is faithful, we split the class $\mathrm{Mor}(\mathbf{C}_{\mathcal{H}})$ into the subclass $\mathbf{C}_{\mathcal{H},\pi}$ of π-factorizable morphisms and its complement $\complement_{\mathrm{Mor}(\mathbf{C}_{\mathcal{H}})}\mathbf{C}_{\mathcal{H},\pi}$. If the images of two π-factorizable morphisms $f : X \times \mathcal{H} \to Y$ and $f : X \times \mathcal{H} \to Y$ are the same, then $\Gamma_\alpha(f) = \Gamma_\alpha(g)$; so then $f = \Gamma_\alpha(f) \circ \pi_X = \Gamma_\alpha(g) \circ \pi_X = g$.

Since the image of every π-factorizable morphisms belongs to **C**, we only have to show that, the restriction of $F_{\mathcal{H}}$ to each homset in $\complement_{\mathrm{Mor}(\mathbf{C}_{\mathcal{H}})}\mathbf{C}_{\mathcal{H},\pi}$ is also injective. Let $f : X \times \mathcal{H} \to Y$ and $g : X \times \mathcal{H} \to Y$ be two morphisms with the same image $\coprod_{\alpha\in\mathcal{H}}\{\breve{h}_\alpha\}$. By definition, for every $(x, \alpha) \in X \times \mathcal{H}$: $f(x, \alpha) = \Gamma_\alpha(f)(x) = \breve{h}_\alpha(x) = \Gamma_\alpha(g)(x) = g(x, \alpha)$; therefore $f = g$. Finally, the image under $F_{\mathcal{H}}$ of each π-factorizable morphism f belongs to Mor(**C**), while the image of every non-π-factorizable one g lies in Cprd($\mathbf{C}, \mathcal{H}$). Since both sets are disjoint, $F_{\mathcal{H}}(f) \neq F_{\mathcal{H}}(g)$.

2) According to the preceding statement, there is the isomorphism $F_{\mathcal{H}} : \mathbf{C}_{\mathcal{H}} \to \mathbf{C}[\mathcal{H}]$; hence we can define $T^\natural_{\mathcal{H}}$ by

$$T^\natural_{\mathcal{H}} = F^{-1}_{\mathcal{H}} \circ T^* \circ F_{\mathcal{H}} \tag{3.22}$$

where $T^* : \mathbf{C}[\mathcal{H}] \to \mathbf{D}[\mathcal{H}]$ is the extension of T defined in Theorem 2.1. Taking into account (3.13), every π-factorizable morphism $f \in \mathrm{Mor}(\mathbf{C}_{\mathcal{H}})$ satisfies the equation,

$$\Gamma_\alpha \circ T^\natural_{\mathcal{H}}(f) = \Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^* \circ F_{\mathcal{H}}(f) = T^* \circ \Gamma_\alpha(f) \tag{3.23}$$

because $\Gamma_\alpha = F_{\mathcal{H}}$. Since f is π-factorizable, $\Gamma_\alpha(f) \in \mathrm{Mor}(\mathbf{C})$, hence $T^* \circ \Gamma_\alpha(f) = T \circ \Gamma_\alpha(f)$. Thus, the former equation leads to

$$\Gamma_\alpha \circ T^\natural_{\mathcal{H}}(f) = T \circ \Gamma_\alpha(f) \tag{3.24}$$

For each non-π-factorizable morphism $f : X \times \mathcal{H} \to Y$,

$$\begin{aligned}\Gamma_\alpha \circ T^\natural_{\mathcal{H}}(f) &= \Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^* \circ F_{\mathcal{H}}(f) = \\ &\Gamma_\alpha \circ F^{-1}_{\mathcal{H}} \circ T^*\Big(\coprod_{\beta\in\mathcal{H}}\{\breve{f}_\beta\}\Big) = \\ &\Gamma_\alpha \circ F^{-1}_{\mathcal{H}}\Big(\coprod_{\beta\in\mathcal{H}}\{T(\breve{f}_\beta)\}\Big) = \Gamma_\alpha(h)\end{aligned} \tag{3.25}$$

where $h : T(X) \times \mathcal{H} \to T(Y)$ is the map defined by

$$\forall (x, \alpha) \in T(X) \times \mathcal{H} : \quad h(x, \alpha) = T(\breve{f}_\alpha)(x).$$

Thus, $\Gamma_\alpha(h) = \breve{h}_\alpha = T(\breve{f}_\alpha) = T(\Gamma_\alpha(f))$. This relation and equation (3.25) lead to equation (3.14).

3) The image of σ under $F_{\mathcal{H}}$ is $\coprod_{\alpha\in\mathcal{H}}\{\sigma_\alpha\}$. Since $T^\natural_{\mathcal{H}} = F^{-1}_{\mathcal{H}} \circ T^* \circ F_{\mathcal{H}}$ and $F_{\mathcal{H}}$ is a category isomorphism, statement **3)** is a consequence of Theorem 2.1.

4) If (X, σ_X) is a $T^\natural_{\mathcal{H}}$-algebra, for every $\alpha \in \mathcal{H}$, its image $\Gamma_\alpha(X, \sigma_X) = (\Gamma_\alpha(X), \Gamma_\alpha(\sigma_X))$ under Γ_α is a T-algebra. By definition, every set $X \in \mathrm{Obj}(\mathbf{C})$ remains unaltered under Γ_α. Accordingly, $(\Gamma_\alpha(X), \Gamma_\alpha(\sigma_X)) = (X, \Gamma_\alpha(\sigma_X))$. In addition, although $\sigma_X : T^\natural_{\mathcal{H}}(X) \times \mathcal{H} \to X$ is a $\mathbf{C}_{\mathcal{H}}$-morphism, its image under Γ_α is an ordinary map. According to statement 2), and taking into account (3.7),

$$\Gamma_\alpha\left(T^\natural_{\mathcal{H}}(X) \xrightarrow{\sigma_X} X\right) = \Gamma_\alpha \circ T^\natural_{\mathcal{H}}(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} \Gamma_\alpha(X) =$$

$$T \circ \Gamma_\alpha(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} \Gamma_\alpha(X) = T(X) \xrightarrow{\Gamma_\alpha(\sigma_X)} X \quad (3.26)$$

therefore, $(X, \Gamma_\alpha(\sigma_X))$ is a T-algebra, where $\Gamma_\alpha(\sigma_X)$ is either the image of α under the map $\breve{\sigma}_X \in (\mathrm{Hom}_{\mathbf{C}}(T(X), X))^{\mathcal{H}}$ whenever σ_X is not π-factorizable, or the map σ^*_X such that $\sigma_X = \sigma^*_X \circ \pi_X$ otherwise. Likewise, if $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is a morphism between two $T^\natural_{\mathcal{H}}$-algebras, the following quadrangle commutes.

$$\begin{array}{ccc} T^\natural_{\mathcal{H}}(X) & \xrightarrow{\sigma_X} & X \\ {\scriptstyle T^\natural_{\mathcal{H}}(f)}\downarrow & & \downarrow{\scriptstyle f} \\ T^\natural_{\mathcal{H}}(Y) & \xrightarrow[\sigma_Y]{} & Y \end{array} \quad (3.27)$$

Consequently, taking into account Statement 2), its image under Γ_α

$$\begin{array}{ccc} T(X) & \xrightarrow{\Gamma_\alpha(\sigma_X)} & X \\ {\scriptstyle T(\Gamma_\alpha(f))}\downarrow & & \downarrow{\scriptstyle \Gamma_\alpha(f)} \\ T(Y) & \xrightarrow[\Gamma_\alpha(\sigma_Y)]{} & Y \end{array} \quad (3.28)$$

is also commutative, and both $(X, \Gamma_\alpha(\sigma_X))$ and $(Y, \Gamma_\alpha(\sigma_Y))$ are ordinary T-algebras. The proof for co-algebras is the dual one.

□

Remark. The main application of the former result consists of considering most T-algebras (co-algebras) as restrictions or particular cases of $T^\natural_{\mathcal{H}}$-algebras (co-algebras) when we observe behavior changes. The members of $\mathcal{H}$ that work as parameters need not be ruled by the axioms of the extended constructs, and remain hidden until we observe either any anomalous event, or some behavior changes. In the following sections we expose two illustrative applications.

4. Bernoulli distribution with hidden parameters.

Probability spaces can be formalized as co-algebras. For instance, let $(\Omega, \mathcal{E}, P)$ be a probability space; where Ω is the set of outcomes, $\mathcal{E}$ the set of events, and $P : \mathcal{E} \to [0, 1]$ the probability assignation. If $T : \mathbf{Set} \to \mathbf{Set}$ is the endofunctor sending each set into $[0, 1]$, and every map $f : X \to Y$ into the identity $\mathrm{id} : [0, 1] \to [0, 1]$, then $P : \mathcal{E} \to T(\mathcal{E}) = [0, 1]$ gives rise to a co-algebra. A map $f : \mathcal{E}_1 \to \mathcal{E}_2$ is a morphism whenever the following quadrangle commutes.

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{P_1} & T(\mathcal{E}_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle T(f)=\mathrm{id}} \\ \mathcal{E}_2 & \xrightarrow[P_2]{} & T(\mathcal{E}_2) \end{array}$$

We can interpret these co-algebras as restrictions of those with hidden parameters, such that the probability assignations P_1 and P_2 depend on some parameter set $\mathcal{H}$. The following paragraphs illustrate these ideas.

Let X be a random variable, with Bernoulli distribution, like tossing a coin n-times. Let (T, P) be the associated co-algebra, where P denotes the probability assignation. Let $S = f_1, f_2, f_3 \ldots f_n$ be the observed relative frequency sequence of the event $X = 1$ (success) in some experiment. Suppose that the sequence S converges in probability to $\frac{1}{2}$, and the relative frequencies satisfy the relation $\forall n \in \mathbb{N} : f_n \leq \frac{1}{2}$. By the *weak law of large numbers* we know that $p = q = \frac{1}{2}$ and both events (success and failure) are equiprobable. Nevertheless, the relation $\forall n \in \mathbb{N} : f_n \leq \frac{1}{2}$ leads to $P(f_n \leq \frac{1}{2}) = 1$. This relation is not a consequence of probability laws. By contrast, it does not satisfy the expected symmetry in equiprobable situations. We can interpret this fact introducing hidden parameters as follows.

We can consider (T, P) as a particular case of an extension $(T^{\natural}_{\mathcal{H}}, \tilde{P})$ with a hidden parameter set $\mathcal{H} = \{\tau, \omega\}$, where the probability assignation is a $\mathbf{Set}_{\mathcal{H}}$-morphism $\tilde{P} : X \times \mathcal{H} \to T(X) = [0, 1]$ defined as follows.

$$\tilde{P}(X, \alpha) = \begin{cases} \frac{1}{2} \text{ if } (X, \alpha) = (0, \tau) \\ \frac{1}{2} \text{ if } (X, \alpha) = (1, \tau) \\ 1 \text{ if } (X, \alpha) = (0, \omega) \\ 0 \text{ if } (X, \alpha) = (1, \omega) \end{cases} \tag{4.1}$$

Now, suppose that

$$\forall n \in \mathbb{N} : \quad \alpha = \begin{cases} \omega \text{ if } n = 1 \\ \tau \text{ if } n > 1 \text{ and } f_{n-1} < \frac{1}{2} \\ \omega \text{ if } n > 1 \text{ and } f_{n-1} = \frac{1}{2} \end{cases} \tag{4.2}$$

With these conditions the relative frequency sequence of the event $X = 1$ converges in probability to $\frac{1}{2}$ and keeps always less than or equal to $\frac{1}{2}$. Notice that the parameter α takes the value ω whenever the event $f_n = \frac{1}{2}$ occurs; otherwise keeps equal to τ.

In the former example, we can see that hidden parameters correspond to "events" or "situations" that can occur in real world phenomena. This example is artificial, but there are natural random phenomena whose probability assignation can be modified by hidden parameters. For instance, the frequency under which a word "w" occurs increases its probability occurrence. However, in smart text, under excessive repetition the probability occurrence of "w" can vanish. Academic style, smartness, and word repetition can be regarded as hidden parameters that modify the occurrence probability of any word.

5. Structured Languages

As in (Palomar Tarancón, 2011), for each nonempty object-class $\mathbf{C}$, we denote by $\mathbf{C}^{\curlyvee}$ the generic object of $\mathbf{C}$. For instance, if $\mathbf{C}$ is the set $\{n \in \mathbb{N} \mid n \equiv 1 \,(\bmod 2)\}$, then $\mathbf{C}^{\curlyvee}$ denotes the concept of odd positive integer. To avoid any exception, we apply the same operator to singletons or one-member classes. The generic object of any singleton $\{O\}$ coincides with its unique member; hence

$$\{O\}^{\curlyvee} = O. \tag{5.1}$$

Definition 5.1. A predicate $P(X) \in \mathbf{Pr}$ is self-contradictory provided that $\neg P(X)$ is a tautology. ◁

It is straightforward consequence of the preceding definition that if $P(X)$ is a tautology, its negation $\neg P(X)$ is self-contradictory. If $P(X)$ is not self-contradictory there is at least one object O such that $P(O)$ is true; otherwise $\neg P(X)$ would be true for every value of X, hence a tautological predicate.

In this section, $\mathbf{Pr}$ denotes a predicate class of higher-order logic, being stable under conjunctions, disjunctions and negations. Likewise, $\mathbf{Mc}(\mathbf{Pr})$ denotes an object class satisfying the following axioms.

Axiom 5.1. *If a predicate $P(X) \in \mathbf{Pr}$ is neither self-contradictory nor tautological, the class $\mathbf{Mc}(\mathbf{Pr})$ contains the generic object $\{O \mid P(O)\}^{\curlyvee}$.*

Axiom 5.2. *For every $O \in \mathbf{Mc}(\mathbf{Pr})$ there is $P(X) \in \mathbf{Pr}$ such that*

$$\{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\} = \{O\}.$$

Definition 5.2. An attributive definition for a member O of $\mathbf{Mc}(\mathbf{Pr})$ is any predicate $P(X) \in \mathbf{Pr}$ such that $O = \{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\}^{\curlyvee}$. If the class $\{Q \in \mathbf{Mc}(\mathbf{Pr}) \mid P(Q)\}$ is a singleton, we say $P(X)$ to be a strictly attributive definition of O. ◁

Remark. In natural languages, most words denote generic objects of equivalence classes. For instance, the word "polygon" denotes a class that contains "triangles" and "quadrangles among others. Each of these words again denotes some object class. Attributive definitions consist of an attribute or property that is stated by a predicate $P(X)$. The defined object O is the generic one of the class that satisfies $P(X)$. Thus, if O_1 is a concretion of O obtained by adding another property $Q(X)$, that is, if O_1 is the generic object of the class $\{R \mid P(R) \wedge Q(R)\}$, then $P(X) \wedge Q(X) \Rightarrow P(X)$.

Lemma 5.1. *Each predicate $P(X) \in \mathbf{Pr}$ that is neither tautological nor self-contradictory, gives rise to a strictly attributive definition for some object $O \in \mathbf{Mc}(\mathbf{Pr})$.*

Proof. Let $P^*(Y, P(X))$ denote the predicate

$$\text{" } Y \text{ is the generic object of the class} \quad \mathbf{C} = \{O \in \mathbf{Mc}(\mathbf{Pr}) \mid P(O)\}."$$

The class $\mathbf{C}$ is nonempty because, by hypothesis, $P(X)$ is not self-contradictory (see Definition 5.1). According to Axiom 5.1 there is the generic object $\mathbf{C}^{\curlyvee}$ in $\mathbf{Mc}(\mathbf{Pr})$, besides, taking into account (5.1),

$$\{O \in \mathbf{Mc}(\mathbf{Pr}) \mid P^*(O, P(X))\}^{\curlyvee} = \{\mathbf{C}^{\curlyvee}\}^{\curlyvee} = \mathbf{C}^{\curlyvee}.$$

Consequently, it is a strictly definition. □

Definition 5.3. The class $\mathbf{Mc}(\mathbf{Pr})$ can be enriched with an order relation $\preceq$ such that, between every couple of objects O_1 and O_2, the relation $O_1 \preceq O_2$ holds whenever there are two attributive definitions $P_{O_1}(X)$ and $P_{O_2}(X)$ for O_1 and O_2, respectively, such that $P_{O_1}(X) \Rightarrow P_{O_2}(X)$. ◁

Enriched with the relation $\preceq$, the class **Mc**(**Pr**) satisfies the structure of a category **Mc**(**Pr**, $\preceq$) such that, for every couple of objects O_1 and O_2, the set $\mathrm{Hom}_{\mathbf{Mc}(\mathbf{Pr},\preceq)}(O_1, O_2)$ either is empty or it is the singleton $\{O_1 \preceq O_2\}$. From now on, we assume that the category **Mc**(**Pr**, $\preceq$) satisfies the following axiom.

Axiom 5.3. *The object-class of* **Mc**(**Pr**, $\preceq$) *contains with each subset* $\{O_i \mid i \in I\}$ *its coproduct* $\coprod_{i\in I} O_i$*; where I is any nonempty index set.*

Notation. For each phrase W in any meaningful language, we denote by $\|W\|$ the meaning associated to W.

Remark. Let $P_1(X)$, $P_2(X)$, and $P(X)$ be attributive definitions for O_1, O_2, and $O_1 \coprod O_2$, respectively. According to the definition of $\preceq$, the following relations are true: $P_1(X) \Rightarrow P(X)$ and $P_2(X) \Rightarrow P(X)$. Thus, $P(X)$ is the more restrictive definition that both objects O_1 and O_2 satisfy. In other words, $O_1 \coprod O_2$ is the more concrete abstraction of both objects O_1 and O_2. For instance,

$$\|\textit{Large positive integer}\| \coprod \|\textit{small positive integer}\| = \|\textit{positive integer}\|.$$

Notation. For every object O in **Mc**(**Pr**) the expression $|O|$ denotes the predicate class $\{P(X) \in \mathbf{Pr} \mid P(O)\}$.

Lemma 5.2. *For every object* $O \in \mathrm{Ob}\,(\mathbf{Mc}(\mathbf{Pr}, \preceq))$ *and each predicate* $Q(X) \in |O|$*, the statement*

$$\forall P(X) \in |O| : \quad Q(X) \Rightarrow P(X) \tag{5.2}$$

is true if and only if $Q(X)$ *is a strictly attributive definition for* O*.*

Proof. First assume $Q(X)$ to be a strictly attributive definition for O, and let $P(X)$ be a member of $|O|$. Suppose that (5.2) is false; hence there is O_1 such that the conjunction $Q(O_1) \wedge (\neg P(O_1))$ is true. Since $Q(X)$ is a strictly attributive definition for O, this relation leads to $O = O_1$ because, by Definition 5.2, the set $\{X \mid Q(X)\}$ must be a singleton. Consequently, these relations lead to $\neg P(O)$, which contradicts the initial assumption $P(X) \in |O|$.

Now suppose that (5.2) holds, and let $Q_1(X)$ be a strictly attributive definition for O. As we have just seen, $Q_1(X) \Rightarrow Q(X)$. Since O must satisfy its own definition $Q_1(X) \in |O|$. As a consequence of (5.2) this membership relation leads to $Q(X) \Rightarrow Q_1(X)$; consequently $Q_1(X) \Leftrightarrow Q(X)$, and $Q(X)$ is also an attributive definition for O. □

Definition 5.4. From now on, we term structured language on a category **Mc**(**Pr**, $\prec$) each 4–tuple $\mathfrak{L} = (A, A^*, A^{**}, M)$ such that,

1. The set A is a finite collection of symbols (alphabet).
2. The set A^* is a partial (syntactic) free-monoid generated by A. We term "word" each member of A^*.
3. The set A^{**} is a partial free-monoid generated by A^*. We say each member of A^{**} to be a phrase.

4. The symbol M denotes a nonempty subset of A^{**} each of its members has a meaning lying in $\mathbf{Mc}(\mathbf{Pr})$. The set A^* contains words denoting the concepts of conjunction, disjunction, and negation. In addition, M is stable under conjunctions, disjunctions and negations.
5. The set M contains with each subset $\{W_i \mid i \in I\}$ a phrase the meaning of which is the coproduct $\coprod_{i\in I}\|W_i\|$, (see Axiom 5.3). ◁

The members of M can be also single words because each meaningful word can be regarded as a one-word phrase. As usual, we term sentence each meaningful phrase. Likewise, statements are truth-valued sentences.

Notation. For each structured language $\mathfrak{L} = (A, A^*, A^{**}, M)$, we denote by $\perp^{\curlyvee}$ a variable ranging over all phrases in A^{**}. This notation allows us to write patterns obtained from any phrase. For instance, consider a phrase $W = w_1w_2 \dots w_iw_{i+1} \dots w_{i+j} \dots w_n$, where the w_i are the involved words. Substituting the sub-phrase $V = w_iw_{i+1} \dots w_{i+j}$ by $\perp^{\curlyvee}$, we obtain the pattern $W_V(\perp^{\curlyvee})$ that sends each phrase $U = u_1, u_2 \dots u_k \in A^{**}$ into

$$W_V(U) = w_1w_2 \dots u_1u_2 \dots u_kw_{j+1} \dots w_n.$$

For instance, let W be the phrase

We can evaluate the area of every polygon.

If we substitute the one-word phrase "*polygon*" by $\perp^{\curlyvee}$, we obtain the pattern

$$W_V(\perp^{\curlyvee}) = \textit{We can evaluate the area of every } \perp^{\curlyvee}.$$

The subscript V in the expression W_V denotes V to be the sub-phrase that we substitute by the variable $\perp^{\curlyvee}$. If U = "*regular triangle,*" then

$$W_V(\textit{regular triangle}) = \textit{We can evaluate the area of every regular triangle.}$$

Definition 5.5. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. A pattern $W_V(\perp^{\curlyvee})$ is continuous provided that for every couple U_1 and U_2 of phrases in M the following conditions hold.

1. If both relations $W_V(U_1) \in M$ and $\|U_2\| \preceq \|U_1\|$ are true, then $W_V(U_2) \in M$.
2. Let $\mathbf{D} = \{U_i \mid i \in I\} \subseteq M$ be a subset with cardinality greater than 1. If a phrase $R \in M$ denotes the object $\coprod_{i\in I}\|U_i\|$, and for every $i \in I$: $W_V(U_i) \in M$, then $W_V(R) \in M$. ◁

Example 5.1. Let $W_V(\perp^{\curlyvee})$ be the English pattern "*The area of every* $\perp^{\curlyvee}$ *is finite.*" Let U_1 denote the word "*triangle*" and U_2 the phrase "*regular triangle.*" If M denotes the class of meaningful English sentences, then the phrase $W_V(U_1)$= *"The area of every triangle is finite"* belongs to M. Likewise, the relation $\|U_2\| \preceq \|U_1\|$ holds because if $\|U_2\|$ is a regular triangle, it is also a triangle. Indeed, $W_V(U_2) \in M$. Finally, $\|U_1\| \coprod \|U_2\| = \|U_1\|$, and by assumption, $W_V(U_1) \in M$.

Since the conjunction of a set of phrases is again a phrase, it is a straightforward consequence that the conjunction of a set of patterns is again a pattern. By definition, there is some symbol or word in each structured language that denotes conjunction. From now on, we denote by the symbol $\mathring{\wedge}$ the conjunction in any structured language. Thus, if the considered language is the English one, $\mathring{\wedge}$ stands for the word "and".

Proposition 5.1. *The conjunction of a set of continuous patterns is again continuous.*

Proof. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. Let $\mathbf{P} = \{W_{V_i}(\bot^\curlyvee) \mid i \in I\}$ a set of patterns in $\mathfrak{L}$ and $\mathbf{P}(\bot^\curlyvee) = \mathring{\bigwedge}_{i \in I} W_{V_i}(\bot^\curlyvee)$ the conjunction of all members of $\mathbf{P}$. Let $U_0 \in M$ and $U_1 \in M$ be two phrases such that $\|U_1\| \preceq \|U_0\|$ and

$$\forall i \in I: \quad W_{V_i}(U_0) \in M \tag{5.3}$$

By continuity,

$$\forall i \in I: \quad W_{V_i}(U_1) \in M \tag{5.4}$$

hence, taking into account Definition 5.4, $\mathbf{P}(U_0) \in M$ and $\mathbf{P}(U_1) \in M$. □

Theorem 5.1. *For every continuous pattern $W_V(\bot^\curlyvee)$ the following statements are true.*

1. *There is a $\preceq$-maximum element in the class*

$$\mathbf{W} = \{\|U\| \mid W_V(U) \in M)\}.$$

2. *If $\|U\|$ is the $\preceq$-maximum element of $\mathbf{W}$, the predicate*

$$P(X) = X \textit{ is the maximum element of } \mathbf{W}$$

is a strictly attributive definition of $\|U\|$, whenever $P(X) \in \mathbf{Pr}$.

Proof.

1. If every element in a chain $\|U_1\| \preceq \|U_2\| \preceq \cdots \preceq \|U_n\|$ lies in $\mathbf{W}$, by Definition 5.5, so does its upper bound $\coprod_{0<i\leq n} \|U_i\|$. Thus, $\mathbf{W}$ satisfies the conditions of Zorn's Lemma. Accordingly, there is, at least, one $\preceq$-maximal element $\|U_1\|$ in $\mathbf{W}$.
   To see that $\|U_1\|$ is the maximum element of $\mathbf{W}$, let $\|U\| \in \mathbf{W}$ be any member. By virtue of both Definition 5.4 and Definition 5.5, there is a phrase R in M such that $\|R\| = \|U\| \coprod \|U_1\|$; hence there are the $\mathbf{Mc}(\mathbf{Pr})$-morphisms $\|U_1\| \preceq \|R\|$ and $\|U\| \preceq \|R\|$. Since $\|U_1\|$ is maximal these relations lead to $\|R\| = \|U_1\|$ and $\|U\| \preceq \|R\| = \|U_1\|$. Accordingly, $\|U_1\|$ is comparable with every member of $\mathbf{W}$.
2. It is a straightforward consequence of the maximum-element uniqueness.

□

Definition 5.6. Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ be a structured language. A pattern class $\mathbf{Pt} = \{W_{i,V_i}(\bot^\curlyvee) \mid i \in I\}$ is compatible provided that there is at least one phrase U in M such that, for every $i \in I$: $W_{i,V_i}(U) \in M$. ◁

Recall that, by virtue of statement 4) in Definition 5.4, the conjunction of all phrases in $\mathbf{Pt}$ again belongs to M.

Notation. By $\in^\partial$ we denote the "sub-phrase/phrase" relationship. For instance, if

$$W = w_1 w_2 \ldots w_i w_{i+1} \ldots w_{i+j} \ldots w_n$$

is a phrase, the following expression denotes the word sequence $w_i w_{i+1} \ldots w_{i+j}$ to be a sub-phrase.

$$w_i w_{i+1} \ldots w_{i+j} \in^\partial w_1 w_2 \ldots w_i w_{i+1} \ldots w_{i+j} \ldots w_n.$$

From now on, for each phrase set A^{**} and every $V \in A^{**}$, the expression A_V^{**} denotes the subset $A_V^{**} = \{W \in A^{**} \mid V \in^\partial W\}$. Likewise, $[A^{**}, M]$ denotes the phrase-set collection

$$[A^{**}, M] = \bigcup_{V \in M} \{X \subseteq A_V^{**} \mid V \in X\} \tag{5.5}$$

Finally, for every couple of phrases V_1 and V_2, the expression $\langle V_1 \leftrightarrows V_2 \rangle : M \to M$ denotes the result of substituting each occurrence of the sub-phrase V_1 in W by one of V_2. If W does not contain any occurrence of V_1, then $\langle V_1 \leftrightarrows V_2 \rangle W = W$. Likewise, the infix operator $\leftrightarrows$ can be used to obtain patterns; for instance $\langle V_1 \leftrightarrows \perp^\curlyvee \rangle W = W_{V_1}(\perp^\curlyvee)$.

Notation. From now on, for each $V \in M$ and every $X \subseteq A_V^{**}$, the expression $\mathrm{Pat}(X)$ denotes the pattern class defined as follows.

$$\mathrm{Pat}(V, X) = \{\langle V \leftrightarrows \perp^\curlyvee \rangle W \mid W \in X\}$$

Proposition 5.2. *If $\mathfrak{L} = (A, A^*, A^{**}, M)$ is a structured language, for every $V \in M$, each subset E of A_V^{**} satisfies the following statements.*

1. *The pattern class $\mathrm{Pat}(V, E) = \{\langle V \leftrightarrows \perp^\curlyvee \rangle W \mid W \in E\}$ is compatible.*
2. *Let E_0 a nonempty subset of E. Let $\mathbf{U}_V(\perp^\curlyvee)$ and $\mathbf{V}_V(\perp^\curlyvee)$ be the conjunctions of the pattern classes $\mathrm{Pat}(V, E)$ and $\mathrm{Pat}(V, E_0)$, respectively. If both patterns $\mathbf{U}_V(\perp^\curlyvee)$ and $\mathbf{V}_V(\perp^\curlyvee)$ are continuous, the maximum elements $\|U\|$ and $\|U_0\|$ of the classes $\mathbf{W} = \{\|X\| \mid \mathbf{U}_V(X) \in M\}$ and $\mathbf{W}_0 = \{\|X\| \mid \mathbf{V}_V(X) \in M\}$ respectively, satisfy the relation $\|U\| \preceq \|U_0\|$.*

Proof. 1. By definition, for each $W \in E$: $W_V(V) = W$; hence

$$\forall W_V \in \mathrm{Pat}(V, E) : \; W_V(V) \in M.$$

2. Since $\mathrm{Pat}(V, E_0)$ is a subset of $\mathrm{Pat}(V, E)$, for each phrase P the relation $\mathbf{U}_V(P) \in M$ leads to $\mathbf{V}_V(P) \in M$; therefore $\|U\|$ belongs to $\mathbf{W}_0$. By assumption, $\|U_0\|$ is the maximum element of the class $\mathbf{W}_0$, then $\|U\| \preceq \|U_0\|$.

□

Lemma 5.3. *For every $E \in [A^{**}, M]$, there is a unique $V \in M$ such that $E \subseteq A_V^{**}$ and $V \in E$.*

Proof. It is a straightforward consequence of (5.5). □

Definition 5.7. For each structured language $\mathfrak{L} = (A, A^*, A^{**}, M)$, the expression $\mathbf{Ph}(\mathfrak{L})$ denotes the small category the object class of which is

$$\mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L})) = [A^{**}, M]$$

For every pair of objects E_1 and E_2, the homset $\mathrm{Hom}_{\mathbf{Ph}(\mathfrak{L})}(E_1, E_2)$ consists of each map $f : E_1 \to E_2$ that satisfies the following condition.

$$\forall W \in E_1 : \quad \langle V_1 \leftrightarrows V_2 \rangle W = f(W) \tag{5.6}$$

where V_1 and V_2 are members of M such that $E_1 \subseteq A^{**}_{V_1}$ and $E_2 \subseteq A^{**}_{V_2}$ ◁

Recall that, by virtue of Lemma 5.3, for every $\mathbf{Ph}(\mathfrak{L})$-object E, there is $V \in M$ such that $E \subseteq A^{**}_V$ and $V \in E$.

The map $\mathfrak{T}_O : \mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L})) \to \mathrm{Ob}\,(\mathbf{Ph}(\mathfrak{L}))$ sending each set $E \in A^{**}_V$ into the singleton $\mathfrak{T}_O(E) = \{V\}$ is the object-map for an endofunctor $\mathfrak{T} : \mathbf{Ph}(\mathfrak{L}) \to \mathbf{Ph}(\mathfrak{L})$ that sends each morphism $f \in \mathrm{Hom}(E_1, E_2)$ into the map $\mathfrak{T}(f) : \{V_1\} \to \{V_2\}$ such that $V_1 \mapsto V_2$. Indeed, this map definition satisfies the condition $\langle V_1 \leftrightarrows V_2 \rangle V_1 = V_2$. We denote this endofunctor by $\mathfrak{T}$.

Proposition 5.3. *Let $\mathfrak{L} = (A, A^*, A^{**}, M)$ a structured language. Let V_1 and V_2 two members of M. If two $\mathfrak{T}$-algebras (E_1, σ_1) and (E_2, σ_2) satisfy the following hypotheses*

1. *There is a morphism $f : (E_1, \sigma_1) \to (E_2, \sigma_2)$.*
2. *The sets E_1 and E_2 are subsets of $A^{**}_{V_1}$ and $A^{**}_{V_2}$, respectively. In addition, all members of both pattern classes*
$$\mathrm{Pat}(V_1, \sigma_1(V_1)) = \{\langle V_1 \leftrightarrows \perp^{\curlyvee} \rangle W \mid W \in \sigma_1(V_1)\}$$
and
$$\mathrm{Pat}(V_2, \sigma_2(V_2)) = \{\langle V_2 \leftrightarrows \perp^{\curlyvee} \rangle W \mid W \in \sigma_2(V_2)\}$$
are continuous.
3. *The objects $\|V_1\|$ and $\|V_2\|$ are the $\preceq$-maximum elements of the object classes $\mathbf{W}_1 = \{\|X\| \mid \mathbf{P}_1(X) \in M\}$ and $\mathbf{W}_2 = \{\|X\| \mid \mathbf{P}_2(X) \in M\}$, respectively; where*
$$\mathbf{P}_1(\perp^{\curlyvee}) = \mathring{\bigwedge}_{W(\perp^{\curlyvee}) \in \mathrm{Pat}\left(V_1, \sigma_1(V_1)\right)} W(\perp^{\curlyvee}),$$
and
$$\mathbf{P}_2(\perp^{\curlyvee}) = \mathring{\bigwedge}_{W(\perp^{\curlyvee}) \in \mathrm{Pat}\left(V_2, \sigma_2(V_2)\right)} W(\perp^{\curlyvee}),$$
respectively.

then the phrases V_1 and V_2 satisfy the relation $\|V_2\| \preceq \|V_1\|$.

Proof. By the definition of $\mathfrak{T}$, and taking into account hypothesis 2), the following relations are true.

$$\begin{cases} \mathfrak{T}(E_1) = \{V_1\} \\ \mathfrak{T}(E_2) = \{V_2\} \\ \sigma_1(V_1) \subseteq E_1 \\ \sigma_2(V_2) \subseteq E_2 \end{cases} \tag{5.7}$$

The existence of the morphism f leads to the relation

$$\forall W \in \sigma_1(V_1): \quad \langle V_1 \leftrightarrows V_2 \rangle W = f(W) \in \sigma_2(V_2) \tag{5.8}$$

therefore $\mathrm{Pat}(V_1, \sigma_1(V_1)) \subseteq \mathrm{Pat}(V_2, \sigma_2(V_2))$. By virtue of Proposition 5.2, this relation leads to $\|V_2\| \preceq \|V_1\|$. □

Remark. By the former proposition we know that $\|U_2\| \preceq \|U_1\|$; accordingly if $P_1(X)$ and $P_2(X)$ are attributive definitions of $\|U_1\|$ and $\|U_2\|$, respectively, the relation $P_2(X) \Rightarrow P_1(X)$ holds (see Definition 5.3). We can deduce this relation, simply, by knowing that substituting V_1 by V_2 in every member of the phrase set $\sigma_1(V_1)$ we obtain a subset of $\sigma_1(V_2)$. This property is a straightforward consequence of the $\mathbf{Ph}(\mathfrak{L})$-morphism definition. Thus, observing occurrences of some sub-phrases in two phrase sets we can find logical implications between attributive definitions of their meanings blindly, that is, without knowing what they mean. Nevertheless, several meanings can be assigned to the same phrase in natural languages or artificial ones, depending on the context, state, style, among other circumstances. Accordingly, contexts, states, styles work as hidden parameters in a set $\mathcal{H}$. Consequently, to apply the method arising from the preceding result, and to interpret sentences in a language properly, we must consider that each $\mathfrak{T}$-algebra (E, σ) is a particular case of a $\mathfrak{T}^{\natural}_{\mathcal{H}}$-algebra; where the members of $\mathcal{H}$ denote states, contexts, styles, frequencies and any other event modifying the meaning of any phrase.

6. Conclusion

Hidden parameters are handled implicitly in Computer Science and Linguistics. We can find noticeable instances almost in each subject. This is a very exciting research field. Theorem 3.1 is the bridge between structured sets, namely, algebras (co-algebras), and any set of hidden parameters that modify their behavior. For instance, the research of those relative frequency anomalies that can be interpreted as the action of hidden parameters is an open problem.

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