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An Application for Certain Subclasses of p-Valent Meromorphic Functions Associated with the Generalized Hypergeometric Function

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Abstract

The main object of this paper is to give an application of a linear operator $H^{m,\mu}_{p,q,s}(\alpha_1)f(z)$ involving the generalized hypergeometric function. We define subclasses of the meromorphic function class $\Sigma_{p,m}$ by means of operator $H^{m,\mu}_{p,q,s}(\alpha_1)f(z)$.

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1. Introduction and definitions

Let $\Sigma_{p,m}$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U\backslash\{0\}$. We also denote $\Sigma_{p,1-p} = \Sigma_p$.
A function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma S^*_p(\alpha)$ of meromorphically p-valent starlike functions of order α in U if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < -\alpha \quad (z \in U; 0 \leqslant \alpha < p). \tag{1.2}$$

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Also a function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphically p-valent convex of order α in U if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) < -\alpha \quad (z \in U; 0 \leqslant \alpha < p). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$f(z) \in \Sigma C_p(\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \Sigma S_p^*(\alpha). \tag{1.4}$$

For a function $f \in \Sigma_{p,m}$, we say that $f \in \Sigma K_p(\beta,\alpha)$ if there exists a function $g \in \Sigma S_p^*(\alpha)$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) < -\beta \quad (z \in U; 0 \leqslant \alpha,\beta < p). \tag{1.5}$$

Functions in the class $\Sigma K_p(\beta,\alpha)$ are called meromorphically p-valent close-to-convex functions of order β and type α. We also say that a function $f \in \Sigma_{p,m}$ is in the class $\Sigma K_p^*(\beta,\alpha)$ of meromorphically quasi-convex functions of order β and type α if there exists a function $g \in \Sigma C_p(\alpha)$ such that

$$\Re\left(\frac{(zf'(z))'}{g'(z)}\right) < -\beta \quad (z \in U; 0 \leqslant \alpha,\beta < p). \tag{1.6}$$

It follows from (1.5) and (1.6) that

$$f(z) \in \Sigma K_p^*(\beta,\alpha) \Leftrightarrow -\frac{zf'(z)}{p} \in \Sigma K_p(\beta,\alpha),$$

where $\Sigma S_p^*(\alpha)$ and $\Sigma C_p(\alpha)$ are, respectively, the classes of meromorphically p-valent starlike functions of order α and meromorphically p-valent convex functions of order α $(0 \leqslant \alpha < p)$(see Aouf (Aouf, 2008) and Frasin (Frasin, 2012)).

For a function $f(z) \in \Sigma_{p,m}$, given by (1.1) and $g(z) \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k,$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$f(z) * g(z) = (f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g * f)(z) \ \ (p \in \mathbb{N}).$$

For real or complex numbers

$$\alpha_1, ..., \alpha_q \ \text{ and } \beta_1, ..., \beta_s \ \ (\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., s),$$

we consider the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, (Kiryakova, 2011, p.19))

$$ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k} . \frac{z^k}{k!} $$

$$ (q \leqslant s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U), $$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$ (\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0; \theta \in \mathbb{C} \backslash \{0\} = \mathbb{C}^*), \\ \theta(\theta - 1)...(\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} $$

Corresponding to the function $\phi_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ given by

$$ \phi_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-p} \; {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) \; , $$

we introduce a function $\phi_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ defined by

$$ \phi_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * \phi_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \frac{1}{z^p(1 - z)^{\mu + p}} $$
$$ (\mu > -p; z \in U^*). $$

We now define a linear operator $H^{m,\mu}_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \Sigma_{p,m} \to \Sigma_{p,m}$ by

$$ H^{m,\mu}_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = \phi_{p,\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z) \tag{1.7} $$

$$ \left(\alpha_i, \beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^-; i = 1, ..., q, \, j = 1, ..., s \, ; \mu > -p, f \in \Sigma_{p,m}; z \in U^*\right). $$

For; convenience, we write

$$ H^{m,\mu}_{p,q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) = H^{m,\mu}_{p,q,s}(\alpha_1) $$

and

$$ H^{1-p,\mu}_{p,q,s}(\alpha_1) = H^{\mu}_{p,q,s}(\alpha_1) \quad (\mu > -p). $$

If $f(z)$ is given by (1.1), then from (1.7), we deduce that

$$ H^{m,\mu}_{p,q,s}(\alpha_1) f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\mu + p)_{p+k} (\beta_1)_{p+k} ... (\beta_s)_{p+k}}{(\alpha_1)_{p+k} ... (\alpha_q)_{p+k}} a_k z^k \quad (\mu > -p; z \in U^*). \tag{1.8} $$

It is easily follows from (1.8) that

$$ z \left(H^{m,\mu}_{p,q,s}(\alpha_1) f(z)\right)' = (\mu + p) H^{m,\mu+1}_{p,q,s}(\alpha_1) f(z) - (\mu + 2p) H^{m,\mu}_{p,q,s}(\alpha_1) f(z). \tag{1.9} $$

From the identity (1.9), we readily have

$$z\left(H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)\right)' = (\mu+p-1)H_{p,q,s}^{m,\mu}(\alpha_1)f(z) - (\mu+2p-1)H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z) \tag{1.10}$$

and

$$z\left(H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)\right)' = (\mu+p+1)H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z) - (\mu+2p+1)H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z). \tag{1.11}$$

The linear operator $H_{p,q,s}^{m,\mu}(\alpha_1)$ was introduced by Patel and Palit (Patel & Palit, 2009) .

We note that the linear operator $H_{p,q,s}^{m,\mu}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator (Choi *et al.*, 2002) for analytic functions and is essentially motivated by the operators defined and studied in (Cho & Noor, 2006) (see also, (Dziok & Srivastava, 1999), (Dziok & Srivastava, 2003), (Srivastava, 2007) and (Srivastava & Karlsson, 1985)).

Specializing the parameters $\mu, \alpha_i (i = 1, 2, ...q), \beta_j (j = 1, 2, ...s), q$ and s we obtain the following :

(i) $H_{p,2,1}^{m,0}(p, p; p)f(z) = H_{p,2,1}^{m,1}(p+1, p; p)f(z) = f(z)$;

(ii) $H_{p,2,1}^{m,1}(p, p; p)f(z) = \frac{2pf(z)+zf'(z)}{p}$;

(iii) $H_{p,2,1}^{m,2}(p+1, p; p)f(z) = \frac{(2p+1)f(z)+zf'(z)}{p+1}$;

(iv) $H_{p,1,1}^{m,1-p}(c+1, 1; c)f(z) = J_{c,p}(f)(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt$ $(c > 0; z \in U^*)$, this integral operator is defined by

$$J_{c,p}(f)(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt \quad (c > 0; f \in \Sigma_{p,m}),$$

(v) $H_{p,2,2}^{m,0}(p+1, p; p)f(z) = \frac{p}{z^{2p}}\int_0^z t^{2p-1}f(t)dt$; $(p \in \mathbb{N}; z \in U^*)$;

(vi) $H_{p,2,1}^{1-p,n}(a, 1; a)f(z) = \frac{1}{z^p(1-z)^{n+p}} = D^{n+p-1}f(z)$ $(n > -p)$, the operator D^{n+p-1} studied by Ganigi and Uralegaddi (Ganigi & Uralegaddi, 1989), Yang (Yang, 1995), Aouf (Aouf, 1993), Aouf and Srivastava (Aouf & Srivastava, 1997) and Uralegaddi and Patil (Uralegaddi & Patil, 1989);

(vii) $H_{p,2,1}^{m,\mu}(c, p+\mu; a)f(z) = L_p(a, c)f(z)$ $(a, c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \mu > -p)$ (see Liu (Liu, 2002));

(viii) $H_{1,2,1}^{o,\mu}(\mu+1, n+1; \mu)f(z) = I_{n,\mu}f(z)$ $(\mu > 0; n > -1)$ (see Yuan et al. (Yuan *et al.*, 2008)).

We also observe that, for $m = 0$, $p = 1$ replacing μ by $\mu-1$, we have the operator $H_{1,\mu,q,s}^{0}(\alpha_1)f(z) = H_{\mu,q,s}(\alpha_1)f(z)$ defined by Cho and Kim (Cho & Kim, 2007).

The object of the present paper is to investigate some properties of meromorphic p- valent functions by the above operator $H_{p,q,s}^{m,\mu}(\alpha_1)f(z)$ given by (1.8).

Definition 1.1. *Let $\mathcal{H}$ the set of complex valued functions $h(r, s, t) : \mathbb{C}^3 \to \mathbb{C}$ such that*

$$h(r, s, t) \text{ is continuous in a domain } D \subset \mathbb{C}^3;$$

$$(1, 1, 1) \in D \quad \text{and} \quad |h(1, 1, 1)| < 1;$$

$$\left|h\left(e^{i\theta}, \frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta, \frac{2}{\mu+p+1}+\frac{\mu+p-1}{\mu+p+1}e^{i\theta}+\right.\right.$$

$$\left.\left.\frac{1}{\mu+p+1}\delta+\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right)\right| \geqslant 1$$

whenever

$$\left(e^{i\theta}, \frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta, \frac{2}{\mu+p+1}+\frac{\mu+p-1}{\mu+p+1}e^{i\theta}+\right.$$

$$\left.\frac{1}{\mu+p+1}\delta+\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right) \in D$$

with $\Re(\beta \geqslant \delta(\delta-1))$ *for real* θ, $\delta \geqslant 1$ *and* $\lambda > 0$.

2. The Main Result

In order to prove our main result, we recall the following lemma due to Miller and Mocanu (Miller & Mocanu, 1978).

Lemma 2.1. *Let* $w(z) = a + w_n z^n + ...$*,be analytic in* $U = \{z : z \in \mathbb{C}$ *and* $|z| < 1\}$ *with* $w(z) \neq a$ *and* $n \geqslant 1$*. If* $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ *and* $|w(z_0)| = \max_{|z|\leqslant r_0} |w(z)|$ *. Then*

$$zw^{'}(z_0) = \delta w(z_0) \tag{2.1}$$

and

$$\Re(\left\{1+\frac{z_0 w^{''}(z_0)}{w^{'}(z_0)}\right\}) \geqslant \delta, \tag{2.2}$$

where δ *is a real number and*

$$\delta \geqslant n\frac{|w(z_0)-a|^2}{|w(z_0)|^2-|a|^2} \geqslant n\frac{|w(z_0)|-|a|}{|w(z_0)|+|a|}.$$

Theorem 2.1. *Let* $h(r, s, t) \in \mathcal{H}$ *and let* $f \in \Sigma_{p,m}$ *satisfies*

$$\left(\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+2}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}\right) \in D \subset \mathbb{C}^3 \tag{2.3}$$

and

$$\left| h\left(\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}, \frac{H^{m,\mu+2}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)} \right) \right| < 1 \tag{2.4}$$

for all $z \in U$ and for some $m \in \mathbb{N}$.Then we have

$$\left| \frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} \right| < 1 \qquad (z \in U; \mu > -p, 0 \leqslant \alpha < p; p \in \mathbb{N}).$$

Proof. Let

$$\frac{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} = w(z). \tag{2.5}$$

Then it follows that $w(z)$ is either analytic or meromorphic in U, $w(0) = 1$ and $w(z) \neq 1$. Differentiating (2.5) logarithmically and multiply by z, we obtain

$$\frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} - \frac{z\left(H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu-1}_{p,q,s}(\alpha_1)f(z)} = \frac{zw'(z)}{w(z)}.$$

Using the identities (1.6) and (1.10) , we have

$$\frac{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} = \frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}. \tag{2.6}$$

Differentiating (2.6) logarithmically and multiply by z, we obtain

$$\frac{z\left(H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu+1}_{p,q,s}(\alpha_1)f(z)} - \frac{z\left(H^{m,\mu}_{p,q,s}(\alpha_1)f(z)\right)'}{H^{m,\mu}_{p,q,s}(\alpha_1)f(z)} = \frac{z\left[\frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}\right]'}{\frac{1}{\mu+p} + \frac{\mu+p-1}{\mu+p}w(z) + \frac{1}{\mu+p}\frac{zw'(z)}{w(z)}} \tag{2.7}$$

$$= \frac{(\mu+p-1)zw'(z) + \left[\frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1 + (\mu+p-1)w(z) + \frac{zw'(z)}{w(z)}}$$

Using the identities (1.9) and (1.11) , we have

$$
\begin{aligned}
(\mu+p+1)\frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)} &= 1+(\mu+p)\frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}+ \\
&\quad \frac{(\mu+p-1)zw'(z)+\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}} \\
&= 1+\left[1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}\right]+ \\
&\quad \frac{(\mu+p-1)zw'(z)+\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{1+(\mu+p-1)w(z)+\frac{zw'(z)}{w(z)}}.
\end{aligned}
$$

We claim that $|w(z)|<1$ for $z\in U$. Otherwise there exists a point $z_0\in U$ such that $\max\limits_{|z|\leqslant r_0}|w(z)| = |w(z_0)|=1$. Letting $w(z_0)=e^{i\theta}$ and using Lemma 2.1 with $a=1$ and $n=1$, we have

$$
\begin{aligned}
\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)} &= e^{i\theta}, \\
\frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)} &= \frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta, \\
\frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)} &= \frac{2}{(\mu+p+1)}+\frac{(\mu+p-1)}{(\mu+p+1)}e^{i\theta}+\frac{1}{(\mu+p+1)}\delta \\
&\quad +\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p+1)(\mu+p-1)e^{i\theta}+(\mu+p+1)\delta},
\end{aligned}
$$

where

$$\beta=\frac{z^2w''(z)}{w(z)} \quad \text{and} \quad \delta\geqslant 1.$$

Further, an application of (2.2) in Lemma 2.1 given $\Re(\beta\geqslant\delta(\delta-1)$. Since $h(r,s,t)\in\mathcal{H}$, we have

$$
\begin{aligned}
&\left|h\left(\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)},\frac{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)},\frac{H_{p,q,s}^{m,\mu+2}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu+1}(\alpha_1)f(z)}\right)\right| \\
&=\left|h\left(e^{i\theta},\frac{1}{\mu+p}+\frac{\mu+p-1}{\mu+p}e^{i\theta}+\frac{1}{\mu+p}\delta,\frac{2}{\mu+p+1}+\frac{\mu+p-1}{\mu+p+1}e^{i\theta}+\right.\right. \\
&\left.\left.\frac{1}{\mu+p+1}\delta+\frac{(\mu+p-1)\delta e^{i\theta}+[\delta+\beta-\delta^2]}{(\mu+p+1)+(\mu+p-1)(\mu+p+1)e^{i\theta}+(\mu+p+1)\delta}\right)\right|\geqslant 1
\end{aligned}
$$

which contradicts the condition (2.4) of Theorem 2.1. Therefore, we conclude that

$$\left|\frac{H_{p,q,s}^{m,\mu}(\alpha_1)f(z)}{H_{p,q,s}^{m,\mu-1}(\alpha_1)f(z)}\right| < 1 \quad (z \in U).$$

The proof is complete. □

Letting $\mu = 1, q = 2, s = 1, \alpha_1 = p+1, \alpha_2 = p$ and $\beta_1 = p$ in Theorem 2.1, we have the following result.

Corollary 2.1. *Let $h(r,s,t) \in \mathcal{H}$ and let $f(z) \in \Sigma_{p,m}$ satisfies*

$$\left(\frac{2pf(z)+zf'(z)}{pf(z)}, \frac{p\left[(2p+1)f(z)+zf'(z)\right]}{(p+1)\left[2pf(z)+zf'(z)\right]},\right.$$

$$\left.\frac{(2p+2)(2p+1)f(z)+4(p+1)zf'(z)+z^2f''(z)}{(p+2)(2p+1)f(z)+zf'(z)}\right) \in D \subset \mathbb{C}^3$$

and

$$\left|h\left(\frac{2pf(z)+zf'(z)}{pf(z)}, \frac{p\left[(2p+1)f(z)+zf'(z)\right]}{(p+1)\left[2pf(z)+zf'(z)\right]},\right.\right.$$

$$\left.\left.\frac{(2p+2)(2p+1)f(z)+4(p+1)zf'(z)+z^2f''(z)}{(p+2)(2p+1)f(z)+zf'(z)}\right)\right| < 1$$

for all $z \in U$. Then we have

$$\left|\frac{2pf(z)+zf'(z)}{pf(z)}\right| < 1 \quad (z \in U).$$

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