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On Uniform h-Stability of Evolution Operators in Banach Spaces

Claudia-Luminiţa Mihiţ[a,*]

[a]*Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, V. Pârvan Blvd. No. 4, 300223 Timişoara, Romania.*

Abstract

The paper treats the general concept of uniform h-stability, as a generalization of uniform exponential stability for evolution operators in Banach spaces.
The main aim is to give necessary and sufficient conditions of Datko-type and Barbashin-type for this property and also criterias for uniform h-stability using Lyapunov functions. As particular cases, we obtain the results for uniform exponential stability.

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1. Preliminaries

One of the most important asymptotic properties studied for evolution operators is the uniform exponential stability. This concept was treated in a large number of papers and of the most important we recall (Coppel, 1965), (Lupa *et al.*, 2010), (Megan *et al.*, 2001), (van Neerven, 1995) and (Stoica & Megan, 2010).

In the last years, are considered more general concepts of stability, as h-stability (see (Megan, 1995)) or (h, k)-stability (see (Fenner & Pinto, 1997), (Megan & Cuc, 1997), (Minda & Megan, 2011)), where h and k are growth rates (i.e. nondecreasing functions with different properties).

In this paper is considered the concept of uniform h-stability, with $h : \mathbb{R}_+ \to [1, +\infty)$ a growth rate (more precisely a nondecreasing function with $\lim_{t\to+\infty} h(t) = +\infty$), for evolution operators in Banach spaces.
Are obtained necessary and sufficient conditions for this notion and as consequences, we emphasize the results for the case of uniform exponential stability.

In what follows, X represents a real or complex Banach space, X^* its topological dual and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. We will denote the norms on X, on X^*

*Corresponding author
Email address: mihit.claudia@yahoo.com (Claudia-Luminiţa Mihiţ)

and on $\mathcal{B}(X)$ by $\|\cdot\|$.
Also, Δ is the set of all the pairs $(t, s) \in \mathbb{R}_+^2$ with $t \geq s$ and I represents the identity operator on X.

Definition 1.1. A mapping $\Phi : \Delta \to \mathcal{B}(X)$ is called *evolution operator* on X if

(eo_1) $\Phi(t, t) = I$, for every $t \geq 0$;
(eo_2) $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$, for all (t, s) and $(s, t_0) \in \Delta$.

We consider $\Phi : \Delta \to \mathcal{B}(X)$ an evolution operator and $h : \mathbb{R}_+ \to [1, +\infty)$ a growth rate.

Definition 1.2. We say that Φ has a *uniform h-growth* if there exists $N \geq 1$ such that for all $(t, s, x) \in \Delta \times X$:

$$h(s)\|\Phi(t, s)x\| \leq Nh(t)\|x\|.$$

If $h(t) = e^{\alpha t}$, with $\alpha > 0$, then we say that Φ has a *uniform exponential growth.*

Definition 1.3. The evolution operator Φ is called *uniformly h-stable* if there exists $S \geq 1$ such that for all $(t, s, x) \in \Delta \times X$:

$$h(t)\|\Phi(t, s)x\| \leq S h(s)\|x\|.$$

In particular, if $h(t) = e^{\alpha t}$, with $\alpha > 0$, then we recover the concept of *uniform exponential stability* and α is called *stability constant*.

Remark. If Φ is uniform h-stable, then it has a uniform h-growth. In general, the converse implication is not valid.

Example 1.1. Considering the evolution operator $\Phi : \Delta \to \mathcal{B}(X)$, defined by

$$\Phi(t, s) = \frac{h(t)}{h(s)}, \quad \text{for all} \quad (t, s) \in \Delta,$$

it is easy to observe that Φ has a uniform h-growth, but Φ is not uniformly h-stable.

Remark. The evolution operator Φ has a uniform h-growth if and only if there exists $N \geq 1$ with

$$h(s)\|\Phi(t, t_0)x_0\| \leq Nh(t)\|\Phi(s, t_0)x_0\|,$$

for all $(t, s), \ (s, t_0) \in \Delta, \ x_0 \in X$.

Remark. Φ is uniformly h-stable if and only if there is $S \geq 1$ such that

$$h(t)\|\Phi(t, t_0)x_0\| \leq S h(s)\|\Phi(s, t_0)x_0\|,$$

for all $(t, s), \ (s, t_0) \in \Delta, \ x_0 \in X$.

Definition 1.4. We say that $\Phi : \Delta \to \mathcal{B}(X)$ is

(i) *strongly measurable* if for all $(s, x) \in \mathbb{R}_+ \times X$ the mapping

$$t \mapsto \|\Phi(t, s)x\| \text{ is measurable on } [s, +\infty);$$

(ii) **-strongly measurable* if for all $(t, x^*) \in \mathbb{R}_+ \times X^*$ the mapping

$$s \mapsto \|\Phi(t, s)^* x^*\| \text{ is measurable on } [0, t].$$

2. Necessary conditions for uniform h-stability

In this section we will denote by $\mathcal{H}$ the set of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there is a constant $M \geq 1$ such that

$$\int_s^{+\infty} \frac{dt}{h(t)} \leq \frac{M}{h(s)}, \quad \text{for all } s \geq 0.$$

Also, $\mathcal{H}_1$ represents the set of the growth rates $h : \mathbb{R}_+ \to [1, +\infty)$ with the property that there exist a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and a constant $M_1 \geq 1$ with

$$\int_s^{+\infty} \frac{h_1(t)}{h(t)} dt \leq M_1 \frac{h_1(s)}{h(s)}, \quad \text{for all } s \geq 0.$$

Remark. Denoting by $\mathcal{E}$ the set of functions $h : \mathbb{R}_+ \to [1, +\infty)$, $h(t) = e^{\alpha t}$, with $\alpha > 0$, it results that $\mathcal{E} \subset \mathcal{H} \cap \mathcal{H}_1$.

Remark. The growth rate $h : \mathbb{R}_+ \to [1, +\infty)$ is in $\mathcal{H}_1$ if and only if there exists a growth rate $h_2 : \mathbb{R}_+ \to [1, +\infty)$, defined by $h_2(t) = \dfrac{h(t)}{h_1(t)}$, for all $t \geq 0$ such that $h_2 \in \mathcal{H}$.

A first result concerning the connections between the uniform exponential stability and uniform h-stability of an evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is

Theorem 2.1. *Following statements are equivalent:*

(i) Φ *is uniformly exponentially stable;*
(ii) *there exists* $h \in \mathcal{H}_1$ *such that* Φ *is uniformly* h*-stable;*
(iii) *there exists* $h \in \mathcal{H}$ *such that* Φ *is uniformly* h*-stable.*

Proof. (1) ⇒ (2). It results for $h(t) = e^{\alpha t}$, with $\alpha > 0$.
(2) ⇒ (3). From the hypothesis, there is a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and $M_1 \geq 1$ with

$$\int_s^{+\infty} \frac{h_1(t)}{h(t)} dt \leq M_1 \frac{h_1(s)}{h(s)}, \quad \text{for all } s \geq 0$$

and using the second *Remark* from this section it follows that $h_2 \in \mathcal{H}$.

Thus, for all $(t, s, x) \in \Delta \times X$ we have

$$h_2(t)\|\Phi(t, s)x\| = \frac{h(t)}{h_1(t)}\|\Phi(t, s)x\| \leq$$

$$\leq S \frac{h(s)}{h_1(t)}\|x\| \leq S h_2(s)\|x\|,$$

which shows that Φ is h_2-stable.
(3) ⇒ (1). It is immediate from the first *Remark* of this section. □

We consider $\Phi : \Delta \to \mathcal{B}(X)$ a strongly measurable evolution operator and a first necessary condition of Datko-type, due to R. Datko ((Datko, 1972)) is

Theorem 2.2. *If $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly h-stable with $h \in \mathcal{H}_1$ then there are a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$ and a constant $D \geq 1$ such that*

$$\int_s^{+\infty} h_1(t) \|\Phi(t, t_0)x_0\| dt \leq D h_1(s) \|\Phi(s, t_0)x_0\|,$$

for all $(t, s), \ (s, t_0) \in \Delta, \ x_0 \in X$.

Proof. It is immediate for $D = M_1 S$, where M_1 and h_1 are given by definition of $\mathcal{H}_1$ and S is given by Definition 1.3. □

Corollary 2.1. *If $\Phi : \Delta \to \mathcal{B}(X)$ is uniformly exponentially stable, then there are the constants $\beta > 0$ and $D \geq 1$ such that*

$$\int_s^{+\infty} e^{\beta t} \|\Phi(t, t_0)x_0\| dt \leq D e^{\beta s} \|\Phi(s, t_0)x_0\|,$$

for all $(t, s), \ (s, t_0) \in \Delta, \ x_0 \in X$.

Proof. It is a particular case of Theorem 2.2. □

Definition 2.1. A mapping $L : \Delta \times X \to \mathbb{R}_+$ is said to be a *h-Lyapunov function for* Φ if

$$L(t, t_0, x_0) + \int_s^t h(\tau) \|\Phi(\tau, t_0)x_0\| d\tau \leq L(s, t_0, x_0),$$

for all $(t, s), \ (s, t_0) \in \Delta, x_0 \in X$.

In particular, if $h(t) = e^{\alpha t}$, with $\alpha > 0$, then the function L is called *exponential Lyapunov function*.

The importance of the Lyapunov functions in the study of the stability property is described for instance in (Barreira & Valls, 2008), (Barreira & Valls, 2013).

Another significant result for the uniform h-stability of an evolution operator is given by

Theorem 2.3. *If the evolution operator Φ is uniformly h-stable with $h \in \mathcal{H}_1$, then there exist a growth rate $h_1 : \mathbb{R}_+ \to [1, +\infty)$, a h_1-Lyapunov function for Φ and $D \geq 1$ such that*

$$L(s, s, x_0) \leq D h_1(s) \|x_0\|,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$.

Proof. Let $L : \Delta \times X \to \mathbb{R}_+,\ L(t,s,x_0) = \int\limits_t^{+\infty} h_1(\tau)\|\Phi(\tau,s)x_0\|d\tau.$

Thus, L is a h_1-Lyapunov function for Φ and using Theorem 2.2 we obtain

$$L(s,s,x_0) = \int\limits_s^{+\infty} h_1(\tau)\|\Phi(\tau,s)x_0\|d\tau \le Dh_1(s)\|x_0\|,$$

for all $(s,x_0) \in \mathbb{R}_+ \times X$. □

In particular, we obtain

Corollary 2.2. *If* $\Phi : \Delta \to \mathcal{B}(X)$ *is uniformly exponentially stable, then there are the constants* $\beta > 0$, $D \ge 1$ *and an exponential Lyapunov function* L *for* Φ *with*

$$L(s,s,x_0) \le De^{\beta s}\|x_0\|,$$

for all $(s,x_0) \in \mathbb{R}_+ \times X$.

We consider now the set $\tilde{\mathcal{H}}$ of the growth rates $h : \mathbb{R}_+ \to [1,+\infty)$ with the property that there is a growth rate $h_1 : \mathbb{R}_+ \to [1,+\infty)$ and a constant $\tilde{M} \ge 1$ with

$$\int\limits_0^t \frac{h(\tau)}{h_1(\tau)}d\tau \le \tilde{M}\frac{h(t)}{h_1(t)}, \quad \text{for all } t \ge 0.$$

Remark. It is easy to see that the functions $h \in \mathcal{E}$ (considered in Remark 2) are in $\tilde{\mathcal{H}}$.

Let $\Phi : \Delta \to \mathcal{B}(X)$ be a $*$-strongly measurable evolution operator. A first result for this type of evolution operators is proved by E. A. Barbashin ((Barbashin, 1967)) in the case of uniform exponential stability.
Concerning the uniform h-stability, we prove

Theorem 2.4. *If* Φ *is uniformly h-stable with* $h \in \tilde{\mathcal{H}}$, *then there is a growth rate* $h_1 : \mathbb{R}_+ \to [1,+\infty)$ *and* $B \ge 1$ *with*

$$\int\limits_0^t \frac{\|\Phi(t,\tau)^*x^*\|}{h_1(\tau)}d\tau \le \frac{B}{h_1(t)}\|x^*\|,$$

for all $(t,x^*) \in \mathbb{R}_+ \times X^*$.

Proof. It results using Definition 1.3 and the definition of $\tilde{\mathcal{H}}$, for $B = S\tilde{M}$. □

As a consequence of the above result, we obtain

Corollary 2.3. *If* Φ *is uniformly exponentially stable, then there are the constants* $\gamma > 0$ *and* $B \ge 1$ *such that*

$$\int\limits_0^t e^{-\gamma\tau}\|\Phi(t,\tau)^*x^*\|d\tau \le Be^{-\gamma t}\|x^*\|,$$

for all $(t,x^*) \in \mathbb{R}_+ \times X^*$.

3. Sufficient conditions for uniform *h*-stability

In what follows, we will denote by $\mathcal{H}_2$ the set of the functions $h : \mathbb{R}_+ \to [1, +\infty)$ with the property

$$\sup_{s\geq 0} \frac{h(s+1)}{h(s)} = M_2 < +\infty.$$

Remark. We observe that all the functions $h \in \mathcal{E}$ (defined in Remark 2) are in $\mathcal{H}_2$, i.e. $\mathcal{E} \subset \mathcal{H}_2$.

We consider $\Phi : \Delta \to \mathcal{B}(X)$ a strongly measurable evolution operator and a sufficient criteria of Datko-type is

Theorem 3.1. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there is $D \geq 1$ such that*

$$\int_s^{+\infty} h(t)\|\Phi(t,t_0)x_0\|dt \leq Dh(s)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta,\ x_0 \in X$, then Φ is uniformly h-stable.

Proof. Let $S = M_2^2 ND$.
Case 1. We consider $(t,s),\ (s,t_0) \in \Delta$ with $t \geq s+1$, $x_0 \in X$. Thus,

$$h(t)\|\Phi(t,t_0)x_0\| \leq \int_{t-1}^{t} h(t)\|\Phi(t,\tau)\| \cdot \|\Phi(\tau,t_0)x_0\|d\tau \leq$$

$$\leq N\int_{t-1}^{t} h(t)\frac{h(t)}{h(\tau)}\|\Phi(\tau,t_0)x_0\|d\tau \leq$$

$$\leq NM_2^2\int_s^{+\infty} h(\tau)\|\Phi(\tau,t_0)x_0\|d\tau \leq S\,h(s)\|\Phi(s,t_0)x_0\|.$$

It results that

$$h(t)\|\Phi(t,t_0)x_0\| \leq S\,h(s)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta$ with $t \geq s+1$, $x_0 \in X$.

Case 2. Let $(t,s),\ (s,t_0) \in \Delta$ with $t \in [s, s+1]$, $x_0 \in X$. We have

$$h(t)\|\Phi(t,t_0)x_0\| \leq h(t)\|\Phi(t,s)\| \cdot \|\Phi(s,t_0)x_0\| \leq$$

$$\leq N\frac{h^2(t)}{h^2(s)}h(s)\|\Phi(s,t_0)x_0\| \leq S\,h(s)\|\Phi(s,t_0)x_0\|.$$

In conclusion,

$$h(t)\|\Phi(t,t_0)x_0\| \leq S\,h(s)\|\Phi(s,t_0)x_0\|,$$

for all $(t,s),\ (s,t_0) \in \Delta$, $x_0 \in X$, which shows that Φ is uniformly h-stable. □

Corollary 3.1. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential growth. If there is $D \geq 1$ such that*

$$\int_s^{+\infty} e^{\alpha t}\|\Phi(t, t_0)x_0\|dt \leq De^{\alpha s}\|\Phi(s, t_0)x_0\|,$$

for all $(t, s), \ (s, t_0) \in \Delta, \ x_0 \in X$, then Φ is uniformly exponentially stable.

Proof. It results from Theorem 3.1. □

Theorem 3.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there exist a h-Lyapunov function for Φ and $D \geq 1$ with*

$$L(s, s, x_0) \leq Dh(s)\|x_0\|,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$, then Φ is uniformly h-stable.

Proof. From Definition 2.1, for $s = t_0$ we obtain

$$\int_s^t h(\tau)\|\Phi(\tau, s)x_0\|d\tau \leq L(s, s, x_0) \leq Dh(s)\|x_0\|,$$

for all $(t, s, x_0) \in \Delta \times X$ and for $t \to +\infty$, it follows that Φ is uniformly h-stable. □

In particular, a sufficient condition for the uniform exponential stability is given by

Corollary 3.2. *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator with uniform exponential growth. If there exist an exponential Lyapunov function for Φ and $D \geq 1$ such that*

$$L(s, s, x_0) \leq De^{\alpha t}\|x_0\|,$$

for all $(s, x_0) \in \mathbb{R}_+ \times X$, then Φ is uniformly exponentially stable.

A sufficient condition of Barbashin-type for the uniform h-stability of a $*$-strongly measurable evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is

Theorem 3.3. *We consider Φ an evolution operator with uniform h-growth and $h \in \mathcal{H}_2$. If there is $B \geq 1$ with*

$$\int_0^t \frac{\|\Phi(t, \tau)^* x^*\|}{h(\tau)}d\tau \leq \frac{B}{h(t)}\|x^*\|,$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$, then Φ is uniformly h-stable.

Proof. We consider $S = NM_2^2 B$.

Let $(t, s) \in \Delta$, $t \geq s + 1$ and $(x, x^*) \in X \times X^*$. Then,

$$h(t)| < x^*, \Phi(t, s)x > | = \int_s^{s+1} h(t)| < \Phi(t, \tau)^* x^*, \Phi(\tau, s)x > |d\tau \leq$$

$$\leq h(t)\int_s^{s+1} ||\Phi(t,\tau)^* x^*|| \cdot ||\Phi(\tau,s)x|| d\tau \leq$$

$$\leq Nh(t)\int_s^{s+1} \frac{||\Phi(t,\tau)^* x^*||}{h(\tau)} \frac{h^2(\tau)}{h^2(s)} h(s) d\tau ||x|| \leq$$

$$\leq S\,h(s)||x|| \cdot ||x^*||.$$

Considering the supremum relative to $||x^*|| \leq 1$ it results that

$$h(t)||\Phi(t,s)x|| \leq S\,h(s)||x||, \text{ for all } t \geq s+1, \ x \in X.$$

Let now $t \in [s, s+1], \ x \in X$. We obtain

$$h(t)||\Phi(t,s)x|| \leq N\frac{h^2(t)}{h(s)}||x|| \leq S\,h(s)||x||,$$

for all $t \in [s, s+1], \ x \in X$.
In conclusion, Φ is uniformly h-stable. □

As a particular case, we obtain

Corollary 3.3. *Let Φ be an evolution operator with uniform exponential growth. If there is $B \geq 1$ with*

$$\int_0^t e^{-\alpha\tau}||\Phi(t,\tau)^* x^*|| d\tau \leq Be^{-\alpha t}||x^*||,$$

for all $(t, x^*) \in \mathbb{R}_+ \times X^*$, then Φ is uniformly exponentially stable.

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