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On Certain Properties for Hadamard Product of Uniformly Univalent Meromorphic Functions with Positive Coefficients

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Abstract

In this paper we study some results concerning the Hadamard product of certain classes related to uniformly starlike and convex univalent meromorphic functions with positive coefficients.

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1. Introduction

Throughout this paper, let the functions of the form

$$\varphi(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n \qquad (c_1 > 0; c_n \geq 0), \tag{1.1}$$

and

$$\psi(z) = d_1 z - \sum_{n=2}^{\infty} d_n z^n \qquad (d_1 > 0; d_n \geq 0) \tag{1.2}$$

which are analytic and univalent in the unit disc

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\};$$

also, let

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_0 > 0; a_n \geq 0), \tag{1.3}$$

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$$f_i(z) = \frac{a_{0,i}}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n \qquad (a_{0,i} > 0; a_{n,i} \geq 0), \tag{1.4}$$

$$g(z) = \frac{b_0}{z} + \sum_{n=1}^{\infty} b_n z^n \qquad (b_0 > 0; b_n \geq 0), \tag{1.5}$$

$$g_j(z) = \frac{b_{0,j}}{z} + \sum_{n=1}^{\infty} b_{n,j} z^n \qquad (b_{0,j} > 0; b_{n,j} \geq 0). \tag{1.6}$$

which are analytic and univalent in the punctured unit disc

$$U^* = \{z : z \in \mathbb{C} \text{ and } 0 < \ |z| < 1\}.$$

A function $f(z) \in \Sigma$ is meromorhpically starlike of order α if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\,(z \in U^*; 0 \leq \alpha < 1). \tag{1.7}$$

A function f of the form (1.3) is said to be in the class $U\Sigma S_0^*(\alpha,\beta)$ of meromorphic uniformly β -starlike functions of order α if it satisfies the condition:

$$-Re\left\{\frac{zf'(z)}{f(z)} + \alpha\right\} > \beta\left|\frac{zf'(z)}{f(z)} + 1\right| (z \in U; 0 \leq \alpha < 1; \beta \geq 0). \tag{1.8}$$

Also, a function f of the form (1.3) is said to be in the class $U\Sigma C_0(\alpha,\beta)$ of meromorphic uniformly β -convex functions of order α if it satisfies the condition:

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)} + \alpha\right\} > \beta\left|2 + \frac{zf''(z)}{f'(z)}\right| (z \in U; 0 \leq \alpha < 1; \beta \geq 0). \tag{1.9}$$

It follows from (1.8) and (1.9) that

$$f \in U\Sigma C_0(\alpha,\beta) \iff -zf' \in U\Sigma S_0^*(\alpha,\beta). \tag{1.10}$$

The classes $U\Sigma S_0^*(\alpha,\beta)$ and $U\Sigma C_0(\alpha,\beta)$ have been studied by (Aouf *et al.*, 2014), (Atshan & Kulkarni, 2007), and others. We note that
(i) $U\Sigma S_0^*(\alpha,0) = S_n^*(\alpha)$ and $U\Sigma C_0(\alpha,0) = C_n(\alpha)$ (see (Aouf & Silverman, 2008), *with* $n = 1$);
(ii) $U\Sigma S_0^*(\alpha,0) = \Sigma_p S_n^*(\alpha,\gamma)$ and $U\Sigma C_0(\alpha,0) = \Sigma_p C_n(\alpha,\gamma)$ (also see (R. M. El-Ashwah & Hassan, 2013), with $n = p = \gamma = 1$);
(iii) $U\Sigma S_0^*(\alpha,0) = \Sigma S_0^*(\alpha)$ and $U\Sigma C_0(\alpha,0) = \Sigma K_0(\alpha,\beta)$ (see (Mogra, 1991)).

Lemma 1.1. *Let the function f defined by (1.3). Then $f \in U\Sigma S_0^*(\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty}[n(1+\beta) + (\alpha+\beta)]a_n \leq (1-\alpha)a_0. \tag{1.11}$$

Lemma 1.2 (3)**.** *Let the function f defined by (1.3). Then $f \in U\Sigma C_0(\alpha,\beta)$ if and only if*

$$\sum_{n=1}^{\infty} n[n(1+\beta)+(\alpha+\beta)]a_n \le (1-\alpha)a_0. \tag{1.12}$$

Definition 1.1. Let the function f defined by (1.3). Then $f \in U\Sigma S_m(\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty} n^m[n(1+\beta)+(\alpha+\beta)]a_n \le (1-\alpha)a_0, \tag{1.13}$$

where $(0 \le \beta < \infty)$, $(0 \le \alpha < 1)$ and m any positive integer number.

We note that $U\Sigma S_1(\alpha,\beta) = U\Sigma C_0(\alpha,\beta)$ and $U\Sigma S_0(\alpha,\beta)$ is equivalent to $U\Sigma S_0^*(\alpha,\beta)$. Further, $U\Sigma S_m(\alpha,\beta) \subset U\Sigma S_r(\alpha,\beta)$ if $m > r \ge 0$, the containment beign proper. Whence, for any positive integer m, we have the inclusion relation

$$U\Sigma S_m(\alpha,\beta) \subset U\Sigma S_{m-1}(\alpha,\beta) \subset ... \subset U\Sigma S_2(\alpha,\beta) \subset U\Sigma C_0(\alpha,\beta)\, U\Sigma S_0^*(\alpha,\beta).$$

Also, we note that for nonnegative real number m the class $U\Sigma S_m(\alpha,\beta)$ is nonempty as the functions of the form

$$f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} \frac{(1-\alpha)a_0}{n^m[n(1+\beta)+(\alpha+\beta)]}\lambda_n z^n,$$

where $a_0 > 0$, and $\sum_{n=1}^{\infty} \lambda_n \le 1$, satisfy the inequality (1.13). For the functions

$$f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \left(a_{n,j} \ge 0;\, j = 1,2\right). \tag{1.14}$$

We denote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of functions $f_1(z)$ and $f_2(z)$, that is

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}z^n. \tag{1.15}$$

Similarly, we can define the Hadamard product of more than two functions. The quasi-Hadamard product of two or more functions $\varphi(z)$ and $\psi(z)$ given by (1.1) and (1.2), (see (Kumar, 1987)).

$$(\varphi * \psi)(z) = c_1 d_1 z - \sum_{n=2}^{\infty} c_n d_n z^n \tag{1.16}$$

In this paper, we can discuss certain results concerning the Hadamard product of functions in the classes $U\Sigma S_0^*(\alpha,\beta)$, $U\Sigma S_m(\alpha,\beta)$ and $U\Sigma C_0(\alpha,\beta)$.

2. Main results

Theorem 2.1. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma C_0(\alpha,\beta)$ for every $i = 1,2,...,m$, and suppose that the functions $g_j(z)$ defined by (1.6) be in the class $U\Sigma S_0^*(\alpha,\beta)$ for every $j = 1,2,...,q$. Then the Hadamard product $(f_1 * f_2 * ... * f_m * g_1 * g_2 * ...g_q)(z)$ belongs to the class $U\Sigma S_{2m+q-1}(\alpha,\beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{q}b_{n,j}\right]\right\} \leq (1-\alpha)\left[\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q}b_{0,i}\right]. \tag{2.1}$$

Since $f_i(z) \in U\Sigma C_0(\alpha,\beta)$, we get

$$\sum_{n=1}^{\infty} n[n(1+\beta)+(\alpha+\beta)]a_{n,i} \leq (1-\alpha)a_{0,i} \quad (i = 1,2,...,m). \tag{2.2}$$

Therefore,

$$a_{n,i} \leq \frac{(1-\alpha)}{n[n(1+\beta)+(\alpha+\beta)]}a_{0,i} \tag{2.3}$$

which implies that

$$a_{n,i} \leq n^{-2}a_{0,i} \quad (i = 1,2,...,m). \tag{2.4}$$

Similarly, for $g_j(z) \in U\Sigma S_0^*(\alpha,\beta)$, we obtain

$$\sum_{n=1}^{\infty}[n(1+\beta)+(\alpha+\beta)]b_{n,j} \leq (1-\alpha)b_{0,j}, \tag{2.5}$$

for $j = 1,2,...,q$. Hence we have

$$b_{n,j} \leq n^{-1}b_{0,j} \quad (j = 1,2,...,q). \tag{2.6}$$

Using (2.4) for $i = 1,2,...,m$, (2.6) for $j = 1,2,...,q-1$, and (2.5) for $j = q$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\prod_{j=1}^{q}b_{n,j}\right]\right\}\\
\leq\ &\sum_{n=1}^{\infty}\left\{n^{2m+q-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-2m}n^{-(q-1)}\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q-1}b_{0,j}\right]b_{n,q}\right\}\\
=\ &\left[\prod_{i=1}^{m-1}a_{0,i}\prod_{j=1}^{q-1}b_{0,j}\right]\sum_{n=1}^{\infty}\left[n\{n(1+\beta)+(\alpha+\beta)\}b_{n,q}\right] \leq (1-\alpha)\prod_{i=1}^{m}a_{0,i}\prod_{j=1}^{q}b_{0,j}.
\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m * g_1 * g_2 * ...g_q)(z) \in U\Sigma S_{2m+q-1}(\alpha,\beta)$. The proof of Theorem 1 is completed. □

Theorem 2.2. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma C_0(\alpha,\beta)$ for every $i = 1,2,...,m$, then the Hadamard product $(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $U\Sigma S_{2m-1}(\alpha,\beta)$.*

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\right]\right\}\leq(1-\alpha)\left[\prod_{i=1}^{m}a_{0,i}\right]. \tag{2.7}$$

Since $f_i(z)\in U\Sigma C_0(\alpha,\beta)$, the inequalities (2.1) and (2.2) hold for every $i = 1,2,...,m$.

Using (2.2) for $i = 1,2,...,m-1$, and (2.1) for $i = 1,2,...,m$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\right]\right\}\\ \leq\ &\sum_{n=1}^{\infty}\left\{n^{2m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-2(m-1)}\prod_{i=1}^{m-1}a_{0,i}\right]a_{n,m}\right\}\\ =\ &\left[\prod_{i=1}^{m-1}a_{0,i}\right]\sum_{n=1}^{\infty}\left[n\{n(1+\beta)+(\alpha+\beta)\}a_{n,m}\right]\leq(1-\alpha)\prod_{i=1}^{m}a_{0,i}.\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m)(z)\in U\Sigma S_{2m-1}(\alpha,\beta)$. The proof of Theorem 2 is completed. □

Theorem 2.3. *Let the functions $f_i(z)$ defined by (1.4) be in the class $U\Sigma S_0^*(\alpha,\beta)$ for every $i = 1,2,...,m$, then the Hadamard product $(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $U\Sigma S_{m-1}(\alpha,\beta)$.*

Proof. Since $f_i(z)\in U\Sigma S_0^*(\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}[n(1+\beta)+(\alpha+\beta)]a_{n,i}\leq(1-\alpha)a_{0,i}, \tag{2.8}$$

for every $i = 1,2,...,m$. Therefore, we obtain $a_{n,i}\leq\frac{(1-\alpha)}{n(1+\beta)+(\alpha+\beta)}a_{0,i}$ which implies that

$$a_{n,i}\leq n^{-1}a_{0,i}\quad(i = 1,2,...,m). \tag{2.9}$$

Using (2.9) for $i = 1,2,...,m-1$, and (2.8) for $i = 1,2,...,m$, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\left\{n^{m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[\prod_{i=1}^{m}a_{n,i}\right]\right\}\\ \leq\ &\sum_{n=1}^{\infty}\left\{n^{m-1}\{n(1+\beta)+(\alpha+\beta)\}\left[n^{-(m-1)}\prod_{i=1}^{m-1}a_{0,i}\right]a_{n,m}\right\}\\ =\ &\left[\prod_{i=1}^{m-1}a_{0,i}\right]\sum_{n=1}^{\infty}\left[\{n(1+\beta)+(\alpha+\beta)\}a_{n,m}\right]\leq(1-\alpha)\prod_{i=1}^{m}a_{0,i}.\end{aligned}$$

Hence $(f_1 * f_2 * ... * f_m)(z)\in U\Sigma S_{m-1}(\alpha,\beta)$, which completes the proof of Theorem 3. □

Remark. Taking $\beta = 0$ in our main results, we obtain the results obtained by Mogra (Mogra, 1991).

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