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On BV_σ I-convergent Sequence Spaces Defined by an Orlicz Function

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Abstract

In this article we study ${}_0BV_\sigma^I(M)$, $BV_\sigma^I(M)$ and ${}_\infty BV_\sigma^I(M)$ sequence spaces with the help of BV_σ space see (Mursaleen, 1983*b*) and an Orlicz function M. we study some topological and algebraic properties of these spaces and prove some inclusion relations.

Keywords: bounded variation, invariant mean, σ-Bounded variation, ideal, filter, Orlicz function, I-convergence, I-null, solid space, sequence algebra, convergence free space.
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1. Introduction and Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers respectively. We denote

$$\omega = \{x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the space of all real or complex sequences. Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively with norm $\|x\| = \sup_k \mid x_k \mid$.

Let v be denote the space of sequences of bounded variation. That is,

$$v = \left\{x = (x_k) : \sum_{k=o}^{\infty} \mid x_k - x_{k-1} \mid < \infty, x_{-1} = 0\right\}, \tag{1.1}$$

v is a Banach Space normed by $\|x\| = \sum_{k=0}^{\infty} \mid x_k - x_{k-1} \mid$ (Mursaleen, 1983*b*). Let σ be an injective mapping of the set of the positive integers into itself having no finite orbits. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or σ-mean if and only if:

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1. $\phi(x) \geq 0$ where the sequence $x = (x_k)$ has $x_k \geq 0$ for all k,
2. $\phi(e) = 1$ where $e = \{1, 1, 1, ...\}$,
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \left\{ x = (x_k) : \lim_{m\to\infty} t_{m,k}(x) = L \text{ uniformly in k}, L = \sigma - \lim x \right\} \tag{1.2}$$

where $m \geq 0, k > 0$.

$$t_{m,k}(x) = \frac{x_k + x_{\sigma(k)} ... + x_{\sigma^m(k)}}{m+1} \text{ and } t_{-1,k} = 0, \tag{1.3}$$

where $\sigma_m(k)$ denote the m^{th}-iterate of $\sigma(k)$ at k. In case σ is the translation mapping, that is, $\sigma(k) = k+1$, σ-mean is called a Banach limit (Banach, 1932) and V_σ, the set of bounded sequences of all whose invariant means are equal, is the set of almost convergent sequences.The special case of (1.2) in which $\sigma(k) = k + 1$ was given by (Lorentz, 1948), (Theorem 1), and that the general result can be proved in a similar way. It is familiar that a Banach limit extends the limit functional on c in the sense that

$$\phi(x) = \lim x, \text{ for all } x \in c. \tag{1.4}$$

Remark. In view of above discussion we have $c \subset V_\sigma$.

Theorem 1.1. *A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in$ c if and only if σ has no finite orbits. That is, if and only if for all $k \geq 0, j \geq 1$, $\sigma^j(k) \neq k$.*

Put

$$\phi_{m,k}(x) = t_{m,k}(x) - t_{m-1,k}(x), \tag{1.5}$$

assuming that $t_{-1,k}(x) = 0$.

A straight forward calculation shows that (Mursaleen, 1983*a*)

$$\phi_{m,k}(x) = \left\{ \begin{array}{ll} \frac{1}{m(m+1)} \sum_{j=1}^{m} j(x_\sigma^j(k) - x_\sigma^{j-1}(k)), & \text{if}(m \geq 1), \\ x_k & \text{if}(m = 0) \end{array} \right\}. \tag{1.6}$$

For any sequence x, y and scalar λ, we have $\phi_{m,k}(x + y) = \phi_{m,k}(x) + \phi_{m,k}(y)$ and $\phi_{m,k}(\lambda x) = \lambda\phi_{m,k}(x)$.

Definition 1.1. A sequence $x \in \ell_\infty$ is of σ-bounded variation if and only if

(i) $\sum_{m=0}^{\infty} | \phi_{m,k}(x) |$ converges uniformly in k.

(ii) $\lim_{m\to\infty} t_{m,k}(x)$, which must exist, should take the same value for all k.

Subsequently invariant means have been studied by (Mursaleen, 1983*b*,*a*; Ahmad & Mursaleen, 1986; Raimi, 1963; Khan & Ebadullah, 2013, 2012; Schafer, 1972) and many others. (Mursaleen, 1983*b*) defined the sequence space BV_σ, the space of all sequences of σ -bounded variation as $BV_\sigma = \{x \in \ell_\infty : \sum_m | \phi_{m,k}(x) | < \infty, \text{uniformly in } k\}$.

Theorem 1.2. *(Fast, 1951) BV_σ is a Banach space normed by* $\| x \| = \sup_k \sum | \phi_{m,k}(x) |$.

Definition 1.2. (see[23]) A function $M : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function if it satisfies the following conditions;
(i) M is continuous, convex and non-decreasing,
(ii) $M(0) = 0$, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Remark. (see (Tripathy & Hazarika, 2011)) If the convexity of an Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Remark. If M is an Orlicz function,then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy Δ_2 − Condition for all values of u if there exists a constant $K > 0$ such that $M(Lu) \leq \mathrm{KL}M(u)$ for all values of $L > 1$(see (Tripathy & Hazarika, 2011)).

(Lindenstrauss & Tzafriri, 1971) used the idea of an Orlicz function to construct the sequence space $\ell_M = \{x \in \omega : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty, \text{for some } \rho > 0\}$. The space ℓ_M becomes a Banach space with the norm

$$\| x \| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{| x_k |}{\rho}) \leq 1\}, \tag{1.7}$$

which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = t^P$ for $1 \leq p < \infty$.

Later on, some Orlicz sequence spaces were investigated by (Hazarika & Esi, 2013; Maddox, 1970; Parshar & Choudhary, 1994; Bhardwaj & Singh, 2000; Et, 2001; Tripathy & Hazarika, 2011) and many others.

Initially, as a generalization of statistical convergence (Fridy, 1985), the notation of ideal convergence (I-convergence) was introduced and studied by (P. Kostyrko & Wilczyński, 2000). Later on, it was studied by (Khan & Ebadullah, 2013), (Hazarika & Esi, 2013; T. Šalát & Ziman, 2004, 2005) and many others.

Here we give some preliminaries about the notion of I-convergence.

Definition 1.3. A sequence x=$(x_k) \in \omega$ is said to be statistically convergent to a limit $L \in \mathbb{C}$ if for every $\epsilon > 0$, we have $\lim_k \frac{1}{k}|\{n \in \mathbb{N} : |x_k - L| \geq \epsilon, n \leq k\}| = 0$, where vertical lines denote the cardinality of the enclosed set.

Definition 1.4. Let $\mathbb{N}$ be the set of natural numbers. Then a family of sets $I \subseteq 2^{\mathbb{N}}$ (power set of $\mathbb{N}$) is said to be an ideal if:
1) I is additive i.e $\forall A, B \in I \Rightarrow A \cup B \in I$,
2) I is hereditary i.e $\forall A \in I$and $B \subseteq A \Rightarrow B \in I$.

Definition 1.5. A non-empty family of sets $£(I) \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if
1) $\Phi \notin £(I)$,
2) $\forall\ A,\ \ B \in £(I)$ we have $A \cap B \in £(I)$,
3) $\forall\ A \in £(\mathrm{I})$ and $A \subseteq B \Rightarrow B \in £(I)$.

Definition 1.6. An Ideal $I \subseteq 2^{\mathbb{N}}$ is called non-trivial if $I \neq 2^{\mathbb{N}}$.

Definition 1.7. A non-trivial ideal $I \subseteq 2^{\mathbb{N}}$ is called admissible if $\{\{x\} : x \in \mathbb{N}\} \subseteq I$.

Definition 1.8. A non-trivial ideal I is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

Definition 1.9. For each ideal I, there is a filter $£(I)$ corresponding to I.
i.e $£(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$, where $K^c = \mathbb{N} \setminus K$.

Definition 1.10. A sequence $x = (x_k) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in I$.
In this case, we write $I - \lim x_k = L$.

Definition 1.11. A sequence $x = (x_k) \in \omega$ is said to be I-null if $L = 0$. In this case, we write $I - \lim x_k = 0$.

Definition 1.12. A sequence $x = (x_k) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$ there exists a number $m = m(\epsilon)$ such that $\{k \in \mathbb{N} : |x_k - x_m| \geq \epsilon\} \in I$.

Definition 1.13. A sequence space E is said to be solid(normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for any sequence(α_k) of scalars with $| \alpha_k |\leq 1$, for all $k \in \mathbb{N}$.

Definition 1.14. A sequence space E is said to be symmetric if $(x_{\pi(k)}) \in E$ whenever $x_k \in E$. where π is a permutation on $\mathbb{N}$.

Definition 1.15. A sequence space is$E$ said to be sequence algebra if $(x_k) * (y_k) = (x_k.y_k) \in E$ whenever $(x_k), (y_k) \in E$.

Definition 1.16. A sequence space E is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$, for all k.

Definition 1.17. Let $K = \{k_1 < k_2 < k_3 < k_4 < k_5 ...\} \subset \mathbb{N}$ and E be a Sequence space. A K-step space of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in \omega : (x_k) \in E\}$.

Definition 1.18. A canonical pre-image of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of preimages all elements in λ_K^E.i.e. y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some $x \in \lambda_K^E$.

Definition 1.19. A sequence space E is said to be monotone if it contains the canonical preimages of its step space.

Remark. If $I = I_f$, the class of all finite subsets of $\mathbb{N}$. Then, I is an admissible ideal in $\mathbb{N}$ and I_f convergence coincides with the usual convergence.

Definition 1.20. If $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. Then, I is an admissible ideal in N and we call the I_δ-convergence as the logarithmic statistical convergence.

Definition 1.21. If $I = I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then, I is an admissible ideal in $\mathbb{N}$ and we call the I_d-convergence as the asymptotic statistical convergence.

Remark. If $I_\delta - \lim x_k = l$, then $I_d - \lim x_k = l$.

The following lemmas remained an important tool for the establishment of some results of this article.

Lemma 1.1. *(Tripathy & Hazarika, 2011). Every solid space is monotone.*

Lemma 1.2. *Let $K \in \pounds(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

Lemma 1.3. *If $I \subseteq 2^{\mathbb{N}}$ and $M \subseteq N$. If $M \notin I$, then $M \cap N \notin I$.*

2. Main Results

Recently (Khan & Ebadullah, 2012) introduced and studied the following sequence space. For $m \geq 0$

$$BV_\sigma^I = \left\{x = (x_k) \in \omega : \{k \in \mathbb{N} :| \phi_{m.k}(x) - L |\geq \epsilon\} \in I, \text{for some L} \in \mathbb{C}\right\}. \tag{2.1}$$

In this article we introduce the following sequence spaces. For $m \geq 0$

$$BV_\sigma^I(M) = \left\{x = (x_k) \in \omega : I - \lim M(\frac{| \phi_{m,k}(x) - L |}{\rho}) = 0, \text{for some L} \in \mathbb{C}, \rho > 0\right\}, \tag{2.2}$$

$${}_0BV_\sigma^I(M) = \left\{x = (x_k) \in \omega : I - \lim M(\frac{| \phi_{m,k}(x) |}{\rho}) = 0, \rho > 0\right\}, \tag{2.3}$$

$${}_\infty BV_\sigma^I(M) = \left\{x = (x_k) \in \omega : \{k \in \mathbb{N} : \exists K > 0 s.t. M(\frac{| \phi_{m,k}(x) |}{\rho}) \geq K\} \in I, \rho > 0\right\}, \tag{2.4}$$

$${}_\infty BV_\sigma(M) = \left\{x = (x_k) \in \omega : \sup M(\frac{| \phi_{m,k}(x) |}{\rho}) < \infty, \rho > 0\right\}. \tag{2.5}$$

We also denote $\mathcal{M}_{BV_\sigma}^I(M) = BV_\sigma^I(M) \cap {}_\infty BV_\sigma(M)$ and ${}_0\mathcal{M}_{BV_\sigma}^I(M) = {}_0BV_\sigma^I(M) \cap {}_\infty BV_\sigma(M)$.

Throughout the article, if required, we denote $\phi_{m,k}(x)=x^{'}$, $\phi_{m,k}(\mathrm{y})=y^{'}$ **and** $\phi_{m,k}(\mathrm{z})=z^{'}$ **where** x, y, z are (x_k), (y_k) **and** (z_k) **respectively.**

Theorem 2.1. *For any Orlicz function M, the classes of sequence ${}_0BV_\sigma^I(M)$, $BV_\sigma^I(M)$, ${}_0\mathcal{M}_{BV_\sigma}^I(M)$ and $\mathcal{M}_{BV_\sigma}^I(M)$ are the linear spaces.*

Proof. We shall prove the result for the space $BV_\sigma^I(M)$, others will follow similarly.
For, let $x = (x_k), y = (y_k) \in BV_\sigma^I(M)$ be any two arbitrary elements and let α, β are scalars.
Now, since $(x_k), (y_k) \in BV_\sigma^I(M) \Rightarrow \exists$ some positive numbers $L_1, L_2 \in \mathbb{C}$ and $\rho_1,\ \rho_2 > 0$ such that

$$I - \lim_k M\Big(\frac{|\ \phi_{m,k}(x) - L_1\ |}{\rho_1}\Big) = 0, \tag{2.6}$$

$$I - \lim_k M\Big(\frac{|\ \phi_{m,k}(y) - L_2\ |}{\rho_2}\Big) = 0 \tag{2.7}$$

$\Rightarrow$ for any given $\epsilon > 0$, the sets

$$A_1 = \Big\{k \in \mathbb{N} : M\Big(\frac{|\ \phi_{m,k}(x) - L_1\ |}{\rho_1}\Big) > \frac{\epsilon}{2}\Big\} \in I, \tag{2.8}$$

$$A_2 = \Big\{k \in \mathbb{N} : M\Big(\frac{|\ \phi_{m,k}(y) - L_2\ |}{\rho_2}\Big) > \frac{\epsilon}{2}\Big\} \in I. \tag{2.9}$$

Let

$$\rho_3 = \max\{2 \mid \alpha \mid \rho_1, 2 \mid \beta \mid \rho_2\}. \tag{2.10}$$

Since, M is non-decreasing and convex function, we have

$$M\Big(\frac{|\ (\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)\ |}{\rho_3}\Big) \leq$$

$$M\Big(\frac{|\ \alpha\ ||\ x'_k - L_1\ |}{\rho_3}) + M(\frac{|\ \beta\ ||\ y'_k - L_2\ |}{\rho_3}\Big) \leq M\Big(\frac{|\ x'_k - L_1\ |}{\rho_1}) + M(\frac{|\ y'_k - L_2\ |}{\rho_2}\Big). \tag{2.11}$$

Therefore, from (2.8), (2.9) and (2.11), we have $\Big\{k \in \mathbb{N} : M\Big(\frac{|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)|}{\rho_3}\Big) > \epsilon\Big\} \subseteq A_1 \cup A_2 \in I$ implies that $\Big\{k \in \mathbb{N} : M\Big(\frac{|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)|}{\rho_3}\Big) > \epsilon\Big\} \in I$. That is, $I - \lim M\Big(\frac{|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)|}{\rho_3}\Big) = 0$. Thus, $\alpha x_k + \beta y_k \in BV_\sigma^I(M)$. But $(x_k), (y_k) \in BV_\sigma^I(M)$ are the arbitrary elements. Therefore, $\alpha x_k + \beta y_k \in BV_\sigma^I(M)$, for all $(x_k), (y_k) \in BV_\sigma^I(M)$ and for all scalars α, β. Hence, $BV_\sigma^I(M)$ is linear. □

Theorem 2.2. *Let M_1 and M_2 be two Orlicz functions and satisfying Δ_2 − Condition, then*
(a) $\mathcal{X}(M_2) \subseteq \mathcal{X}(M_1 M_2)$,
(b) $\mathcal{X}(M_1) \cap (M_2) \subseteq \mathcal{X}(M_1 + M_2)$ for $\mathcal{X} = {}_0BV_\sigma^I$, BV_σ^I, ${}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{M}_{BV_\sigma}^I$.

Proof. (a) Let $x = (x_k) \in {}_0BV_\sigma^I(M_2)$ be any arbitrary element$\Rightarrow \exists\, \rho > 0$ such that

$$I - \lim M_2\Big(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\Big) = 0, \tag{2.12}$$

i.e.

$$I - \lim M_2\Big(\frac{|\ x'_k\ |}{\rho}\Big) = 0. \tag{2.13}$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_1(t) < \epsilon,\ 0 \le t \le \delta$. Let us write $y_k = M_2(\frac{|x'_k|}{\rho})$ and consider

$$\lim_k M_1(y_k) = \lim_{y_k \le \delta, k\in\mathbb{N}} M_1(y_k) + \lim_{y_k > \delta, k\in\mathbb{N}} M_1(y_k). \tag{2.14}$$

Now, since M_1 is an Orlicz function ,we have $M_1(\lambda x) \le \lambda M_1(x)$ for all λ with $0 < \lambda < 1$. Therefore,

$$\lim_{y_k \le \delta, k\in\mathbb{N}} M_1(y_k) \le M_1(2) \lim_{y_k \le \delta, k\in\mathbb{N}} (y_k). \tag{2.15}$$

For $y_k > \delta$, we have $y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$. Now, since M_1 is non-decreasing and convex, it follows that

$$M_1(y_k) < M_1(1 + \frac{y_k}{\delta}) < \frac{1}{2}M_1(2) + \frac{1}{2}M_1(\frac{2y_k}{\delta}). \tag{2.16}$$

Again, since M_1 satisfies Δ_2 − Condition, we have $M_1(y_k) < \frac{1}{2}K\frac{(y_k)}{\delta}M_1(2) + \frac{1}{2}K\frac{(y_k)}{\delta}M_1(2)$. Thus, $M_1(y_k) < K\frac{(y_k)}{\delta}M_1(2)$. Hence,

$$\lim_{y_k > \delta, k\in\mathbb{N}} M_1(y_k) \le \max\{1, K\delta^{-1}M_1(2) \lim_{y_k > \delta, k\in\mathbb{N}} (y_k). \tag{2.17}$$

Therefore, from (2.12), (2.13) and (2.14), we have $I{-}\lim_k M_1(y_k) = 0$, i.e. $I{-}\lim_k M_1M_2(\frac{|\phi_{m,k}(x)|}{\rho}) = 0$, implies that $(x_k) \in {}_0BV_\sigma^I(M_1M_2)$. Thus, ${}_0BV_\sigma^I(M_2) \subseteq {}_0BV_\sigma^I(M_1M_2)$. Hence, $\mathcal{X}(M_2) \subseteq \mathcal{X}(M_1M_2)$ for $\mathcal{X} = {}_0BV_\sigma^I$. For $\mathcal{X} = BV_\sigma^I, \mathcal{X} = {}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{X} = \mathcal{M}_{BV_\sigma^I}$ the inclusions can be established similarly.

(b). Let $x = (x_k) \in {}_0BV_\sigma^I(M_1) \cap {}_0BV_\sigma^I(M_2)$. Let $\epsilon > 0$ be given. Then there exists $\rho > 0$ such that the sets $I - \lim M_1\left(\frac{|\phi_{m,k}(x)|}{\rho}\right) = 0$ and $I - \lim M_2\left(\frac{|\phi_{m,k}(x)|}{\rho}\right) = 0$. Therefore, $I - \lim M_1 + M_2\left(\frac{|\phi_{m,k}(x)|}{\rho}\right) = I - \lim M_1\left(\frac{|\phi_{m,k}(x)|}{\rho}\right) + I - \lim M_2\left(\frac{|\phi_{m,k}(x)|}{\rho}\right)$ implies that $I - \lim M_1 + M_2\left(\frac{|\phi_{m,k}(x)|}{\rho}\right) = 0$. Thus, $x = (x_k) \in {}_0BV_\sigma^I(M_1 + M_2)$ Hence, ${}_0BV_\sigma^I(M_1) \cap {}_0BV_\sigma^I(M_2) \subseteq {}_0BV_\sigma^I(M_1 + M_2)$. For $\mathcal{X} = BV_\sigma^I$, $\mathcal{X} = {}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{X} = \mathcal{M}_{BV_\sigma}^I$ the inclusions are similar. □

For $M_2(x) = (x)$ and $M_1(x) = M(x)$, $\forall\ x \in [0, \infty)$, we have the following corollary.

Corollary 2.1. *$\mathcal{X} \subseteq \mathcal{X}(M)$ for $\mathcal{X} = {}_0BV_\sigma^I$, BV_σ^I, ${}_0\mathcal{M}_{BV_\sigma}^I$ and $\mathcal{M}_{BV_\sigma}^I$.*

Theorem 2.3. *For any orlicz function M, the spaces ${}_0BV_\sigma^I(M)$ and ${}_0\mathcal{M}_{BV_\sigma}^I$ are solid and monotone.*

Proof. Here we consider ${}_0BV_\sigma^I(M)$ and for ${}_0\mathcal{M}_{BV_\sigma}^I$ the proof shall be similar. For, let $(x_k) \in_0 BV_\sigma^I(M)$ be any arbitrary element. $\Rightarrow$ $\exists\rho > 0$ such that $I - \lim_k M(\frac{|\phi_{m,k}(x)|}{\rho}) = 0$. Let (α_k) be a sequence of scalars such that $|\ \alpha_k\ | \le 1$, for all $k \in \mathbb{N}$.

Now, since M is an Orlicz function. Therefore,

$$M\left(\frac{|\ \alpha_k\phi_{m,k}(x)\ |}{\rho}\right) \le |\ \alpha_k\ |\ M\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right) \le M\left(\frac{|\ \phi_{m,k}(x)\ |}{\rho}\right)$$

$$\Rightarrow M\left(\frac{|\ \alpha_k\phi_{m,k}(x)\ |}{\rho}\right) \le M\left(\frac{|\ \phi_{m.k}(x)\ |}{\rho}\right), \text{for all } \ k \in \mathbb{N},$$

implies that $I - \lim_k M(\frac{|\alpha_k \phi_{m,k}(x)|}{\rho}) = 0$.

Thus, $(\alpha_k x_k) \in_0 BV_\sigma^I(M)$. Hence ${}_0BV_\sigma^I(M)$ is solid. Therefore, by lemma 1.1, ${}_0BV_\sigma^I(M)$ is monotone. Hence the result. □

Theorem 2.4. *For any orlicz function M, the spaces $BV_\sigma^I(M)$ and $\mathcal{M}_{BV_\sigma}^I$ are neither solid nor monotone in general.*

Proof. Here we give counter example for the establishment of this result. For, let us consider $I = I_f$ and $M(x) = x$, for all $x \in [0, \infty)$. Consider, the K-step space $B_K(M)$ of $B(M)$ as follows. Let$(x_k) \in B(M)$ and $(y_k) \in B_K(M)$ be such that

$$y_k = \begin{cases} x_k, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence (x_k) defined as $x_k = 1$, for all k $\in \mathbb{N}$, then $x_k \in BV_\sigma^I(M)$ and $\mathcal{M}_{BV_\sigma}^I$ but its K-step space pre-image does not belong to $BV_\sigma^I(M)$ and $\mathcal{M}_{BV_\sigma}^I$. Thus, $BV_\sigma^I(M)$ and $\mathcal{M}_{BV_\sigma}^I$ are not monotone and hence by lemma(I) they are not solid. □

Theorem 2.5. *For an Orlicz function M, the spaces ${}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$ are not convergence free.*

Proof. Let $I = I_f$ and $M(x) = x$ for all x $\in [0, \infty)$. Consider the sequences (x_k) and (y_k) defined as follows.

$$x_k = \frac{1}{k} \text{ and } y_k = k, \text{ for all } k \in \mathbb{N}.$$

Then, (x_k) belongs to both $\in {}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$ but (y_k) does not belongs to both ${}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$.

Hence, the spaces ${}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$ are not convergence free. □

Theorem 2.6. *For an Orlicz function M, the spaces ${}_0BV_\sigma^I(M)$ and $BV_\sigma^I(M)$ are sequence algebra.*

Proof. Here we consider ${}_0BV_\sigma^I(M)$. For the other one, result is similar.
Let $x = (x_k), y = (y_k) \in {}_0BV_\sigma^I(M)$ be any two arbitrary elements.
$\Rightarrow \exists\, \rho_1, \rho_2 > 0$ such that

$$I - \lim_k M(\frac{|\phi_{m,k}(x)|}{\rho_1}) = 0$$

and

$$I - \lim_k M(\frac{|\phi_{m,k}(y)|}{\rho_2}) = 0.$$

Let $\rho = \rho_1 \rho_2 > 0$. Then, it is obvious that $I - \lim_k M(\frac{|\phi_{m,k}(x)\phi_{m,k}(y)|}{\rho}) = 0$ implies that $(x_k.y_k) = (x_k y_k) \in {}_0BV_\sigma^I(M)$. Hence, ${}_0BV_\sigma^I(M)$ is a Sequence algebra. □

Theorem 2.7. *Let M be an Orlicz function. Then, ${}_0BV_\sigma^I(M) \subseteq BV_\sigma^I(M) \subseteq {}_\infty BV_\sigma^I(M)$.*

Proof. Let M be an Orlicz function. Then, we have to show that ${}_0BV_\sigma^I(M) \subseteq BV_\sigma^I(M) \subseteq {}_\infty BV_\sigma^I(M)$. Firstly, ${}_0BV_\sigma^I(M) \subseteq BV_\sigma^I(M)$ is obvious.

Now, let $x = (x_k) \in BV_\sigma^I(M)$ be any arbitrary element $\Rightarrow \exists \rho > 0$ such that $I\text{-}\lim_k M(\frac{|\phi_{m,k}(x)-L|}{\rho}) = 0$ for some $L \in \mathbb{N}$.

Now, $M(\frac{|\phi_{m,k}(x)|}{2\rho}) \leq \frac{1}{2}M(\frac{|\phi_{m,k}(x)-L|}{\rho}) + \frac{1}{2}M(\frac{|L|}{\rho})$. Taking supremum over k to both sides, we have $x = (x_k) \in {}_\infty BV_\sigma^I(M)$. Hence, ${}_0BV_\sigma^I(M) \subseteq BV_\sigma^I(M) \subseteq {}_\infty BV_\sigma^I(M)$. □

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