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Exponential Stability versus Polynomial Stability for Skew-Evolution Semiflows in Infinite Dimensional Spaces

Codruţa Stoica[a,*], Diana Borlea[b]

[a] *"Aurel Vlaicu" University of Arad, Department of Mathematics and Computer Science, 2 E. Drăgoi Str., RO-310300 Arad, Romania.*

[b] *West University of Timişoara, Faculty of Mathematics and Computer Science, 4 V. Pârvan Bd., RO-300223 Timişoara, Romania.*

Abstract

As the dynamical systems that model processes issued from engineering, economics or physics are extremely complex, of great interest is to study the solutions of differential equations by means of associated evolution families. In this paper we emphasize some notions of asymptotic stability for skew-evolution semiflows on Banach spaces, such as exponential and polynomial stability, in a nonuniform setting. Examples for every concept and connections between them are also presented, as well as some characterizations.

Keywords: Skew-evolution semiflow, exponential stability, Barreira-Valls exponential stability, polynomial stability.
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1. Preliminaries

The theory of asymptotic properties for evolution equations has witnessed lately an explosive development. We intend to emphasize in our paper a framework which enables us to obtain characterizations in a unitary approach for the asymptotic stability on Banach spaces. The notion of skew-evolution semiflow, introduced in (Megan & Stoica, 2008), is more appropriate for the study in the nonuniform case. They depend on three variables, making thus possible the generalization for skew-product semiflows and evolution operators, which depend only on two. Hence, the study of asymptotic behaviors for skew-evolution semiflows in the nonuniform setting arises as natural, relative to the third variable. The notion has proved itself of interest in the development of the stability theory, in a uniform as well as in a nonuniform setting, being already adopted by some researchers, as, for example, A.J.G. Bento and C.M. Silva (see (Bento & Silva, 2012)), P. Viet Hai (see (Hai, 2010) and (Hai, 2011)) and T. Yue, X.Q. Song and D.Q. Li (see (Yue *et al.*, 2014)), which have contributed to the expansion of the concept of skew-evolution semiflows and deepened the study of their asymptotic behaviors and applications. Some properties for skew-evolution semiflows are defined and characterized in (Stoica, 2010).

*Corresponding author
Email addresses: `codruta.stoica@uav.ro` (Codruţa Stoica), `dianab268@yahoo.com` (Diana Borlea)

The definitions of various types of stability are illustrated by examples and the connections between them are emphasized. Our aim is also to give some integral characterizations for them. We present a concept of nonuniform exponential stability, given and studied by L. Barreira and C. Valls in (Barreira & Valls, 2008), which we call *"Barreira-Valls exponential stability"*. In this paper, some generalizations for the results obtained in the uniform setting in (Stoica & Megan, 2010) are proved in the nonuniform case.

2. Skew-evolution semiflows

This section gives the notion of skew-evolution semiflow on a Banach space, defined by means of an evolution semiflow and of an evolution cocycle.

Let (X, d) be a metric space, V a Banach space and V^* its topological dual. Let $\mathcal{B}(V)$ be the space of all V-valued bounded operators defined on V. The norm of vectors on V and on V^* and of operators on $\mathcal{B}(V)$ is denoted by $\|\cdot\|$. I is the identity operator. Let us denote $Y = X \times V$ and $T = \left\{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\right\}$.

Definition 2.1. A mapping $\varphi : T \times X \to X$ is said to be *evolution semiflow* on X if following properties are satisfied:

(es_1) $\varphi(t, t, x) = x,\ \forall (t, x) \in \mathbb{R}_+ \times X$;

(es_2) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x),\ \forall (t, s), (s, t_0) \in T,\ \forall x \in X$.

Definition 2.2. A mapping $\Phi : T \times X \to \mathcal{B}(V)$ is called *evolution cocycle* over an evolution semiflow φ if it satisfies following properties:

(ec_1) $\Phi(t, t, x) = I,\ \forall t \geq 0,\ \forall x \in X$;

(ec_2) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall (t, s), (s, t_0) \in T, \forall x \in X$.

Let Φ be an evolution cocycle over an evolution semiflow φ. The mapping

$$C : T \times Y \to Y,\ C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v) \tag{2.1}$$

is called *skew-evolution semiflow* on Y.

Example 2.1. Let us denote $C = C(\mathbb{R}, \mathbb{R})$ the set of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. For every $x, y \in C$, we define

$$d_n(x, y) = \sup_{t \in [-n, n]} |x(t) - y(t)|.$$

The set C is metrizable with respect to the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}.$$

We consider for every $n \in \mathbb{N}^*$ a decreasing function

$$x_n : \mathbb{R}_+ \to \left(\frac{1}{2n+1}, \frac{1}{2n}\right), \text{ with the property } \lim_{t \to \infty} x_n(t) = \frac{1}{2n+1}.$$

We denote

$$x_n^s(t) = x_n(t + s),\ \forall t, s \geq 0.$$

Let X be the closure in C of the set $\{x_n^s, n \in \mathbb{N}^*, s \in \mathbb{R}_+\}$. The mapping

$$\varphi : T \times X \to X,\ \varphi(t, s, x) = x_{t-s},\ \text{where } x_{t-s}(\tau) = x(t - s + \tau),\ \forall \tau \geq 0,$$

is an evolution semiflow on X. Let us consider the Banach space $V = \mathbb{R}^p$, $p \geq 1$, with the norm $\left\|(v_1, ..., v_p)\right\| = |v_1| + ... + |v_p|$. Then the mapping

$$\Phi : T \times X \to \mathcal{B}(V),\ \Phi(t, s, x)v = \left(e^{\alpha_1 \int_s^t x(\tau - s) d\tau} v_1, ..., e^{\alpha_p \int_s^t x(\tau - s) d\tau} v_p\right),$$

where $(\alpha_1, ..., \alpha_p) \in \mathbb{R}^p$ is fixed, is an evolution cocycle over the evolution semiflow φ and $C = (\varphi, \Phi)$ is a skew-evolution semiflow on Y.

Example 2.2. For $X = \mathbb{R}_+$, the mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t, s, x) = x$ is an evolution semiflow. For every evolution cocycle Φ over φ, we obtain that the mapping $E_\Phi : T \to \mathcal{B}(V)$, $E_\Phi(t, s) = \Phi(t, s, 0)$ is an evolution operator on V.

Example 2.3. If $C = (\varphi, \Phi)$ denotes a skew-evolution semiflow and $\alpha \in \mathbb{R}$ a parameter, then $C_\alpha = (\varphi, \Phi_\alpha)$, where

$$\Phi_\alpha : T \times X \to \mathcal{B}(V),\ \Phi_\alpha(t, t_0, x) = e^{\alpha(t - t_0)} \Phi(t, t_0, x), \tag{2.2}$$

is also a skew-evolution semiflow, called the *α-shifted skew-evolution semiflow*.

3. Exponential stability

In this section we consider several concepts of exponential stability for skew-evolution semiflows. Some connections between these concepts are established. We will emphasize that they are not equivalent.

The nonuniform exponential stability is given by

Definition 3.1. A skew-evolution semiflow $C = (\varphi, \Phi)$ is *exponentially stable* $(e.s.)$ if there exist a mapping $N : \mathbb{R}_+ \to [1, \infty)$ and a constant $\alpha > 0$ such that, for all $(t, s) \in T$, following relation takes place:

$$\|\Phi(t, t_0, x)v\| \leq N(s) e^{-\alpha t} \|\Phi(s, t_0, x)v\|,\ \forall (x, v) \in Y. \tag{3.1}$$

A concept of nonuniform exponential stability, which we will name *"Barreira-Valls exponential stability"*, is given by L. Barreira and C. Valls in (Barreira & Valls, 2008) for evolution equations.

Definition 3.2. A skew-evolution semiflow $C = (\varphi, \Phi)$ is *Barreira-Valls exponentially stable* $(BV.e.s.)$ if there exist some constants $N \geq 1$, $\alpha > 0$ and β such that, for all $(t, s), (s, t_0) \in T$, the relation holds:

$$\|\Phi(t, t_0, x)v\| \leq N e^{-\alpha t} e^{\beta s} \|\Phi(s, t_0, x)v\|,\ \forall\ (x, v) \in Y. \tag{3.2}$$

The asymptotic property of nonuniform stability is considered in

Definition 3.3. A skew-evolution semiflow $C = (\varphi, \Phi)$ is *stable* $(s.)$ if there exists a mapping $N : \mathbb{R}_+ \to [1, \infty)$ such that, for all $(t, s), (s, t_0) \in T$, the relation is true:

$$\|\Phi(t, t_0, x)v\| \leq N(s) \|\Phi(s, t_0, x)v\|,\ \forall\ (x, v) \in Y. \tag{3.3}$$

Let us remind the property of exponential growth for skew-evolution semiflows, given by

Definition 3.4. A skew-evolution semiflow $C = (\varphi, \Phi)$ has *exponential growth* (*e.g.*) if there exist two mappings $M, \omega : \mathbb{R}_+ \to [1, \infty)$, ω nondecreasing, such that, for all $(t, s), (s, t_0) \in T$, we have:

$$||\Phi(t, t_0, x)v|| \le M(s)e^{\omega(s)(t-s)} \, ||\Phi(s, t_0, x)v|| \, , \; \forall \, (x, v) \in Y. \tag{3.4}$$

Remark. The relations concerning the previously defined asymptotic properties for skew-evolution semiflows are given by

$$(BV.e.s.) \Longrightarrow (e.s.) \Longrightarrow (s.) \Longrightarrow (e.g.) \tag{3.5}$$

The reciprocal statements are not true, as shown in what follows.

The following example presents a skew-evolution semiflow which is exponentially stable but not Barreira-Valls exponentially stable.

Example 3.1. Let $X = \mathbb{R}_+$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t, s, x) = x$ is an evolution semiflow on $\mathbb{R}_+$.

Let us consider a continuous function $u : \mathbb{R}_+ \to [1, \infty)$ with

$$u(n) = e^{n \cdot 2^{2n}} \text{ and } u\left(n + \frac{1}{2^{2n}}\right) = e^4.$$

We define

$$\Phi_u(t, s, x)v = \frac{u(s)e^s}{u(t)e^t} v, \text{ where } (t, s) \in T, \; (x, v) \in Y.$$

As following relation

$$||\Phi_u(t, s, x)v|| \le u(s)e^s e^{-t} \, ||v||$$

holds for all $(t, s, x, v) \in T \times Y$, it results that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is exponentially stable.

Let us now suppose that the skew-evolution semiflow $C_u = (\varphi, \Phi_u)$ is Barreira-Valls exponentially stable. Then, according to Definition 3.2, there exist $N \ge 1$, $\alpha > 0, \beta > 0$ and $t_1 > 0$ such that

$$\frac{u(s)e^s}{u(t)e^t} \le N e^{-\alpha t} e^{\beta s}, \; \forall t \ge s \ge t_1.$$

For $t = n + \frac{1}{2^{2n}}$ and $s = n$ it follows that

$$e^{n\left(2^{2n}+1\right)} \le N e^{n + \frac{1}{2^{2n}} + 4} e^{-\alpha\left(n + \frac{1}{2^{2n}}\right)} e^{\beta n},$$

which is equivalent with

$$e^{n\left(2^{2n}-\beta\right)} \le N e^{\frac{1}{2^{2n}} + 4 - \alpha\left(n + \frac{1}{2^{2n}}\right)}.$$

For $n \to \infty$, a contradiction is obtained, which proves that C_u is not Barreira-Valls exponentially stable.

There exist skew-evolution semiflows that are stable but not exponentially stable, as results from the following

Example 3.2. Let us consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and

$$u : \mathbb{R}_+ \to [1, \infty) \text{ with the property } \lim_{t \to \infty} \frac{u(t)}{e^t} = 0.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t,s,x)v = \frac{u(s)}{u(t)}v$$

is an evolution cocycle. As $|\Phi_u(t,s,x)v| \le u(s)|v|$, $\forall (t,s,x,v) \in T \times Y$, it follows that $C_u = (\varphi, \Phi_u)$ is a stable skew-evolution semiflow, for every evolution semiflow φ.

On the other hand, if we suppose that C_u is exponentially stable, according to Definition 3.1, there exist a mapping $N : \mathbb{R}_+ \to [1,\infty)$ and a constant $\alpha > 0$ such that, for all $(t,s),(s,t_0) \in T$, we have

$$\|\Phi_u(t,t_0,x)v\| \le N(s)e^{-\alpha t} \|\Phi_u(s,t_0,x)v\|,\ \forall\,(x,v) \in Y.$$

It follows that

$$\frac{u(s)}{N(s)} \le \frac{u(t)}{e^{\alpha t}}.$$

For $t \to \infty$ we obtain a contradiction, and, hence, C_u is not exponentially stable.

Following example gives a skew-evolution semiflow that has exponential growth but is not stable.

Example 3.3. We consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and

$$u : \mathbb{R}_+ \to [1,\infty) \text{ with the property } \lim_{t\to\infty} \frac{e^t}{u(t)} = \infty.$$

The mapping

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t,s,x)v = \frac{u(s)e^t}{u(t)e^s}v$$

is an evolution cocycle. We have $|\Phi(t,s,x)v| \le u(s)e^{t-s}|v|$, $\forall (t,s,x,v) \in T \times Y$. Hence, $C_u = (\varphi, \Phi_u)$ is a skew-evolution semiflow, over every evolution semiflow φ, and has exponential growth.

Let us suppose that C_u is stable. According to Definition 3.3, there exists a mapping $N : \mathbb{R}_+ \to [1,\infty)$ such that $u(s)e^t \le N(s)u(t)e^s$, for all $(t,s) \in T$. If $t \to \infty$, a contradiction is obtained. Hence, C_u is not stable.

4. Polynomial stability

In this section, we introduce a new concept of nonuniform stability for skew-evolution semiflows, given by the next

Definition 4.1. A skew-evolution semiflow $C = (\varphi, \Phi)$ is called *polynomially stable* (*p.s.*) if there exist a mapping $N : \mathbb{R}_+ \to [1,\infty)$ and a constant $\gamma > 0$ such that:

$$\|\Phi(t,s,x)v\|\,ds \le N(s)(t-s)^{-\gamma}\,\|v\|\,, \tag{4.1}$$

for all $t > s \ge 0$ and all $(x,v) \in Y$.

Remark. If a skew-evolution semiflow C is exponentially stable, then it is polynomially stable.

$$(e.s.) \Longrightarrow (p.s.)$$

The reciprocal statement is not true, as shown in

Example 4.1. Let $X = \mathbb{R}_+$, $V = \mathbb{R}$ and the mapping $u : \mathbb{R}_+ \to \mathbb{R}$ given by $u(t) = t + 1$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\varphi(t, s, x) = x$ is an evolution semiflow on $\mathbb{R}_+$. We consider

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v.$$

Then, as we have

$$|\Phi_u(t, s, x)v| \leq \frac{s^2}{t}|v| = s\frac{s}{t}|v|,\ \forall t \geq s \geq 1 = t_0,\ \forall (x, v) \in Y,$$

it follows that $C = (\varphi, \Phi)$ is a Barreira-Valls polynomially stable skew-evolution semiflow.

If we suppose that C is exponentially stable, according to Definition 3.1, there exist $N : \mathbb{R}_+ \to [1, \infty)$ and $\alpha > 0$ such that

$$\frac{s+1}{t+1} \leq N(s)e^{-\alpha t},\ \forall t \geq s \geq t_0,$$

which is equivalent with

$$\frac{e^{\alpha t}}{t+1} \leq \frac{N(t_0)}{t_0+1},\ \forall t \geq t_0,$$

and which, for $t \to \infty$, leads to a contradiction. Hence, C is not exponentially stable.

Remark. For $\alpha \geq \beta$ in Definition 3.2, a Barreira-Valls exponentially stable skew-evolution semiflow C is polynomially stable.

$$(B.V.e.s.) \Longrightarrow (p.s.)$$

Example 4.2. Let us consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and the mapping $u : \mathbb{R}_+ \to \mathbb{R}$ given by $u(t) = t^2 + 1$. The mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\varphi(t, s, x) = t - s + x$ is an evolution semiflow on $\mathbb{R}_+$. We define

$$\Phi_u : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R}),\ \Phi_u(t, s, x)v = \frac{u(s)}{u(t)}v.$$

Then, as the relation

$$|\Phi_u(t, s, x)v| \leq (s^2 + 1)(t - s)^{-2}|v|,\ \forall t > s \geq 0,\ \forall (x, v) \in Y$$

holds, it follows that $C = (\varphi, \Phi)$ is a polynomially stable skew-evolution semiflow. On the other hand, C is not Barreira-Valls exponentially stable.

A similar concept to the nonuniform exponential growth can be considered the following nonuniform asymptotic property, given by

Definition 4.2. A skew-evolution semiflow $C = (\varphi, \Phi)$ has *polynomial growth* (*p.g.*) if there exist two mappings $M, \gamma : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that:

$$\|\Phi(t, s, x)v\| \leq M(s)(t - s)^{\gamma(s)} \|v\|, \tag{4.2}$$

for all $t > s \geq 0$ and all $(x, v) \in Y$.

Remark. If a skew-evolution semiflow C has polynomial growth, then it has exponential growth.

$$(p.g.) \Longrightarrow (e.g.)$$

In order to obtain an integral characterization for the property of nonuniform polynomial stability for skew-evolution semiflows, we introduce the following concept, given by

Definition 4.3. A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be $*$-*strongly measurable* ($* - s.m.$) if for every $(t, t_0, x, v^*) \in T \times X \times V^*$ the mapping defined by $s \mapsto \|\Phi(t, s, \varphi(s, t_0, x))^* v^*\|$ is measurable on $[t_0, t]$.

A particular class of $*$-strongly measurable skew-evolution semiflows is given by the next

Definition 4.4. A $*$-strongly measurable skew-evolution semiflow $C = (\varphi, \Phi)$ is called $*$-*integrally stable* ($* - i.s.$) if there exists a nondecreasing mapping $B : \mathbb{R}_+ \to [1, \infty)$ such that:

$$\int_s^t \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| d\tau \leq B(s) \left\|v^*\right\|, \tag{4.3}$$

for all $(t, s) \in T$, all $x \in X$ and all $v^* \in V^*$ with $\|v^*\| \leq 1$.

Theorem 4.3. *Let $C = (\varphi, \Phi)$ be a $*$-strongly measurable skew-evolution semiflow with polynomial growth. If C is $*$-integrally stable, then C is stable.*

Proof. Let us consider the function

$$\gamma_1 : \mathbb{R}_+ \to \mathbb{R}_+,\ \gamma_1(t) = \frac{1}{1 + \gamma(t)},$$

where the mapping γ is given by Definition 4.2. We remark that for $t \geq s + 1$ we have

$$\int_s^t (\tau - s)^{-\gamma(s)} d\tau = \int_0^{t-s} u^{-\gamma(s)} du \geq \int_0^1 u^{-\gamma(s)} du = \gamma_1(s).$$

Hence, it follows that

$$\gamma_1(s) | < v^*, \Phi(t, s, x)v > | \leq$$
$$\leq \int_s^t (\tau - s)^{-\gamma(s)} \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| \|\Phi(\tau, s, x)v\| \, d\tau \leq$$
$$\leq M(s) \|v\| \int_s^t \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| d\tau \leq M(s)B(s) \|v\| \left\|v^*\right\|,$$

where the existence of function M is assured by Definition 4.2. We obtain

$$\|\Phi(t, s, x)v\| \leq M_1(s) \|v\|, \ \forall t \geq s + 1 > s \geq 0, \ \forall (x, v) \in Y,$$

where we have denoted

$$M_1(s) = \frac{M(s)B(s)}{\gamma(s)}, \ s \geq 0.$$

On the other hand, for $t \in [s, s + 1)$, we have

$$\|\Phi(t, s, x)v\| \leq M(s)(t - s)^{\gamma(s)} \|v\| \leq M(s) \|v\|,$$

and, hence, it follows that

$$\|\Phi(t, s, x)v\| \leq [M(s) + M_1(s)] \|v\|, \ \forall (t, s) \in T, \ \forall (x, v) \in Y,$$

which proves that the skew-evolution semiflow C is stable. □

The main result of this section is the following

Theorem 4.4. *Let $C = (\varphi, \Phi)$ be a $*$-strongly measurable skew-evolution semiflow with polynomial growth. If C is $*$-integrally stable, then C is polynomially stable.*

Proof. As the skew-evolution semiflow $C = (\varphi, \Phi)$ is $*$-integrally stable, according to Theorem 4.3, it follows that there exists a mapping $M_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|< v^*, \Phi(t, s, x)v >| = |< \Phi(t, \tau, \varphi(\tau, s, x))^* v^*, \Phi(\tau, s, x)v >| \leq$$

$$\leq \|\Phi(\tau, s, x)v\| \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| \leq M_2(s) \|v\| \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\|.$$

By integrating on $[s, t]$ we obtain for $(x, v) \in Y$ and $v^* \in V^*$ with $\|v^*\| \leq 1$

$$(t - s)| < v^*, \Phi(t, s, x)v > | \leq M_2(s) \|v\| \int_s^t \left\|\Phi(t, \tau, \varphi(\tau, s, x))^* v^*\right\| d\tau \leq$$

$$\leq M_2(s)B(s) \|v\| \left\|v^*\right\|,$$

which implies

$$(t - s) \|\Phi(t, s, x)v\| \leq M_2(s)B(s) \|v\|.$$

Hence, following relation

$$\|\Phi(t, s, x)v\| \leq M_2(s)B(s)(t - s)^{-1} \|v\|$$

holds for all $(t, s) \in T$ and all $(x, v) \in Y$.

Finally, it results that the skew-evolution semiflow $C = (\varphi, \Phi)$ is polynomially stable. □

Remark. In (Stoica & Megan, 2010), a variant of Theorem 4.4 for the case of uniform exponential stability is proved, as a generalization of a well known theorem of E.A. Barbashin, given in (Barbashin, 1967) for differential systems and of a result obtained in (Buşe *et al.*, 2007) by C. Buşe, M. Megan, M. Prajea and P. Preda for evolution operators. We remark that, in the nonuniform setting, the property of $*$-integral stability only implies the polynomial stability.

Remark. The reciprocal of Theorem 4.4 is not true. The skew-evolution semiflow given in Example 4.2 is polynomially stable but not $*$-strongly measurable. If we suppose that C is $*$-strongly measurable, we have

$$\int_s^t \frac{\tau^2 + 1}{t^2 + 1} d\tau = \frac{t - s}{t^2 + 1} \left(1 + \frac{t^2 + ts + s^2}{3}\right) \leq N(s).$$

For $t \to \infty$, a contradiction is obtained.

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