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Some Inequalities Involving Fuzzy Complex Numbers

Sanjib Kumar Datta[a,*], Tanmay Biswas[b], Samten Tamang[a]

[a]*Department of Mathematics, University of Kalyani,Kalyani, Dist-Nadia,PIN- 741235, West Bengal, India.*
[b]*Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.- Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India.*

Abstract

In this paper we wish to establish a few inequalities related to fuzzy complex numbers which extend some standard results.

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1. Introduction, Definitions and Notations

The idea of fuzzy subset μ of a set X was primarily introduced by L.A. Zadeh (Zadeh, 1965) as a function $\mu : X \to [0, 1]$. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Among the various types of fuzzy sets, those which are defined on the universal set of complex numbers are of particular importance. They may, under certain conditions, be viewed as fuzzy complex numbers.

A fuzzy set z_f is defined by its membership function $\mu\left(z \mid z_f\right)$ which is a mapping from the complex numbers $\mathbb{C}$ into $[0, 1]$ where z is a regular complex number as $z = x + iy$, is called a fuzzy complex number if it satisfies the following conditions :

1. $\mu\left(z \mid z_f\right)$ is continuous;

2. An α-cut of z_f which is defined as $z_f^{\alpha} = \left\{z \mid \mu\left(z \mid z_f\right) > \alpha\right\}$, where $0 \leq \alpha < 1$, is open, bounded, connected and simply connected; and

3. $z_f^{1} = \left\{z \mid \mu\left(z \mid z_f\right) = 1\right\}$ is non-empty, compact, arcwise connected and simply connected.

(For detail on the set z_f as mentioned above, one may see (Buckley, 1989)).

Using this concept of fuzzy complex numbers, J. J. Buckley (Buckley, 1989) shown that fuzzy complex numbers is closed under the basic arithmetic operations. In paper (Buckley, 1989) we

*Corresponding author
Email addresses: `sanjib_kr_datta@yahoo.co.in` (Sanjib Kumar Datta), `tanmaybiswas_math@rediffmail.com` (Tanmay Biswas), `samtentamang@yahoo.in` (Samten Tamang)

also see the development of fuzzy complex numbers by defining addition and multiplication from the extension principle which has been shown in terms of α-cuts.

We now review some definitions used in this paper.

Definition 1.1. (Buckley, 1989)The complex conjugate $\bar{z}_f$ of z_f is defined as

$$\mu\left(z \mid \bar{z}_f\right) = \mu\left(\bar{z} \mid z_f\right),$$

where $\bar{z} = x - iy$ is the complex conjugate of $z = x + iy$. The complex conjugate $\bar{z}_f$ of a fuzzy complex number z_f is also a fuzzy complex number because the mapping $z = x + iy \to \bar{z} = x - iy$ is continuous.

Definition 1.2. (Buckley, 1989)The modulus $\left|z_f\right|$ of a fuzzy complex number z_f is defined as

$$\mu\left(r \mid \left|z_f\right|\right) = \sup\left\{\mu\left(z \mid z_f\right) \mid |z| = r\right\},$$

where r is the modulus of z.

Similarly we may define the modulus of a real fuzzy number R_f as follows:

$$\mu\left(|a| \mid \left|R_f\right|\right) = \sup\{\mu\left(a \mid R_f\right) \mid |a| = a \text{ if } a > 0, |a| = 0 \text{ if } a = 0 \text{ and } |a| = -a \text{ if } a < 0\}.$$

Now in the following, we define two special types of fuzzy complex numbers z_f^n and nz_f of the fuzzy complex number z_f, for any complex number $z \in z_f$ and $n \in R$.

Definition 1.3. Fuzzy complex numbers z_f^n and nz_f are defined as

$$\mu(z \mid z_f^n) = \mu(z^n \mid z_f)$$

and

$$\mu(z \mid nz_f) = \mu(n.z \mid z_f).$$

In particular when $n = 2$, we have

$$\mu(z \mid z_f^2) = \mu(z^2 \mid z_f) \quad \text{and} \quad \mu(z \mid 2z_f) = \mu(2.z \mid z_f)\ .$$

It can be easily verified that

$$z_f^2 \neq z_f.z_f \qquad \text{and} \qquad 2z_f \neq z_f + z_f \text{ but } 2(z_{f_1} + z_{f_2}) = 2z_{f_1} + 2z_{f_2}.$$

From the definition of fuzzy complex number one may easily verify that z_f^n and nz_f are also fuzzy complex numbers when z_f is a fuzzy complex number. It should be noted that the significance of Definition 1.3 is completely different from the definitions of additions and multiplications of fuzzy complex numbers as mentioned in (Buckley, 1989).

In this paper we wish to establish a few standard inequalities related to fuzzy complex numbers.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *(Buckley, 1989) Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers. Suppose $A = z_{f_1} + z_{f_2}$ and $M = z_{f_1}.z_{f_2}$ respectively. Then for $0 \leq \alpha \leq 1$, $A^\alpha = S^\alpha$ and $M^\alpha = P^\alpha$ holds where*

$$S^\alpha = \left\{z_{f_1} + z_{f_2} \mid (z_1, z_2) \in z_{f_1}^\alpha \times z_{f_2}^\alpha\right\}$$

and

$$P^\alpha = \left\{z_{f_1}.z_{f_2} \mid (z_1, z_2) \in z_{f_1}^\alpha \times z_{f_2}^\alpha\right\}.$$

Also $z_{f_1} + z_{f_2}$ and $z_{f_1}.z_{f_2}$ are fuzzy complex numbers.

The following lemma may be deduced in the line of Lemma 2.1 and so its proof is omitted.

Lemma 2.2. *Let $z_{f_1}, z_{f_2}, z_{f_3}, ..., z_{f_n}$ be any n number of fuzzy complex numbers. Also let $A = z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}$ and $M = z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}$ respectively. Then for $0 \leq \alpha \leq 1$, $A^\alpha = S^\alpha$ and $M^\alpha = P^\alpha$ holds where*

$$S^\alpha = \left\{z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n} \mid (z_1, z_2, z_3, ..., z_n) \in z_{f_1}^\alpha \times z_{f_2}^\alpha \times z_{f_3}^\alpha \times ... \times z_{f_n}^\alpha\right\}$$

and

$$P^\alpha = \left\{z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n} \mid (z_1, z_2, z_3, ..., z_n) \in z_{f_1}^\alpha \times z_{f_2}^\alpha \times z_{f_3}^\alpha \times ... \times z_{f_n}^\alpha\right\}.$$

Lemma 2.3. *(Buckley, 1989) If z_f is any fuzzy complex number then*

$$\left|z_f\right|^\alpha = \left|z_f^\alpha\right|$$

where $0 \leq \alpha \leq 1$ and $\left|z_f\right|$ is a truncated real fuzzy number.

Lemma 2.4. *(Kaufmann & Gupta, 1985) If M and N be any two real fuzzy numbers then*

$$(M + N)^\alpha = M^\alpha + N^\alpha$$

and if $M \geq 0, N \geq 0$ then

$$(M \cdot N)^\alpha = M^\alpha \cdot N^\alpha.$$

Lemma 2.5. *(Buckley, 1989) Let $\bar{z}_f$ be a fuzzy complex conjugate number of a fuzzy complex number z_f. Then*

$$\bar{z}_f^\alpha = \overline{z_f^\alpha}$$

where $0 \leq \alpha \leq 1$.

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let* z_{f_1} *and* z_{f_2} *be any two fuzzy complex numbers. Then*

$$\left|z_{f_1} - z_{f_2}\right| \geq \left|z_{f_1}\right| - \left|z_{f_2}\right| .$$

Proof. The meaning of the inequality is that the interval $\left|z_{f_1} - z_{f_2}\right|^{\alpha}$ is greater that or equal to the interval $\left(\left|z_{f_1}\right| - \left|z_{f_2}\right|\right)^{\alpha}$ for $0 \leq \alpha \leq 1$.

Now from Lemma 2.1 and Lemma 2.3, we get that

$$\left|z_{f_1} - z_{f_2}\right|^{\alpha} = \left|\left(z_{f_1} - z_{f_2}\right)^{\alpha}\right| = \left|z_{f_1}^{\alpha} - z_{f_2}^{\alpha}\right| = \left\{|z_1 - z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.1}$$

Again in view of Lemma 2.4, we obtain from Lemma 2.3 that

$$\left(\left|z_{f_1}\right| - \left|z_{f_2}\right|\right)^{\alpha} = \left|z_{f_1}\right|^{\alpha} - \left|z_{f_2}\right|^{\alpha} = \left|z_{f_1}^{\alpha}\right| - \left|z_{f_2}^{\alpha}\right| = \left\{|z_1| - |z_2| \mid z_i \in z_{f_i}^{\alpha}, i = 1, 2\right\}. \tag{3.2}$$

Hence the result follows from (3.1) and (3.2) and in view of

$$|z_1 - z_2| \geq |z_1| - |z_2| .$$

This proves the theorem. □

J. J. Buckley (Buckley, 1989) proved the following results:

Theorem A (Buckley, 1989) *Let* z_{f_1} *and* z_{f_2} *be any two fuzzy complex numbers. Then*

$$(1).\quad \left|z_{f_1} - z_{f_2}\right| \leq \left|z_{f_1}\right| + \left|z_{f_2}\right| \quad \textit{and} \quad (2).\quad \left|z_{f_1}.z_{f_2}\right| = \left|z_{f_1}\right|\left|z_{f_2}\right| .$$

But he (Buckley, 1989) remained silent about the question when the equality holds in the inequality (1) of Theorem A. In the next two theorems, we wish to generalise the results of Theorem A and find out the condition for which $\left|z_{f_1} - z_{f_2}\right| = \left|z_{f_1}\right| + \left|z_{f_2}\right|$ holds respectively.

Theorem 3.2. *Let* $z_{f_1}, z_{f_2}, z_{f_3}, ..., z_{f_n}$ *be any n number of fuzzy complex numbers. Then*

$$(i).\quad \left|z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}\right| \leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + ... + \left|z_{f_n}\right| \ \textit{and}$$

$$(ii).\quad \left|z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}\right| = \left|z_{f_1}\right|\left|z_{f_2}\right|\left|z_{f_3}\right|...\left|z_{f_n}\right| .$$

Proof. In view of Lemma 2.1, it follows from Theorem A that

$$\begin{aligned}\left|z_{f_1} + z_{f_2} + z_{f_3} + ... + z_{f_n}\right| &\leq \left|z_{f_1}\right| + \left|z_{f_2} + z_{f_3} + ... + z_{f_n}\right| \\ &\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3} + ... + z_{f_n}\right| \\ &\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + \left|z_{f_4} + ... + z_{f_n}\right| \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \left|z_{f_1}\right| + \left|z_{f_2}\right| + \left|z_{f_3}\right| + ... + \left|z_{f_n}\right| .\end{aligned}$$

This proves the first part of the theorem.

Similarly with the help of Lemma 2.1 and the equality $\left|z_{f_1}.z_{f_2}\right| = \left|z_{f_1}\right|\left|z_{f_2}\right|$, one can easily establish the second part of the theorem. □

Remark. In view of Lemma 2.2, Lemma 2.3 and Lemma 2.4 it can also be shown that the intervals $\left|z_{f_1}+z_{f_2}+z_{f_3}+...+z_{f_n}\right|^{\alpha}$ and $\left|z_{f_1}.z_{f_2}.z_{f_3}...z_{f_n}\right|^{\alpha}$ are less than or equal to the intervals $\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|+\left|z_{f_3}\right|+...+\left|z_{f_n}\right|\right)^{\alpha}$ and $\left(\left|z_{f_1}\right|\left|z_{f_2}\right|\left|z_{f_3}\right|...\left|z_{f_n}\right|\right)^{\alpha}$ respectively in Theorem 3.2 for $0\le\alpha\le 1$.

Theorem 3.3. *Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers such that $\left|z_{f_1}+z_{f_2}\right|=\left|z_{f_1}\right|+\left|z_{f_2}\right|$ then either $\arg z_1-\arg z_2$ is an even multiple of π or $\frac{z_1}{z_2}$ is a positive real number where z_1 and z_2 are any two members of z_{f_1} and z_{f_2} respectively.*

Proof. The meaning of the equality is that the interval $\left|z_{f_1}+z_{f_2}\right|^{\alpha}$ is equal to the interval $\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)^{\alpha}$ for $0\le\alpha\le 1$.
Thus $\left|z_{f_1}+z_{f_2}\right|=\left|z_{f_1}\right|+\left|z_{f_2}\right|$ i.e, $\left|z_{f_1}+z_{f_2}\right|^{\alpha}=\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)^{\alpha}$ i.e., $\left|z_{f_1}+z_{f_2}\right|^{\alpha}=\left(\left|z_{f_1}\right|^{\alpha}+\left|z_{f_2}\right|^{\alpha}\right)$ i.e, $\left|z^{\alpha}_{f_1}+z^{\alpha}_{f_2}\right|=\left|z^{\alpha}_{f_1}\right|+\left|z^{\alpha}_{f_2}\right|$ i.e., $|z_1+z_2|=|z_1|+|z_2|\mid z_i\in z^{\alpha}_{f_i}, i=1,2$; which is only possible when either $\arg z_1-\arg z_2$ is an even multiple of π or $\frac{z_1}{z_2}$ is a positive real number. Hence the theorem follows. □

Theorem 3.4. *If z_{f_1} and z_{f_2} are any two fuzzy complex numbers with $\left|z_{f_1}+z_{f_2}\right|=\left|z_{f_1}-z_{f_2}\right|$, then $\arg z_1$ and $\arg z_2$ differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ where z_1 and z_2 are any two members of z_{f_1} and z_{f_2} respectively.*

Proof. The meaning of the equality is that the α-cuts of $\left|z_{f_1}+z_{f_2}\right|$ is equal to the corresponding α-cuts of $\left|z_{f_1}-z_{f_2}\right|$ for $0\le\alpha\le 1$.
Now in view of Lemma 2.1 and Lemma 2.3, we obtain that

$$\left|z_{f_1}+z_{f_2}\right|^{\alpha}=\left|\left(z_{f_1}+z_{f_2}\right)^{\alpha}\right|=\left|z^{\alpha}_{f_1}+z^{\alpha}_{f_2}\right|=\left\{|z_1+z_2|\mid z_i\in z^{\alpha}_{f_i}, i=1,2\right\}. \tag{3.3}$$

Similarly,

$$\left|z_{f_1}-z_{f_2}\right|^{\alpha}=\left|\left(z_{f_1}-z_{f_2}\right)^{\alpha}\right|=\left|z^{\alpha}_{f_1}-z^{\alpha}_{f_2}\right|=\left\{|z_1-z_2|\mid z_i\in z^{\alpha}_{f_i}, i=1,2\right\}. \tag{3.4}$$

Therefore from (3.3) and (3.4) it follows that $\left|z_{f_1}+z_{f_2}\right|=\left|z_{f_1}-z_{f_2}\right|$ which implies that $|z_1+z_2|=|z_1-z_2|\mid z_i\in z^{\alpha}_{f_i}, i=1,2$ which is only possible when $\arg z_1$ and $\arg z_2$ differ by $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.
Thus the theorem is established. □

Theorem 3.5. *Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers. Then*

$$\left|z_{f_1}\pm z_{f_2}\right|\ge\left|\left|z_{f_1}\right|-\left|z_{f_2}\right|\right|.$$

Proof. For $0\le\alpha\le 1$, we have

$$\left|z_{f_1}\pm z_{f_2}\right|^{\alpha}=\left|\left(z_{f_1}\pm z_{f_2}\right)^{\alpha}\right|=\left|z^{\alpha}_{f_1}\pm z^{\alpha}_{f_2}\right|=\left\{|z_1\pm z_2|\mid z_i\in z^{\alpha}_{f_i}, i=1,2\right\}. \tag{3.5}$$

We also deduce that

$$\left|\left|z_{f_1}\right|-\left|z_{f_2}\right|\right|^{\alpha}=\left|\left(\left|z_{f_1}\right|-\left|z_{f_2}\right|\right)^{\alpha}\right|=\left|\left|z_{f_1}\right|^{\alpha}-\left|z_{f_2}\right|^{\alpha}\right|=\left|\left|z^{\alpha}_{f_1}\right|-\left|z^{\alpha}_{f_2}\right|\right|=\left\{||z_1|-|z_2||\mid z_i\in z^{\alpha}_{f_i}, i=1,2\right\}. \tag{3.6}$$

Hence the theorem follows from (3.5) and (3.6) and in view of the following inequality :

$$|z_1\pm z_2|\ge||z_1|-|z_2||.$$

□

Theorem 3.6. *If z_{f_1} and z_{f_2} are any two fuzzy complex numbers, then*

$$2\left|z_{f_1}+z_{f_2}\right| \geq \left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right| .$$

Proof. In order to prove this theorem, we wish to show that the interval $\left(2\left|z_{f_1}+z_{f_2}\right|\right)^{\alpha}$ is greater than or equal to the interval $\left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|\right\}^{\alpha}$ for $0 \leq \alpha \leq 1$.

From Lemma 2.3 and Lemma 2.4, we get that

$$\left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|\right\}^{\alpha} = \left\{\left(\left|z_{f_1}\right|+\left|z_{f_2}\right|\right)^{\alpha}\left|\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right|^{\alpha}\right\} = \left\{\left(\left|z_{f_1}\right|^{\alpha}+\left|z_{f_2}\right|^{\alpha}\right)\left|\left(\frac{z_{f_1}}{\left|z_{f_1}\right|}+\frac{z_{f_2}}{\left|z_{f_2}\right|}\right)^{\alpha}\right|\right\}$$

$$= \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(\frac{z_{f_1}}{\left|z_{f_1}\right|}\right)^{\alpha}+\left(\frac{z_{f_2}}{\left|z_{f_2}\right|}\right)^{\alpha}\right|\right\} = \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(z_{f_1}\left|z_{f_1}\right|^{-1}\right)^{\alpha}+\left(z_{f_2}\left|z_{f_2}\right|^{-1}\right)^{\alpha}\right|\right\}$$

$$= \left\{\left(\left|z_{f_1}^{\alpha}\right|+\left|z_{f_2}^{\alpha}\right|\right)\left|\left(z_{f_1}^{\alpha}\left|z_{f_1}^{\alpha}\right|^{-1}\right)+\left(z_{f_2}^{\alpha}\left|z_{f_2}^{\alpha}\right|^{-1}\right)\right|\right\} = \left\{\left(|z_1|+|z_2|\right)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}. \quad (3.7)$$

Since

$$2\left|z_1+z_2\right| \geq \left(|z_1|+|z_2|\right)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|,$$

in view of Definition 1.2 and Definition 1.3, the theorem follows from (3.3) and (3.7). □

Theorem 3.7. *Let z_{f_1} and z_{f_2} be any two fuzzy complex numbers. Then*

$$\left|\left(z_{f_1}+z_{f_2}\right)^2\right|+\left|\left(z_{f_1}-z_{f_2}\right)^2\right| = \left(2\left|z_{f_1}^2\right|-2\left|z_{f_2}^2\right|\right).$$

Proof. In view of Lemma 2.1, Lemma 2.3 and Lemma 2.4, we get for $0 \leq \alpha \leq 1$ that

$$\left(\left|\left(z_{f_1}+z_{f_2}\right)^2\right|+\left|\left(z_{f_1}-z_{f_2}\right)^2\right|\right)^{\alpha} = \left(\left|\left(z_{f_1}+z_{f_2}\right)^2\right|^{\alpha}+\left|\left(z_{f_1}-z_{f_2}\right)^2\right|^{\alpha}\right) = \left|\left(\left(z_{f_1}+z_{f_2}\right)^2\right)^{\alpha}\right|+\left|\left(\left(z_{f_1}-z_{f_2}\right)^2\right)^{\alpha}\right|$$
$$= \left(\left|\left(z_{f_1}^{\alpha}+z_{f_2}^{\alpha}\right)^2\right|+\left|\left(z_{f_1}^{\alpha}-z_{f_2}^{\alpha}\right)^2\right|\right) = \left\{\left|(z_1+z_2)^2\right|+\left|(z_1-z_2)^2\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}$$
$$= \left\{|z_1+z_2|^2+|z_1-z_2|^2 \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}. \quad (3.8)$$

Analogously we also see that

$$\left(2\left|z_{f_1}^2\right|-2\left|z_{f_2}^2\right|\right)^{\alpha} = \left(2\left|z_{f_1}^2\right|^{\alpha}-2\left|z_{f_2}^2\right|^{\alpha}\right) = \left(2\left|\left(z_{f_1}^2\right)^{\alpha}\right|-2\left|\left(z_{f_2}^2\right)^{\alpha}\right|\right) = \left(2\left|\left(z_{f_1}^{\alpha}\right)^2\right|-2\left|\left(z_{f_2}^{\alpha}\right)^2\right|\right)$$
$$= \left\{2\left|z_1^2\right|-2\left|z_2^2\right| \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\} = \left\{2|z_1|^2-2|z_2|^2 \mid z_i \in z_{f_i}^{\alpha}, i=1,2\right\}. \quad (3.9)$$

Now in the line of Definition 1.3, it follows from (3.8)and (3.9) that the corresponding α-cuts are equal. Hence the theorem follows as we obtain the equality of the two real fuzzy numbers. □

In the next theorem we establish a few properties of fuzzy complex conjugate numbers depending on the concept of it.

Theorem 3.8. *Let $\bar{z}_f$ be a fuzzy complex conjugate number of a fuzzy complex number z_f. Then*
(1). $\bar{\bar{z}}_f = z_f$, (2). $\overline{\left(z_{f_1} \pm z_{f_2}\right)} = \bar{z}_{f_1} \pm \bar{z}_{f_2}$, (3). $\overline{\left(z_{f_1} . z_{f_2}\right)} = \bar{z}_{f_1} \cdot \bar{z}_{f_2}$,
(4). $\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)} = \frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}$ *and* (5). $\left|z_f\right| = \left|\bar{z}_f\right|$.

Proof. In view of Lemma 2.5 and for $0 \le \alpha \le 1$, we obtain that

$$\left(\bar{\bar{z}}_f\right)^\alpha = \overline{\left(\bar{z}_f\right)^\alpha} = \overline{\overline{z_f^\alpha}} = \left\{\bar{\bar{z}} \mid \text{ for all } z \in z_f^\alpha\right\}.$$

Again

$$z_f^\alpha = \left\{z \mid \mu\left(z \mid z_f\right) > \alpha\right\} = \left\{z \mid \text{ for all } z \in z_f^\alpha\right\}.$$

Since $\bar{\bar{z}} = z$, the first part of the theorem follows from above.

For the second part of the theorem, we have to prove that the α-cuts of $\overline{\left(z_{f_1} \pm z_{f_2}\right)}$ are equal to the corresponding α-cuts of $\bar{z}_{f_1} \pm \bar{z}_{f_2}$.
Now it follows from Lemma 2.1 and Lemma 2.5 that

$$\left(\overline{\left(z_{f_1} \pm z_{f_2}\right)}\right)^\alpha = \overline{\left(z_{f_1} \pm z_{f_2}\right)^\alpha} = \overline{\left(z_{f_1}^\alpha \pm z_{f_2}^\alpha\right)} = \left\{\overline{z_1 \pm z_2} \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}$$

and

$$\left(\bar{z}_{f_1} \pm \bar{z}_{f_2}\right)^\alpha = \left(\bar{z}_{f_1}\right)^\alpha \pm \left(\bar{z}_{f_2}\right)^\alpha = \left(\overline{z_{f_1}^\alpha}\right) \pm \left(\overline{z_{f_2}^\alpha}\right) = \left\{\bar{z}_1 \pm \bar{z}_2 \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}.$$

Thus the second part of the theorem is established in view of $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

We also observe that

$$\overline{\left(z_{f_1} . z_{f_2}\right)}^\alpha = \overline{\left(z_{f_1} . z_{f_2}\right)^\alpha} = \overline{\left(z_{f_1}^\alpha . z_{f_2}^\alpha\right)} = \left\{\overline{z_1 . z_2} \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}. \tag{3.10}$$

We may also see that

$$\left(\bar{z}_{f_1} . \bar{z}_{f_2}\right)^\alpha = \left(\bar{z}_{f_1}\right)^\alpha . \left(\bar{z}_{f_2}\right)^\alpha = \overline{z_{f_1}^\alpha} . \overline{z_{f_2}^\alpha} = \left\{\bar{z}_1 . \bar{z}_2 \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}. \tag{3.11}$$

Now from (3.10) and (3.11), we obtain that the corresponding α-cuts are equal. This proves the third part of the theorem.

For the fourth part of the theorem, we deduce that

$$\left(\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)}\right)^\alpha = \overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)^\alpha} = \overline{\left(z_{f_1} . z_{f_2}^{-1}\right)^\alpha} = \overline{z_{f_1}^\alpha . \left(z_{f_2}^{-1}\right)^\alpha} = \overline{\left(z_{f_1}^\alpha . \left(z_{f_2}^\alpha\right)^{-1}\right)} = \left\{\overline{\frac{z_1}{z_2}} \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}$$

and

$$\left(\frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}\right)^\alpha = \left(\bar{z}_{f_1} . \bar{z}_{f_2}^{-1}\right)^\alpha = \left(\bar{z}_{f_1}\right)^\alpha . \left(\bar{z}_{f_2}^{-1}\right)^\alpha = \overline{z_{f_1}^\alpha} . \left(\left(\overline{z_{f_2}}\right)^\alpha\right)^{-1} = \overline{z_{f_1}^\alpha} \cdot \left(\left(\overline{z_{f_2}^\alpha}\right)\right)^{-1} = \left\{\frac{\bar{z}_1}{\bar{z}_2} \mid z_i \in z_{f_i}^\alpha, i = 1, 2\right\}.$$

Hence the α-cuts of $\overline{\left(\frac{z_{f_1}}{z_{f_2}}\right)}$ are equal to the corresponding α-cuts of $\frac{\bar{z}_{f_1}}{\bar{z}_{f_2}}$ which implies that the two fuzzy complex numbers are equal. Thus the fourth part of the theorem follows.
Again we have from Lemma 2.3 and Lemma 2.5 that

$$\left|z_f\right|^\alpha = \left|z_f^\alpha\right| = \left\{|z| \mid \text{ for all } z \in z_f\right\}$$

and

$$\left(\left|\bar{z}_f\right|\right)^\alpha = \left|\left(\bar{z}_f\right)^\alpha\right| = \left|\bar{z}_f^\alpha\right| = \left\{\left|\bar{z}\right| \mid \text{ for all } z \in z_f\right\}.$$

Consequently the last part of the theorem follows in view of $|z| = \left|\bar{z}\right|$. □

4. Open Problem

As open problems, there are several scopes to investigate the theory of analyticity and singularity in case of functions of fuzzy complex variables; and analogously entire or meromorphic functions of fuzzy complex variables may be defined. Naturally, the theory of different aspects of growth properties of entire and meromorphic functions, comparative growth estimates of iterated entire functions, results related to exponent of convergence of zeros of entire functions of fuzzy complex variables etc. may also be studied afresh.

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References

Buckley, J.J. (1989). Fuzzy complex numbers. *Fuzzy Sets and Systems* **33**(3), 333 – 345.

Kaufmann, A. and M. Gupta (1985). *Introduction to Fuzzy Arithmetic: Theory and Applications*. Van Nostrand Reinhold, New York.

Zadeh, L.A. (1965). Fuzzy sets. *Information and Control* **8**(3), 338 – 353.