Some Fixed Point Theorems for Ordered F-generalized Contractions in 0-f-orbitally Complete Partial Metric Spaces

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Abstract

We prove some fixed point theorems for ordered F-generalized contractions in ordered 0-f-orbitally complete partial metric spaces. Our results generalize some well-known results in the literature, in particular the recent result of Wardowski [Fixed Point Theory Appl. 2012:94 (2012)] from metric spaces to ordered 0-f-orbitally complete partial metric spaces. Some examples are given which illustrate the new results.

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1. Introduction

In 1994, Matthews (Matthews, 1994) introduced the notion of a partial metric space, as a part of the study of denotational semantics of dataflow networks. In a partial metric space, the usual distance was replaced by partial metric, with an interesting property of "nonzero self distance" of points. Also, the convergence of sequences in this space was defined in such a way that the limit of a convergent sequence need not be unique. Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verifications. Later on, several authors generalized the results of Matthews (see, for example, (Ahmad *et al.*, 2012; Bari *et al.*, 2012; Kadelburg *et al.*, 2013; Nashine *et al.*, 2012; Vetro & Radenović, 2012)). O'Neill (O'Neill, 1996) generalized the concept of partial metric space a bit further by admitting negative distances. The partial metric defined by O'Neill is called dualistic partial metric. Heckmann (Heckmann,

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1999) generalized it by omitting small self-distance axiom. The partial metric defined by Heckmann is called the weak partial metric. Romaguera (Romaguera, 2010) introduced the notions of 0-Cauchy sequences and 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

The existence of fixed points of mappings in partially ordered sets was investigated by Ran and Reurings (Ran & Reurings, 2004) and then by Nieto and Rodríguez-Lopez (Nieto & Lopez, 2005, 2007). In these papers, some results on the existence of a unique fixed point for nondecreasing mappings were applied to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Later on, these results were generalized by several authors in different spaces.

Recently, Wardowski (Wardowski, 2012) has introduced a new concept of F-contraction and proved a fixed point theorem which generalizes Banach contraction principle in a different direction than in the known results from the literature in complete metric spaces.

In this paper, we prove some fixed point theorems for ordered F-generalized contractions in ordered 0-f-orbitally complete partial metric spaces. The results of this paper generalize and extend the results of Wardowski (Wardowski, 2012), Ran and Reurings (Ran & Reurings, 2004), Nieto and Rodríguez-Lopez (Nieto & Lopez, 2005, 2007), Altun et al. (Altun *et al.*, 2010), Ćirić (Ćirić, 1971, 1972) and some other well-known results in the literature. Some examples are given which illustrate our results.

2. Preliminaries

First we recall some definitions and properties of partial metric spaces (see, e.g., (Matthews, 1994; Oltra & Valero, 2004; O'Neill, 1996; Romaguera, 2010, 2011)).

Definition 2.1. A partial metric on a nonempty set X is a function $p: X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Also, every metric space is a partial metric space, with zero self distance.

Example 2.1. If $p: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x, y) = \max\{x, y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.

For some more examples of partial metric spaces, we refer to (Aydi *et al.*, 2012) and the references therein.

Each partial metric on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \epsilon): x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all

$x \in X$ and $\epsilon > 0$. A mapping $f : X \to X$ is continuous if and only if, whenever a sequence $\{x_n\}$ in X converging with respect to τ_p to a point $x \in X$, the sequence $\{fx_n\}$ converges with respect to τ_p to $fx \in X$.

Theorem 2.1. *(Matthews, 1994) For each partial metric $p : X \times X \to \mathbb{R}^+$ the pair (X, d) where, $d(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, is a metric space.*

Here (X, d) is called the induced metric space and d is the induced metric. In further discussion, unless something else is specified, (X, d) will represent the induced metric space.

Let (X, p) be a partial metric space.

(1) A sequence $\{x_n\}$ in (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to \infty} p(x_n, x)$.
(2) A sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \to \infty} p(x_n, x_m)$.
(3) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.
(4) A sequence $\{x_n\}$ in (X, p) is called a 0-Cauchy sequence if $\lim_{n,m \to \infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 2.1. *(Matthews, 1994; Oltra & Valero, 2004; Romaguera, 2010, 2011) Let (X, p) be a partial metric space and $\{x_n\}$ be any sequence in X.*

(i) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d).*
(ii) *(X, p) is complete if and only if the metric space (X, d) is complete. Furthermore, $\lim_{n \to \infty} d(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.*
(iii) *Every 0-Cauchy sequence in (X, p) is Cauchy in (X, d).*
(iv) *If (X, p) is complete then it is 0-complete.*

The converse assertions of (iii) and (iv) do not hold. Indeed, the partial metric space $(\mathbb{Q} \cap \mathbb{R}^+, p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x, y) = \max\{x, y\}$, provides an easy example of a 0-complete partial metric space which is not complete. Also, it is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

The proof of the following lemma is easy and for details we refer to (Karapınar, 2012) and the references therein.

Lemma 2.2. *Assume $x_n \to z$ as $n \to \infty$ in a partial metric space (X, p) such that $p(z, z) = 0$. Then $\lim_{n \to \infty} p(x_n, y) = p(z, y)$ for all $y \in X$.*

The notion of orbital completeness of metric spaces was introduced in (Ćirić, 1971) and adapted to partial metric spaces in (Karapınar, 2012) as follows:

Let (X, p) be a partial metric space and $f : X \to X$ be a mapping. For any $x \in X$, the set $O(x) = \{x, fx, f^2x, \ldots\}$ is called the orbit of f at point x. (X, p) is called f-orbitally complete if every Cauchy sequence in $O(x)$ converges in (X, p).

Now, we define 0-f-orbital completeness of a partial metric space.

Definition 2.2. Let (X, p) be a partial metric space and $f: X \to X$ be a mapping. (X, p) is said to be 0-f-orbitally complete, if every 0-Cauchy sequence in $O(x) = \{x, fx, f^2x, \ldots\}$, $x \in X$, converges with respect to τ_p to a point $u \in X$ such that $p(u, u) = 0$.

Note that every complete partial metric space is 0-complete, and every 0-complete partial metric space is 0-f-orbitally complete for every $f: X \to X$. But, the converse assertions need not hold as shown by the following example.

Example 2.2. Let $X = \mathbb{R}^+ \cap (\mathbb{Q} \setminus \{1\})$ and $p: X \times X \to \mathbb{R}^+$ be defined by

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1); \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

Define $f: X \to X$ by $fx = \frac{x}{2}$ for all $x \in X$. Then (X, p) is a partial metric space. Note that (X, p) is not complete because the induced metric space (X, d), where

$$d(x, y) = \begin{cases} 2|x - y|, & \text{if } x, y \in [0, 1); \\ |x - y|, & \text{otherwise,} \end{cases}$$

is not complete. Also (X, p) is not 0-complete. Indeed, for $x_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$, we observe that, $p(x_n, x_m) = |\frac{1}{n} - \frac{1}{m}| \to 0$ as $n \to \infty$. But, there is no $u \in X$ such that $\lim_{n \to \infty} p(x_n, u) = p(u, u) = 0$. Now, it is easy to see that (X, p) is 0-f-orbitally complete.

Consider, together with Wardowski in (Wardowski, 2012), the following properties for a mapping $F: \mathbb{R}^+ \to \mathbb{R}$:

(F1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$, $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;

(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

We denote the set of all functions satisfying properties (F1)–(F3), by $\mathcal{F}$.

For examples of functions $F \in \mathcal{F}$, we refer to (Wardowski, 2012). Wardowski defined in (Wardowski, 2012) F-contractions as follows:

Let (X, ρ) be a metric space. A mapping $T: X \to X$ is said to be an F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$, $\rho(Tx, Ty) > 0$ we have

$$\tau + F(\rho(Tx, Ty)) \le F(\rho(x, y)).$$

Similarly, we adopt the following definitions.

Definition 2.3. Let X be a nonempty set, $\le$ a partial order relation defined on X and p be a partial metric on X (then, $(X, \le, p)$ is called an ordered partial metric space). A map $f: X \to X$ is called:

1. an ordered F-contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \le y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \le F(p(x, y)). \tag{2.1}$$

2. an ordered F-weak contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \leq y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \leq F(\max\{p(x, y), p(x, fx), p(y, fy)\}). \tag{2.2}$$

If inequality (2.2) is satisfied for all $x, y \in X$, then f is called an F-weak contraction;

3. an ordered F-generalized contraction if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with $x \leq y$ and $p(fx, fy) > 0$ we have

$$\tau + F(p(fx, fy)) \leq F(\max\{p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2}\}). \tag{2.3}$$

If inequality (2.3) is satisfied for all $x, y \in X$, then f is called an F-generalized contraction.

The following example shows that the class of F-contractions in partial metric spaces is more general than that in metric spaces.

Example 2.3. Let $X = \mathbb{R}^+$ and $p: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Note that the metric induced by p (as well as the usual metric) on X is given by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $f: X \to X$ by

$$fx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1); \\ 0, & \text{if } x = 1. \end{cases}$$

Then for $x = 1, y = \frac{9}{10}$ there is no $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\tau + F(d(fx, fy)) \leq F(d(x, y)).$$

On the other hand, for $\tau = \log 2$ and $F(\alpha) = \log \alpha + \alpha$, it is easy to see that f is an F-contraction in (X, p).

3. Main results

The following is our first main result.

Theorem 3.1. *Let $(X, \leq, p)$ be an ordered partial metric space and $f: X \to X$ be an ordered F-generalized contraction for some $F \in \mathcal{F}$. If (X, p) is 0-f-orbitally complete and the following conditions hold:*

 (i) *f is nondecreasing with respect to "$\leq$", that is, if $x \leq y$ then $fx \leq fy$;*
 (ii) *there exists $x_0 \in X$ such that $x_0 \leq fx_0$;*
 (iii)   (a) *f is continuous, or*
         (b) *F is continuous and for every nondecreasing sequence $\{x_n\}$, $x_n \to u \in X$ as $n \to \infty$ implies $x_n \leq u$ for all $n \in \mathbb{N}$.*

Then f has a fixed point $u \in X$. Furthermore, the fixed point of f is unique if and only if the set of all fixed points of f is well-ordered.

Proof. First, we shall show the existence of fixed point of f. Let $x_0 \in X$ be the point given by (ii). We define a sequence $\{x_n\}$ in X by $x_{n+1} = f x_n$ for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ then x_{n_0} is a fixed point of f. Therefore, assume that $x_{n+1} \neq x_n$ for all $n \geq 0$. As, $x_0 \leq f x_0$ we have $x_0 \leq x_1$, and since f is nondecreasing with respect to $\leq$, we have $f x_0 \leq f x_1$ that is $x_1 \leq x_2$. Similarly, we obtain $x_n \leq x_{n+1}$ for all $n \geq 0$. Also, f is an ordered F-generalized contraction therefore, for any $n \in \mathbb{N}$ it follows from (2.3) and symmetry of p that

$$
\begin{aligned}
\tau + F(p(f x_n, f x_{n-1})) &= \tau + F(p(f x_{n-1}, f x_n)) \\
&\leq F(\max\{p(x_n, x_{n-1}), p(x_n, f x_n), p(x_{n-1}, f x_{n-1}), \\
&\qquad \frac{p(x_n, f x_{n-1}) + p(x_{n-1}, f x_n)}{2}\}) \\
&= F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), p(x_{n-1}, x_n), \\
&\qquad \frac{p(x_n, x_n) + p(x_{n-1}, x_{n+1})}{2}\}) \\
&\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), \\
&\qquad \frac{p(x_{n-1}, x_n) + p(x_{n+1}, x_n)}{2}\}).
\end{aligned}
$$

Note that, for any $a, b \in \mathbb{R}^+$ we have $\max\{a, b, \frac{a+b}{2}\} = \max\{a, b\}$, therefore it follows from the above inequality that

$$
\begin{aligned}
\tau + F(p(x_{n+1}, x_n)) &\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}) \\
F(p(x_{n+1}, x_n)) &\leq F(\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}) - \tau.
\end{aligned}
\tag{3.1}
$$

If, $\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n+1})$ then from (3.1) we have

$$
F(p(x_{n+1}, x_n)) \leq F(p(x_n, x_{n+1})) - \tau < F(p(x_n, x_{n+1})),
$$

a contradiction. Therefore, $\max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\} = p(x_n, x_{n-1})$ and from (3.1) we have

$$
F(p(x_{n+1}, x_n)) \leq F(p(x_n, x_{n-1})) - \tau \quad \text{for all} \;\; n \in \mathbb{N}.
\tag{3.2}
$$

Setting $p_n = p(x_{n+1}, x_n)$ it follows by successive applications of (3.2) that

$$
F(p_n) \leq F(p_{n-1}) - \tau \leq F(p_{n-2}) - 2\tau \leq \cdots \leq F(p_0) - n\tau.
\tag{3.3}
$$

From (3.3) we have $\lim_{n \to \infty} F(p_n) = -\infty$, and since $F \in \mathcal{F}$ we must have

$$
\lim_{n \to \infty} p_n = 0.
\tag{3.4}
$$

Again, as $F \in \mathcal{F}$ there exists $k \in (0, 1)$ such that

$$
\lim_{n \to \infty} (p_n)^k F(p_n) = 0.
\tag{3.5}
$$

From (3.3) we have

$$
(p_n)^k [F(p_n) - F(p_0)] \leq -n\tau(p_n)^k \leq 0.
$$

Letting $n \to \infty$ in the above inequality and using (3.4) and (3.5) we obtain

$$\lim_{n \to \infty} n(p_n)^k = 0. \tag{3.6}$$

It follows from (3.6) that there exists $n_1 \in \mathbb{N}$ such that $n(p_n)^k < 1$ for all $n > n_1$, that is,

$$p_n \le \frac{1}{n^{1/k}} \quad \text{for all} \quad n > n_1. \tag{3.7}$$

Let $m, n \in \mathbb{N}$ with $m > n > n_1$. Then it follows from (3.7) that

$$\begin{aligned}
p(x_n, x_m) &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\
&\quad - [p(x_n, x_n) + p(x_{n+1}, x_{n+1}) + \cdots + p(x_{m-1}, x_{m-1})] \\
&\le p_n + p_{n+1} + \cdots \\
&\le \frac{1}{n^{1/k}} + \frac{1}{(n+1)^{1/k}} + \cdots \\
&= \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned}$$

As $k \in (0, 1)$, the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$ converges, so it follows from the above inequality that $\lim_{n \to \infty} p(x_n, x_m) = 0$, that is, the sequence $\{x_n\}$ is a 0-Cauchy sequence in $O(x_0) = \{x_0, fx_0, f^2x_0, \ldots\}$. Therefore, by 0-$f$-orbital completeness of (X, p), there exists $u \in X$ such that

$$\lim_{n \to \infty} p(x_n, u) = \lim_{n,m \to \infty} p(x_n, x_m) = p(u, u) = 0. \tag{3.8}$$

We shall show that u is a fixed point of f. For this, we consider two cases.

Case I: Suppose (a) is satisfied, that is, f is continuous. Then using (3.8) and Lemma 2.2, we obtain

$$p(u, fu) = \lim_{n \to \infty} p(x_n, fu) = \lim_{n \to \infty} p(fx_{n-1}, fu) = p(fu, fu).$$

Suppose that $p(fu, fu) > 0$. Then as $u \preceq u$, using (2.3) we obtain

$$\begin{aligned}
\tau + F(p(fu, fu)) &\le F(\max\{p(u, u), p(u, fu), p(u, fu), \frac{p(u, fu) + p(u, fu)}{2}\}) \\
&= F(\max\{p(u, u), p(u, fu)\}) \\
&= F(p(u, fu)),
\end{aligned}$$

that is, $F(p(fu, fu)) < F(p(u, fu))$ and from $F \in \mathcal{F}$ we have $p(fu, fu) < p(u, fu) = p(fu, fu)$, a contradiction. Therefore, $p(fu, fu) = p(u, fu) = 0$, that is, $fu = u$, so u is a fixed point of f.

Case II: Suppose that (b) is satisfied. Then we consider two subcases.

(i): For each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $p(x_{k_n+1}, fu) = 0$ and $k_n > k_{n-1}$, where $k_0 = 1$. Then, using Lemma 2.2, we have

$$p(u, fu) = \lim_{n \to \infty} p(x_{k_n+1}, fu) = 0.$$

Therefore, $fu = u$, that is, u is a fixed point of f.

(ii): There exists $n_2 \in \mathbb{N}$ such that $p(x_n, fu) \neq 0$ for all $n > n_2$. In this case, since $\{x_n\}$ is a nondecreasing sequence and $x_n \to u$ as $n \to \infty$, we have $x_n \preceq u$ for all $n \in \mathbb{N}$. Therefore, using (2.3) we obtain

$$\tau + F(p(x_{n+1}, fu)) = \tau + F(p(fx_n, fu))$$
$$\leq F(\max\{p(x_n, u), p(x_n, fx_n), p(u, fu), \frac{p(x_n, fu) + p(u, fx_n)}{2}\})$$
$$\leq F(\max\{p(x_n, u), p(x_n, x_{n+1}), p(u, fu),$$
$$\frac{p(x_n, u) + p(u, fu) + p(u, x_{n+1})}{2}\}).$$

From (3.8), there exists $n_3 \in \mathbb{N}$ such that, for all $n > n_3$ we have

$$\max\{p(x_n, u), p(x_n, x_{n+1}), p(u, fu), \frac{p(x_n, u) + p(u, fu) + p(u, x_{n+1})}{2}\} = p(u, fu),$$

so, for $n > \max\{n_2, n_3\}$ we obtain

$$\tau + F(p(x_{n+1}, fu)) \leq F(p(u, fu)).$$

As F is continuous, letting $n \to \infty$ in the above inequality and using (3.8) and Lemma 2.2 we obtain

$$\tau + F(p(u, fu)) \leq F(p(u, fu)),$$

a contradiction. Therefore, we must have $p(u, fu) = 0$ that is $fu = u$. Thus u is a fixed point of f.

Suppose that the set of fixed points of f is well-ordered and $u, v \in F_f$ with $p(u, v) > 0$, where F_f denotes the set of all fixed points of f. As F_f is well-ordered, let $u \preceq v$. Then from (2.3) we obtain

$$\tau + F(p(u, v)) = \tau + F(p(fu, fv))$$
$$\leq F(\max\{p(u, v), p(u, fu), p(v, fv), \frac{p(u, fv) + p(v, fu)}{2}\})$$
$$\leq F(\max\{p(u, v), p(u, u), p(v, v), p(v, u)\})$$
$$\leq F(p(u, v)),$$

a contradiction. Similarly, for $v \preceq u$ we get a contradiction. Therefore, the fixed point of f is unique. For the converse, if the fixed point of f is unique then F_f, being a singleton, is well-ordered. $\qquad\square$

The following corollaries are immediate consequences of the above theorem.

Corollary 3.1. *Let $(X, \preceq, p)$ be an ordered partial metric space and $f : X \to X$ be an ordered F-contraction. Let (X, p) is 0-f-orbitally complete and the following conditions hold:*

(i) f is nondecreasing with respect to "$\preceq$", that is, if $x \preceq y$ then $fx \preceq fy$;

  (ii) *there exists $x_0 \in X$ such that $x_0 \preceq f x_0$;*

 (iii)   *(a)* *f is continuous, or*

        (b) *F is continuous and for every nondecreasing sequence $\{x_n\}$ such that $x_n \to u \in X$ as $n \to \infty$ it follows that $x_n \preceq u$ for all $n \in \mathbb{N}$.*

Then f has a fixed point $u \in X$. Furthermore, the fixed point of f is unique if and only if the set of all fixed points of f is well-ordered.

Corollary 3.2. *Let $(X, \preceq, p)$ be an ordered partial metric space and $f : X \to X$ be an ordered F-weak contraction. If (X, p) is 0-f-orbitally complete and the following conditions hold:*

  (i) *f is nondecreasing with respect to "$\preceq$", that is, if $x \preceq y$ then $f x \preceq f y$;*

 (ii) *there exists $x_0 \in X$ such that $x_0 \preceq f x_0$;*

(iii)   *(a)* *f is continuous, or*

        (b) *F is continuous and for every nondecreasing sequence $\{x_n\}$ such that $x_n \to u \in X$ as $n \to \infty$ it follows that $x_n \preceq u$ for all $n \in \mathbb{N}$.*

Then f has a fixed point $u \in X$. Furthermore, the fixed point of f is unique if and only if the set of all fixed points of f is well-ordered.

Remark. We note that every metric space is a partial metric space with zero self distance. Therefore we can replace the partial metric p by a metric ρ in Theorem 3.1. Also, after this replacement, the 0-f-orbital completeness reduces to orbital completeness of the metric space. Therefore, by this replacement in Theorem 3.1, we obtain the fixed point result for ordered F-generalized contraction in orbitally complete metric spaces.

    In the above theorems the fixed point of the self map f is the limit of a 0-Cauchy sequence and due to 0-f-orbital completeness of the space this limit has zero self distance. The next theorem shows that, if an ordered F-generalized contraction has a fixed point then its self distance must be zero, that is, it does not depend on the properties of space such as completeness etc.

Theorem 3.2. *Let $(X, \preceq, p)$ be an ordered partial metric space and $f : X \to X$ be an ordered F-generalized contraction. If f has a fixed point u then $p(u, u) = 0$.*

Proof. Suppose that $u \in F_f$ and $p(u, u) > 0$. Then, it follows from (2.3) that

$$
\begin{aligned}
\tau + F(p(u, u)) = \tau + F(p(fu, fu)) \\
\leq F(\max\{p(u, u), p(u, fu), p(u, fu), \frac{p(u, fu) + p(u, fu)}{2}\}) \\
= F(p(u, u)).
\end{aligned}
$$

As $\tau > 0$, the above inequality yields a contradiction. Therefore, we have $p(u, u) = 0$ for all $u \in F_f$.   □

    The following example illustrates our results.

Example 3.1. Let $X = [0, 2] \cap (\mathbb{Q} \setminus \{1\})$ and define $p\colon X \times X \to \mathbb{R}^+$ by

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1); \\ 0, & \text{if } x = y = 2; \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

Then (X, p) is a partial metric space. Define a partial order relation "$\preceq$" on X by

$$\preceq \ = \ \{(x, y)\colon x, y \in [0, 1), y \le x\} \cup \{(x, y)\colon x, y \in (1, 2), y \le x\} \cup \{(2, 2)\}.$$

Define $f\colon X \to X$ by

$$fx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1); \\ \frac{1}{4}, & \text{if } x \in (1, 2); \\ 2, & \text{if } x = 2. \end{cases}$$

Then it is easy to see that (X, p) is a 0-f-orbitally complete partial metric space. Let $F(\alpha) = \log \alpha$ for all $\alpha \in \mathbb{R}^+$. Then f satisfies all the conditions of Corollary 3.1 (except that the set of fixed points of f is well-ordered) with $\tau \le \log 2$. Note that, $F_f = \{0, 2\}$ with $p(0, 0) = p(2, 2) = 0$ and $(2, 0), (0, 2) \notin \preceq$. Now, the metric d induced by p is given by

$$d(x, y) = \begin{cases} 2|x - y|, & \text{if } x, y \in [0, 1); \\ |x - y|, & \text{otherwise,} \end{cases}$$

and (X, d) is not complete. Similarly, if ρ is the usual metric on X then (X, ρ) is not complete, therefore the results from metric cases are not applicable here. This example shows also that an ordered F-contraction may not be an F-contraction (not even an F-generalized contraction). Indeed, for $x \in [0, 1)$, $y = 2$ there exists no $F \in \mathcal{F}$ and $\tau > 0$ such that

$$\tau + F(p(fx, fy)) \le F(\max\{p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2}\}).$$

Therefore, f is not an F-generalized contraction in (X, p). Similarly, for $x = 0$, $y = 2$ one can see that f is not an F-generalized contraction in (X, d) and (X, ρ).

In the following theorem the conditions on self map f, "nondecreasing", continuous and 0-f-orbital completeness of space, are replaced by another condition on self map f.

Theorem 3.3. *Let $(X, \preceq, p)$ be an ordered partial metric space and $f\colon X \to X$ be an ordered F-generalized contraction. Let there exists $u \in X$ such that $u \preceq fu$ and $p(u, fu) \le p(x, fx)$ for all $x \in X$. Then f has a fixed point $u \in X$. Furthermore, the fixed point of f is unique if and only if the set of all fixed points of f is well-ordered.*

Proof. Let $G(x) = p(x, fx)$ for all $x \in X$. Then by assumption we have

$$G(u) \le G(x) \text{ for all } x \in X. \tag{3.9}$$

We shall show that $fu = u$. Suppose that $G(u) = p(u, fu) > 0$. Then since $u \preceq fu$, it follows from (2.3) that

$$F(G(fu)) = F(p(fu, ffu))$$
$$\leq F(\max\{p(u, fu), p(u, fu), p(fu, ffu), \frac{p(u, ffu) + p(fu, fu)}{2}\}) - \tau$$
$$\leq F(\max\{p(u, fu), p(fu, ffu), \frac{p(u, fu) + p(fu, ffu)}{2}\}) - \tau$$
$$= F(\max\{G(u), G(fu), \frac{G(u) + G(fu)}{2}\}) - \tau$$
$$= F(\max\{G(u), G(fu)\}) - \tau.$$

If $\max\{G(u), G(fu)\} = G(fu)$, then it follows from the above inequality that $F(G(fu)) < F(G(fu))$, a contradiction. If $\max\{G(u), G(fu)\} = G(u)$, then again we obtain $F(G(fu)) < F(G(u))$ and $F \in \mathcal{F}$ so $G(fu) < G(u)$, a contradiction. Thus, we must have $G(u) = p(u, fu) = 0$, that is $fu = u$ and so u is a fixed point of f.

The necessary and sufficient condition for the uniqueness of fixed point follows from a similar process as used in Theorem 3.1. □

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