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Fractional Order Differential Equations Involving Caputo Derivative

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Abstract

In this paper, the Banach contraction principle and Schaefer theorem are applied to establish new results for the existence and uniqueness of solutions for some Caputo fractional differential equations. Some examples are also discussed to illustrate the main results.

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1. Introduction

The theory of fractional differential equations has excited in recent years a considerable interest both in mathematics and in applications, (see (Bengrine & Dahmani, 2012; Delbosco & Rodino, 1996; Diethelm & Walz, 1998; El-Sayed, 1998)). In particular, existence and uniqueness of solutions for fractional differential equations have attracted the attention of many mathematicians (Diethelm & Ford, 2002; Houas & Dahmani, 2013; Zhang, 2011; Ntouyas, 2012; Su, 2009; Yang, 2012; Zhang, 2011).
This paper deals with the existence and uniqueness of solutions to the following problem

$$\begin{aligned} D^{\alpha}x(t)+f\left(t,y(t),D^{\delta}y(t)\right)&=0, t\in J,\\ D^{\beta}y(t)+g\left(t,x(t),D^{\sigma}x(t)\right)&=0, t\in J,\\ x(0)=y(0)=0, x(1)-\lambda_1 x(\eta)=0, y(1)-\lambda_1 y(\eta)&=0,\\ x''(0)=y''(0)=0, x''(1)-\lambda_2 x''(\xi)=0, y''(1)-\lambda_2 y''(\xi)&=0, \end{aligned} \tag{1.1}$$

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where $D^{\alpha}, D^{\beta}, D^{\delta}$ and D^{σ}, are the Caputo fractional derivatives, $3 < \alpha, \beta \leq 4, \delta \leq \alpha - 1, \sigma \leq \beta - 1, 0 < \xi, \eta < 1$, $J = [0,1]$, λ_1, λ_2 are real constants satisfying $\lambda_1 \eta \neq 1, \lambda_2 \xi \neq 1$ and f, g are two functions which will be specified later.
This paper is organized as follows: In section 2, we present some preliminaries and lemmas. In section 3, we present our main results for the existence and uniqueness of solutions of (1.1). In section 4, some examples are treated to illustrate our results.

2. Preliminaries

To present our main results, we need the the following two definitions:

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[0, \infty[$ is defined as:

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0, \tag{2.1}$$

$$J^0 f(t) = f(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u} u^{\alpha-1} du$.

Definition 2.2. The fractional derivative of $f \in C^n([0, \infty[)$ in the Caputo's sense is defined as:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, n-1 < \alpha, n \in N^*. \tag{2.2}$$

More details about fractional calculus can be found in (Mainardi, 1997; Podlubny *et al.*, 2002).
We need also to introduce the spaces:
$X = \{x : x \in C([0,1]), D^{\sigma} x \in C([0,1])\}$ and $Y = \{y : y \in C([0,1]), D^{\delta} y \in C([0,1])\}$. For these spaces, we associate respectively the norms $\| x \|_X = \| x \| + \| D^{\sigma} x \|$; $\| x \| = \sup_{t \in J} |x(t)|$, $\| D^{\sigma} x \| = \sup_{t \in J} \left| D^{\sigma} x(t) \right|$ and $\| y \|_Y = \| y \| + \| D^{\delta} y \|$; $\| y \| = \sup_{t \in J} |y(t)|$, $\| D^{\delta} y \| = \sup_{t \in J} \left| D^{\delta} y(t) \right|$. It is clear that, $(X, \| . \|_X)$ and $(Y, \| . \|_Y)$, are two Banach spaces.
Also, $(X \times Y, \|(x,y)\|_{X \times Y})$ is a Banach space. Its norm is given by $\|(x,y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$.

The following lemmas are crucial for our main results (Kilbas & Marzan, 2005; Lakshmikantham & Vatsala, 2008):

Lemma 2.1. *For $\alpha > 0$, the general solution of the equation $D^{\alpha} x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.3}$$

where $c_i \in \mathbb{R}, i = 0, 1, 2, .., n-1, n = [\alpha] + 1$.

Lemma 2.2.

$$J^{\alpha} D^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, \tag{2.4}$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1$.

We prove also the following lemma which is needed to present the integral solution for the problem (1.1):

Lemma 2.3. *Let $h \in C([0,1])$, $t \in J$, $3 < \alpha \leq 4$. Then the solution of the equation*

$$D^{\alpha} x(t) + h(t) = 0, \tag{2.5}$$

where,

$$\begin{aligned} x(0) &= 0, x(1) - \lambda_1 x(\eta) = 0, \\ x''(0) &= 0, x''(1) - \lambda_2 x''(\xi) = 0 \end{aligned} \tag{2.6}$$

is given by the following expression

$$\begin{aligned} x(t) = &-\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h\left(s, y(s), D^{\delta} y(s)\right) ds \\ &+\frac{\lambda_1 t}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} h\left(s, y(s), D^{\delta} y(s)\right) ds \\ &-\frac{1}{(\lambda_1 \eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h\left(s, y(s), D^{\delta} y(s)\right) ds \\ &+\frac{\left(\lambda_2 - \lambda_2\lambda_1\eta^3\right)t + (\lambda_2\lambda_1\eta - \lambda_2)t^3}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^{\xi} (\xi - s)^{\alpha-3} h\left(s, y(s), D^{\delta} y(s)\right) ds \\ &-\frac{\left(1 - \lambda_1\eta^3\right)t + (\lambda_1\eta - 1)t^3}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} h\left(s, y(s), D^{\delta} y(s)\right) ds. \end{aligned} \tag{2.7}$$

Proof: Let $c_i \in \mathbb{R}, i = 0,1,2,3$. Then by lemmas 2.1, 2.2, the general solution of (2.5) can be written as:

$$x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds - c_0 - c_1 t - c_2 t^2 - c_3 t^3. \tag{2.8}$$

Using (2.6), we immediately get $c_0 = c_2 = 0$. On the other hand, we have

$$\begin{aligned} c_1 = &-\frac{\lambda_1}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^{\eta} (\eta - s)^{\alpha-1} h(s)\, ds + \frac{1}{(\lambda_1\eta - 1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\, ds \\ &-\frac{\lambda_2\left(1 - \lambda_1\eta^3\right)}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^{\xi} (\xi - s)^{\alpha-3} h(s)\, ds \\ &+\frac{(1 - \lambda_1\eta)}{6(\lambda_1\eta - 1)(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} h(s)\, ds. \end{aligned} \tag{2.9}$$

To obtain the value of c_3, we remarque that

$$c_3 = -\frac{\lambda_2}{6(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^{\xi} (\xi - s)^{\alpha-3} h(s)\, ds + \frac{1}{6(\lambda_2\xi - 1)\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-3} g(s)\, ds. \tag{2.10}$$

Finally, substituting the values of c_1 and c_3 in (2.8), we obtain (2.7).

3. Main results

We begin by introducing the quantities:

$$\begin{aligned}
N_1 &= \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)},\\
N_2 &= \frac{1}{\Gamma(\alpha-\sigma+1)} + \frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} + \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\\
&\quad+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)},\\
N_3 &= \frac{|\lambda_1\eta-1|+|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)} + \frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)},\\
N_4 &= \frac{1}{\Gamma(\beta-\delta+1)} + \frac{|\lambda_1|\eta^{\beta}+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)} + \frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\\
&\quad+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}.
\end{aligned}$$

We impose also the hypotheses:

$(H1)$: The functions f , g : $[0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous.

$(H2)$: There exist non negative functions $a_i, b_i \in C([0,1])$, $i = 1,2$ such that for all $t \in [0,1]$ and $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, the inequalities

$$\begin{aligned}
|f(t,x_1,y_1) - f(t,x_2,y_2)| &\leq a_1(t)|x_1-x_2| + b_1(t)|y_1-y_2|, \qquad (3.1)\\
|g(t,x_1,y_1) - g(t,x_2,y_2)| &\leq a_2(t)|x_1-x_2| + b_2(t)|y_1-y_2|,
\end{aligned}$$

are valid, and

$$\omega_1 = \sup_{t\in J} a_1(t), \omega_2 = \sup_{t\in J} b_1(t), \varpi_1 = \sup_{t\in J} a_2(t), \varpi_2 = \sup_{t\in J} b_2(t).$$

$(H3)$: There exist positive constants L_1 and L_2 such that

$$|f(t,x,y)| \leq L_1, |g(t,x,y)| \leq L_2 \text{ for each } t \in J \text{ and all } x,y \in \mathbb{R}.$$

Our first main result is given by the following theorem:

Theorem 3.1. *Assume that (H2) holds and suppose that*

$$(N_1+N_2)(\omega_1+\omega_2) + (N_3+N_4)(\varpi_1+\varpi_2) < 1. \qquad (3.2)$$

Then the problem (1.1) has a unique solution on J.

Proof: We apply Banach fixed point theorem. So, we consider the operator $\phi : X\times Y \to X\times Y$ defined by:

$$\phi(x,y)(t) := (\phi_1 y(t), \phi_2 x(t)), \qquad (3.3)$$

where

$$\begin{aligned}
\phi_1 y(t) \quad : \quad &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y(s), D^{\delta}y(s)\right) ds \\
&+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^{\eta} (\eta-s)^{\alpha-1} f\left(s, y(s), D^{\delta}y(s)\right) ds \\
&-\frac{1}{(\lambda_1\eta-1)\Gamma(\alpha)}\int_0^{1} (1-s)^{\alpha-1} f\left(s, y(s), D^{\delta}y(s)\right) ds \\
&+\frac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{\xi} (\xi-s)^{\alpha-3} f\left(s, y(s), D^{\delta}y(s)\right) ds \\
&-\frac{\left(1-\lambda_1\eta^3\right)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\alpha-2)}\int_0^{1} (1-s)^{\alpha-3} f\left(s, y(s), D^{\delta}y(s)\right) ds,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\phi_2 x(t) \quad : \quad &= -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g\left(s, x(s), D^{\sigma}x(s)\right) ds \\
&+\frac{\lambda_1 t}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^{\eta} (\eta-s)^{\alpha-1} g\left(s, x(s), D^{\sigma}x(s)\right) ds \\
&-\frac{1}{(\lambda_1\eta-1)\Gamma(\beta)}\int_0^{1} (1-s)^{\alpha-1} g\left(s, x(s), D^{\sigma}x(s)\right) ds \\
&+\frac{\left(\lambda_2-\lambda_2\lambda_1\eta^3\right)t+(\lambda_2\lambda_1\eta-\lambda_2)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{\xi} (\xi-s)^{\alpha-3} g\left(s, x(s), D^{\sigma}x(s)\right) ds \\
&-\frac{\left(1-\lambda_1\eta^3\right)t+(\lambda_1\eta-1)t^3}{6(\lambda_1\eta-1)(\lambda_2\xi-1)\Gamma(\beta-2)}\int_0^{1} (1-s)^{\alpha-3} g\left(s, x(s), D^{\sigma}x(s)\right) ds.
\end{aligned} \tag{3.5}$$

And we shall prove that ϕ is a contraction mapping.

Let $(x, y), (x_1, y_1) \in X \times Y$. Then, for each $t \in J$, we have:

$$\begin{aligned}
|\phi_1 y(t) - \phi_1 y_1(t)| \leq &\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\
&+\tfrac{|\lambda_1|t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta} (\eta-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\
&+\tfrac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{1} (1-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\
&+\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi} (\xi-s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds \\
&+\tfrac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{1} (1-s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right) - f\left(s, y_1(s), D^{\delta}y_1(s)\right)\right| ds.
\end{aligned} \tag{3.6}$$

Thanks to $(H2)$, we obtain

$$|\phi_1 y(t) - \phi_1 y_1(t)| \leq \frac{(|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1)\left(\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \frac{\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right]\left(\omega_1\|y-y_1\|+\omega_2\left\|D^\delta y-D^\delta y_1\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}. \tag{3.7}$$

Consequently,

$$|\phi_1 y(t) - \phi_1 y_1(t)| \leq N_1(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right), \tag{3.8}$$

Hence,

$$\|\phi_1(y) - \phi_1(y_1)\| \leq N_1(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right). \tag{3.9}$$

We have also,

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \leq{}& \frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds \\ &+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds \\ &+\frac{t^{1-\sigma}}{|(\lambda_1\eta-1)|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1 (1-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds \\ &+\left(\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right)\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds \\ &+\left(\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right)\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right) - f\left(s,y_1(s),D^\delta y_1(s)\right)\right|ds.\end{aligned} \tag{3.10}$$

By $(H2)$, yields

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \leq{}& \frac{(\omega_1+\omega_2)\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{\Gamma(\alpha-\sigma+1)} \\ &+\frac{(\omega_1+\omega_2)[|\lambda_1|\eta^\alpha+1]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ &+\frac{(\omega_1+\omega_2)\left[\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|\right]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} \\ &+\frac{(\omega_1+\omega_2)\left[|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|\right]\left(\|y-y_1\|+\left\|D^\delta y-D^\delta y_1\right\|\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}.\end{aligned} \tag{3.11}$$

This implies that,

$$\begin{aligned}\left|D^\sigma\phi_1 y(t) - D^\sigma\phi_1 y_1(t)\right| \leq{}& \left[\frac{(\omega_1+\omega_2)}{\Gamma(\alpha-\sigma+1)} + \frac{\%(\omega_1+\omega_2)[|\lambda_1|\eta^\alpha+1]}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right]\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right) \\ &+\left[\begin{array}{c}\frac{(\omega_1+\omega_2)\left[\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} \\ +\frac{(\omega_1+\omega_2)\left[|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|\right]}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\end{array}\right]\left(\|y-y_1\| + \left\|D^\delta y - D^\delta y_1\right\|\right).\end{aligned} \tag{3.12}$$

Therefore,

$$\left|D^{\sigma}\phi_1 y(t) - D^{\sigma}\phi_1 y_1(t)\right| \leq N_2(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.13}$$

Consequently,

$$\left\|D^{\sigma}\phi_1(y) - D^{\sigma}\phi_1(y_1)\right\| \leq N_2(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.14}$$

By (3.9) and (3.14), we can write

$$\|\phi_1(y) - \phi_1(y_1)\|_X \leq (N_1+N_2)(\omega_1+\omega_2)\left(\|y-y_1\| + \left\|D^{\delta}y - D^{\delta}y_1\right\|\right). \tag{3.15}$$

With the same arguments as before, we have

$$\|\phi_2(x) - \phi_2(x_1)\|_Y \leq (N_3+N_4)(\varpi_1+\varpi_2)\left(\|x-x_1\| + \left\|D^{\sigma}x - D^{\sigma}x_1\right\|\right). \tag{3.16}$$

Using (3.15) and (3.16), we can state that

$$\|\phi(x,y) - \phi(x_1,y_1)\|_{X\times Y} \leq \left[\begin{array}{c}(N_1+N_2)(\omega_1+\omega_2)\\ +(N_3+N_4)(\varpi_1+\varpi_2)\end{array}\right](\|(x-x_1, y-y_1)\|_{X\times Y}). \tag{3.17}$$

Thanks to (3.2), we conclude that ϕ is contraction. As a consequence of Banach fixed point theorem, we deduce that ϕ has a unique fixed point which is a solution of (1.1).

The second main result is based on Schaefer theorem. We have:

Theorem 3.2. *Suppose that (H1) and (H3) are satisfied. Then, the problem* (1.1) *has at least one solution on J.*

Proof: A: Thanks to (H1), we can state that the operator ϕ is continuous on $X\times Y$.

B: We will prove that ϕ maps bounded sets into bounded sets in $X\times Y$.

So, taking $\rho > 0$, and $(x,y) \in B_{\rho} := \{(x,y) \in X\times Y; \|(x,y)\|_{X\times Y} \leq \rho\}$, then for each $t \in J$, we have:

$$\begin{aligned}|\phi_1 y(t)| \leq\ & \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds \\ &+\tfrac{|\lambda_1|t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds \\ &+\tfrac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds \\ &+\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds \\ &+\tfrac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\%\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s, y(s), D^{\delta}y(s)\right)\right| ds.\end{aligned} \tag{3.18}$$

The condition $(H3)$ implies that

$$|\phi_1 y(t)| \leq \frac{L_1(|\lambda_1\eta-1||\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{L_1\left[\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left(\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|\right)\right]}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}$$
$$\leq L_1\left[\begin{array}{c}\frac{|\lambda_1\eta-1||\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}\\ +\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\end{array}\right]. \tag{3.19}$$

Then,

$$\|\phi_1(y)\| \leq L_1 N_1. \tag{3.20}$$

For D^σ, we have the following inequalities

$$\begin{aligned}\left|D^\sigma\phi_1 y(t)\right| \leq &\frac{1}{\Gamma(\alpha-\sigma)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\
&+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta(\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\
&+\frac{t^{1-\sigma}}{|(\lambda_1\eta-1)|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^1(1-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\
&+\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\sigma)}\\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)}\end{array}\right)\int_0^\xi(\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds\\
&+\frac{1}{\Gamma(\alpha-2)}\left(\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{\%6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(2-\sigma)}\\ +\frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(4-\sigma)}\end{array}\right)\int_0^1(1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta y(s)\right)\right|ds.\end{aligned} \tag{3.21}$$

By $(H3)$ again, yields the following formula

$$\begin{aligned}\left|D^\sigma\phi_1 y(t)\right| &\leq L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right]\\
&\quad+L_1\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\end{array}\right]\\
&\leq L_1\left[\begin{array}{c}\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\\ +\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{\%|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\end{array}\right].\end{aligned} \tag{3.22}$$

Hence, we can write

$$\left\|D^\sigma\phi_1(y)\right\| \leq L_1 N_2. \tag{3.23}$$

Using (3.20) and (3.23), we obtain

$$\|\phi_1(y)\|_X \leq L_1(N_1+N_2). \tag{3.24}$$

As before, we obtain

$$\|\phi_2(x)\|_Y \leq L_2(N_3 + N_4). \tag{3.25}$$

By (3.24) and (3.25), we get

$$\|\phi(x,y)\|_{X\times Y} \leq L_1(N_1 + N_2) + L_2(N_3 + N_4). \tag{3.26}$$

Therefore,

$$\|\phi(x,y)\|_{X\times Y} < \infty. \tag{3.27}$$

C: Now, we prove the equi-continuity of ϕ.
Let us take $(x,y) \in B_\rho$, and $t_1, t_2 \in J$, with $t_1 < t_2$. We have:

$$\begin{aligned}
|\phi_1 y(t_2) - \phi_1 y(t_1)| \leq\ & \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right)\left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
&+\tfrac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
&+\tfrac{|\lambda_1|(t_2-t_1)}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{\eta}(\eta-s)^{\alpha-1}\left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
&+\tfrac{(t_1-t_2)}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^{1}(1-s)^{\alpha-1}\left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
&+\tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|(t_2-t_1)+|\lambda_2\lambda_1\eta-\lambda_2|\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{\xi}(\xi-s)^{\alpha-3}\left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
&+\tfrac{\left|1-\lambda_1\eta^3\right|(t_1-t_2)+|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^{1}(1-s)^{\alpha-3}\left|f\left(s, y(s), D^\delta y(s)\right)\right| ds \\
\leq\ & \tfrac{L_1\left(t_1^\alpha-t_2^\alpha+2\left(t_2^\alpha-t_1^\alpha\right)\right)}{\Gamma(\alpha+1)} + \tfrac{L_1|\lambda_1|\eta^\alpha(t_2-t_1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \tfrac{L_1(t_1-t_2)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\
&+\tfrac{L_1\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}(t_2-t_1)+L_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}\left(t_2^3-t_1^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)} \\
&+\tfrac{L_1\left|1-\lambda_1\eta^3\right|(t_1-t_2)+L_1|\lambda_1\eta-1|\left(t_1^3-t_2^3\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}.
\end{aligned} \tag{3.28}$$

Therefore,

$$\begin{aligned}
|\phi_1 y(t_2) - \phi_1 y(t_1)| \leq\ & L_1\left[\tfrac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \tfrac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2-t_1) \\
&+L_1\left[\tfrac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)} + \tfrac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_1-t_2) \\
&+L_1\left[\tfrac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right]\left(t_2^3-t_1^3\right) + \tfrac{L_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right) \\
&+\tfrac{L_1}{\Gamma(\alpha+1)}\left(t_1^\alpha-t_2^\alpha\right) + \tfrac{2L_1}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha.
\end{aligned} \tag{3.29}$$

We have also

$$
\begin{aligned}
\left|D^{\sigma}\phi_1 y(t_2) - D^{\sigma}\phi_1 y(t_1)\right| \leq{}& \frac{1}{\Gamma(\alpha-\sigma)} \int_0^{t_1} \left((t_1 - s)^{\alpha-\sigma-1} - (t_2 - s)^{\alpha-\sigma-1}\right) \left|f\left(s, y(s), D^{\delta} y(s)\right)\right| ds \\
&+ \frac{1}{\Gamma(\alpha-\sigma)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-\sigma-1} \left|f\left(s, y(s), D^{\delta} y(s)\right)\right| ds \\
&+ \frac{|\lambda_1|\left(t_2^{1-\sigma} - t_1^{1-\sigma}\right)}{|\lambda_1\eta - 1|\Gamma(\alpha)\Gamma(2-\sigma)} \int_0^{\eta} (\eta - s)^{\alpha-1} \left|f\left(s, y(s), D^{\delta} y(s)\right)\right| ds \\
&+ \frac{\left(t_1^{1-\sigma} - t_2^{1-\sigma}\right)}{|\lambda_1\eta - 1|\Gamma(\alpha)\Gamma(2-\sigma)} \int_0^{1} (1 - s)^{\alpha-1} \left|f\left(s, y(s), D^{\delta} y(s)\right)\right| ds \\
&+ \left[\begin{array}{c} \frac{\left|\lambda_2 - \lambda_2\lambda_1\eta^3\right|\left(t_2^{1-\sigma} - t_1^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ + \frac{|\lambda_2\lambda_1\eta - \lambda_2|\left(t_2^{3-\sigma} - t_1^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)} \end{array}\right] \int_0^{\xi} (\xi - s)^{\alpha-3} \left|f\left(s, y(s), D^{\delta} y(s)\right)\right| ds \\
&+ \left[\begin{array}{c} \frac{\left|1 - \lambda_1\eta^3\right|\left(t_1^{1-\sigma} - t_2^{1-\sigma}\right)}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ + \frac{|\lambda_1\eta - 1|\left(t_1^{3-\sigma} - t_2^{3-\sigma}\right)}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \end{array}\right] \int_0^{1} (1 - s)^{\alpha-3} \left|f\left(s, y(s), D^{\delta} y(s)\right)\right| ds.
\end{aligned}
\tag{3.30}
$$

The condition $(H3)$ implies that

$$
\begin{aligned}
\left|D^{\sigma}\phi_1 y(t_2) - D^{\sigma}\phi_1 y(t_1)\right| \leq{}& \frac{L_1}{\Gamma(\alpha-\sigma+1)} \left(t_1^{\alpha-\sigma} - t_2^{\alpha-\sigma} + 2(t_2 - t_1)^{\alpha-\sigma}\right) \\
+ L_1 \left[\begin{array}{c} \frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ + \frac{\left|\lambda_2 - \lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} \end{array}\right] \left(t_2^{1-\sigma} - t_1^{1-\sigma}\right) &+ L_1 \left[\begin{array}{c} \frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)} \\ + \frac{\left|1 - \lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)} \end{array}\right] \left(t_1^{1-\sigma} - t_2^{1-\sigma}\right) \\
+ \frac{L_1|\lambda_2\lambda_1\eta - \lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)} \left(t_2^{3-\sigma} - t_1^{3-\sigma}\right) &+ \frac{L_1|\lambda_1\eta - 1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)} \left(t_1^{3-\sigma} - t_2^{3-\sigma}\right).
\end{aligned}
\tag{3.31}
$$

The inequalities (3.29) and (3.31) imply that:

$$\begin{aligned}
\|\phi_1 y(t_2)-\phi_1 y(t_1)\|_X &\le L_1\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_2-t_1)\\
&+L_1\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right](t_1-t_2)+L_1\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\right]\left(t_2^3-t_1^3\right)\\
&+\frac{L_1|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\left(t_1^3-t_2^3\right)+\frac{L_1}{\Gamma(\alpha+1)}\left(t_1^\alpha-t_2^\alpha+2(t_2-t_1)^\alpha\right)\\
&+\frac{L_1}{\Gamma(\alpha-\sigma+1)}\left(t_1^{\alpha-\sigma}-t_2^{\alpha-\sigma}+2(t_2-t_1)^{\alpha-\sigma}\right)+L_1\left[\begin{array}{c}\frac{|\lambda_1|\eta^\alpha+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\\+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\end{array}\right]\left(t_2^{1-\sigma}-t_1^{1-\sigma}\right)\\
&+L_1\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}\right]\left(t_1^{1-\sigma}-t_2^{1-\sigma}\right)\\
&+\frac{L_1|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_2^{3-\sigma}-t_1^{3-\sigma}\right)+\frac{L_1|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\left(t_1^{3-\sigma}-t_2^{3-\sigma}\right).
\end{aligned} \tag{3.32}$$

With the same arguments as before, we can write

$$\begin{aligned}
\|\phi_2 x(t_2)-\phi_2 x(t_1)\|_Y &\le L_2\left[\frac{|\lambda_1\eta-1|+|\lambda_1|\eta^\beta}{|\lambda_1\eta-1|\Gamma(\beta+1)}+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right](t_2-t_1) \quad (3.33)\\
&+L_2\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\beta+1)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right](t_1-t_2)+L_2\left[\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\right]\left(t_2^3-t_1^3\right)\\
&+\frac{L_2|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)}\left(t_1^3-t_2^3\right)+\frac{L_2}{\Gamma(\beta+1)}\left(t_1^\beta-t_2^\beta+2(t_2-t_1)^\beta\right)\\
&+\frac{L_2}{\Gamma(\alpha-\beta+1)}\left(t_1^{\beta-\delta}-t_2^{\beta-\delta}+2(t_2-t_1)^{\beta-\delta}\right)+L_2\left[\begin{array}{c}\frac{|\lambda_1|\eta^\beta+1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)}\\+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\beta-2}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\end{array}\right]\left(t_2^{1-\delta}-t_1^{1-\delta}\right)\\
&+L_2\left[\frac{1}{|\lambda_1\eta-1|\Gamma(\beta+1)\Gamma(2-\delta)}+\frac{\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(2-\delta)}\right]\left(t_1^{1-\delta}-t_2^{1-\delta}\right)\\
&+\frac{L_2|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\beta-2}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_2^{3-\delta}-t_1^{3-\delta}\right)+\frac{L_2|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\beta-1)\Gamma(4-\delta)}\left(t_1^{3-\delta}-t_2^{3-\delta}\right).
\end{aligned}$$

Thanks to (3.32) and (3.33), we can state that $\|\phi(x,y)(t_2)-\phi(x,y)(t_1)\|_{X\times Y}\to 0$ as $t_2\to t_1$. Combining **A, B, C** and using Arzela-Ascoli theorem, we conclude that ϕ is completely continuous operator.

D: We shall show that

$$\Omega := \{(x,y)\in X\times Y, (x,y)=\mu\phi(x,y), 0<\mu<1\}, \tag{3.34}$$

is a bounded set.

Let $(x,y)\in\Omega$, then $(x,y)=\mu\phi(x,y)$, for some $0<\mu<1$. Thus, for each $t\in J$, we have:

$$y(t)=\mu\phi_1 y(t), x(t)=\mu\phi_2 x(t).$$

Therefore,

$$\begin{aligned}\frac{1}{\mu}|y(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{|\lambda_1|t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{t}{|\lambda_1\eta-1|\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t+|\lambda_2\lambda_1\eta-\lambda_2|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{\left|1-\lambda_1\eta^3\right|t+|\lambda_1\eta-1|t^3}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds.\end{aligned} \tag{3.35}$$

Thanks to $(H3)$, we can write

$$\begin{aligned}\frac{1}{\mu}|y(t)| \leq\ & \frac{L_1(|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ &+\frac{L_1\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}.\end{aligned} \tag{3.36}$$

Thus,

$$|y(t)| \leq \mu L_1\left[\begin{array}{c}\frac{(|\lambda_1\eta-1|+|\lambda_1|\eta^\alpha+1)}{|\lambda_1\eta-1|\Gamma(\alpha+1)} \\ +\frac{\left(\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|+|\lambda_2\lambda_1\eta-\lambda_2|\right)\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|+|\lambda_1\eta-1|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)}\end{array}\right]. \tag{3.37}$$

Hence,

$$|y(t)| \leq \mu N_1 L_1, t \in J. \tag{3.38}$$

On the other hand,

$$\begin{aligned}\frac{1}{\mu}\left|D^\sigma y(t)\right| \leq\ & \frac{1}{\Gamma(\alpha-\delta)}\int_0^t (t-s)^{\alpha-\sigma-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{|\lambda_1|t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\frac{t^{1-\sigma}}{|\lambda_1\eta-1|\Gamma(\alpha)\Gamma(2-\sigma)}\int_0^\eta (\eta-s)^{\alpha-1}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\left[\begin{array}{c}\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_2\lambda_1\eta-\lambda_2|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^\xi (\xi-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds \\ &+\left[\begin{array}{c}\frac{\left|1-\lambda_1\eta^3\right|t^{1-\sigma}}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(2-\sigma)} \\ +\frac{|\lambda_1\eta-1|t^{3-\sigma}}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-2)\Gamma(4-\sigma)}\end{array}\right]\int_0^1 (1-s)^{\alpha-3}\left|f\left(s,y(s),D^\delta(s)\right)\right|ds.\end{aligned} \tag{3.39}$$

Thanks to $(H3)$, we have

$$\begin{aligned}\frac{1}{\mu}\left|D^{\sigma}y(t)\right| \leq\ & L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right]\\ &+L_1\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right].\end{aligned} \tag{3.40}$$

Therefore,

$$\begin{aligned}\left|D^{\sigma}y(t)\right| \leq\ & \mu L_1\left[\frac{1}{\Gamma(\alpha-\sigma+1)}+\frac{|\lambda_1|\eta^{\alpha}+1}{|\lambda_1\eta-1|\Gamma(\alpha+1)\Gamma(2-\sigma)}\right]\\ &+\mu L_1\left[\frac{\left|\lambda_2-\lambda_2\lambda_1\eta^3\right|\xi^{\alpha-2}+\left|1-\lambda_1\eta^3\right|}{6|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(2-\sigma)}+\frac{|\lambda_2\lambda_1\eta-\lambda_2|\xi^{\alpha-2}+|\lambda_1\eta-1|}{|\lambda_1\eta-1||\lambda_2\xi-1|\Gamma(\alpha-1)\Gamma(4-\sigma)}\right].\end{aligned} \tag{3.41}$$

Thus,

$$\left|D^{\beta}y(t)\right| \leq \mu L_1 N_2, t \in J. \tag{3.42}$$

From (3.38) and (3.42) , we get

$$\|y\|_X \leq \mu L_1 (N_1 + N_2). \tag{3.43}$$

Analogously, we can obtain

$$\|x\|_Y \leq \mu L_2 (N_3 + N_4). \tag{3.44}$$

It follows from (3.43) and (3.4) that

$$\|(x, y)\|_{X\times Y} \leq \mu L_1 (N_1 + N_2) + \mu L_2 (N_3 + N_4). \tag{3.45}$$

Hence,

$$\|\phi (x, y)\|_{X\times Y} < \infty. \tag{3.46}$$

This shows that the set Ω is bounded. Thanks to **A,B,C** and **D**, we conclude that ϕ has at least one fixed point. Theorem 3.2 is thus proved.

4. Examples

Example 4.1. Let us consider the coupled equations:

$$\begin{aligned}&D^{\frac{7}{2}}x(t)+\frac{|y(t)|}{7\left(\pi t^2+3\right)^2(2+|y(t)|)}+\frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)|\left|D^{\frac{7}{3}}y(t)\right|}{7\pi(1+e^t)^2\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)}=0, t\in[0,1],\\ &D^{\frac{11}{3}}y(t)+\frac{3\pi|x(t)|}{\left(5e^{t^2}+3\sqrt{\pi}\right)(1+|x(t)|)}+\frac{\pi e^{-2\pi t}\left|D^{\frac{5}{2}}x(t)\right|}{5\left(t+3\sqrt{\pi}\right)^2\left(1+\left|D^{\frac{5}{2}}x(t)\right|\right)}=0, t\in[0,1],\\ &x(0)=0, x(1)-\tfrac{3}{4}x\left(\tfrac{1}{3}\right)=0, y(0)=0, y(1)-\tfrac{3}{4}y\left(\tfrac{1}{3}\right)=0,\\ &x''(0)=0, x''(1)-\tfrac{4}{5}x''\left(\tfrac{2}{3}\right)=0, y''(0)=0, y''(1)-\tfrac{4}{5}y''\left(\tfrac{2}{3}\right)=0.\end{aligned} \tag{4.1}$$

It is clear that

$$\begin{aligned} f(t,x,y) &= \frac{|x|}{7\left(\pi t^2+3\right)^2(2+|x|)} + \frac{\sqrt{\pi}e^{-\pi t}|\cos(\pi t)||y|}{7\pi(1+e^t)^2(2+|y|)}, t\in[0,1], x,y\in\mathbb{R}, \\ g(t,x,y) &= \frac{3\pi|x|}{\left(5e^{t^2}+3\sqrt{\pi}\right)(1+|x|)} + \frac{\pi e^{-2\pi t}|y|}{5\left(t+3\sqrt{\pi}\right)^2(1+|y|)}, t\in[0,1], x,y\in\mathbb{R}. \end{aligned}$$

For $x, y, x_1, y_1 \in \mathbb{R}, t \in [0,1]$, we have

$$|f(t,x,y) - f(t,x_1,y_1)| \le \frac{1}{7\left(\pi t^2+3\right)^2}|x-x_1| + \frac{\sqrt{\pi}e^{-\pi t}}{7\pi(1+e^t)^2}|y-y_1|,$$

and

$$|g(t,x,y) - g(t,x_1,y_1)| \le \frac{3\pi}{\left(5e^{t^2}+3\sqrt{\pi}\right)}|x-x_1| + \frac{\pi e^{-2\pi t}}{5\left(t+3\sqrt{\pi}\right)^2}|y-y_1|.$$

Hence,

$$a_1(t) = \frac{1}{7\left(\pi t^2+3\right)^2}, b_1(t) = \frac{\sqrt{\pi}e^{-\pi t}}{7\pi(1+e^t)^2},$$

and

$$a_2(t) = \frac{3\pi}{5e^{t^2}+3\sqrt{\pi}}, b_2(t) = \frac{\pi e^{-2\pi t}}{5\left(t+3\sqrt{\pi}\right)^2}.$$

These imply that

$$\begin{aligned} \omega_1 &= \sup_{t\in[0,1]} a_1(t) = \frac{1}{63}, \omega_2 = \sup_{t\in[0,1]} b_1(t) = \frac{\sqrt{\pi}}{28\pi}, \\ \varpi_1 &= \sup_{t\in[0,1]} a_2(t) = \frac{3\pi}{5+3\sqrt{\pi}}, \varpi_2 = \sup_{t\in[0,1]} b_2(t) = \frac{1}{45}, \end{aligned}$$

$$N_1 = 1,08935, N_2 = 3,444, N_3 = 0,77571, N_4 = 2,51754,$$

and,

$$(N_1+N_2)(\omega_1+\omega_2) + (N_3+N_4)(\varpi_1+\varpi_2) = 0,16329 + 0,36466 = 0,52795 < 1.$$

So by Theorem 3.1, the problem (4.1) has a unique solution (x, y) on $[0,1]$.

Example 4.2. The following example illustrates Theorem 3.2. We take:

$$\begin{cases} D^{\frac{15}{4}}x(t) + \frac{1}{\left(t^2+1\right)\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)} + \frac{2e^{-t}|\cos(ty(t))|}{7(1+e^t)^2\left(2+\left|D^{\frac{7}{3}}y(t)\right|\right)} = 0, t \in [0,1], \\ D^{\frac{10}{3}}y(t) + \frac{1}{\left(e^{t^2}+1\right)(1+|x(t)|)} + \frac{e^{-t}}{(t+1)^2\left(1+\left|D^{\frac{5}{2}}x(t)\right|\right)} = 0, t \in [0,1], \\ x(0) = 0, x(1) - \frac{3}{4}x\left(\frac{1}{3}\right) = 0, y(0) = 0, y(1) - \frac{3}{4}y\left(\frac{1}{3}\right) = 0, \\ x''(0) = 0, x''(1) - \frac{4}{5}x''\left(\frac{2}{3}\right) = 0, y''(0) = 0, y''(1) - \frac{4}{5}y''\left(\frac{2}{3}\right) = 0. \end{cases} \tag{4.2}$$

We have

$$f(t,x,y) = \frac{1}{\left(t^2+1\right)(2+|y|)} + \frac{2e^{-t}|\cos(tx)|}{7(1+e^t)^2(2+|y|)}$$

and

$$g(t,x,y) = \frac{1}{\left(e^{t^2}+1\right)(1+|x|)} + \frac{e^{-t}}{(t+1)^2(1+|y|)}$$

So by Theorem 3.2, the problem (4.2) has at least one solution on $[0,1]$.

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