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Reich Type Contractions on Cone Rectangular Metric Spaces Endowed with a Graph

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Abstract

In this paper we prove some fixed point theorems for Reich type contractions on cone rectangular metric spaces endowed with a graph without assuming the normality of cone. The results of this paper extends and generalize several known results from metric, rectangular metric, cone metric and cone rectangular metric spaces in cone rectangular metric spaces endowed with a graph. Some examples are given which illustrate the results.

Keywords: Graph, cone rectangular metric space, Reich type contraction, fixed point.
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1. Introduction

In 1906, the French mathematician M. Fréchet Fréchet (1906) introduced the concept of metric spaces. After the work of Fréchet several authors generalized the concept of metric space by applying the conditions on metric function. In this sequel, Branciari Branciari (2000) introduced a class of generalized (rectangular) metric spaces by replacing triangular inequality of metric spaces by similar one which involves four or more points instead of three and improved Banach contraction principle Banach (1922) in such spaces. The result of Branciari is generalized and extended by several authors (see, for example, Flora *et al.* (2009); Bari & Vetro (2012); Chen (2012); Işik & Turkoglu (2013); Lakzian & Samet (2012); Arshad *et al.* (2013); Malhotra *et al.* (2013*a*,*b*) and the references therein).

Let (X, d) be a metric space and $T: X \to X$ be a mapping. Then T is called a Banach contraction if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \text{for all} \quad x, y \in X. \tag{1.1}$$

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Banach contraction principle ensures the existence of a unique fixed point of a Banach contraction on a complete metric space.

Kannan Kannan (1968) introduced the following contractive condition: there exists $\lambda \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \quad \text{for all} \quad x, y \in X. \tag{1.2}$$

Reich Reich (1971) introduced the following contractive condition: there exist nonnegative constants λ, μ, δ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \leq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty) \quad \text{for all} \quad x, y \in X. \tag{1.3}$$

Examples show that (see Kannan (1968); Reich (1971)) the conditions of Banach and Kannan are independent of each other while the condition of Reich is a proper generalization of conditions of Banach and Kannan.

On the other hand, the study of abstract spaces and the vector-valued spaces can be seen in Kurepa (1934, 1987); Rzepecki (1980); Lin (1987); Zabreǐko (1997). L.G. Huang and X. Zhang Huang & Zhang (2007) reintroduced such spaces under the name of cone metric spaces and generalized the concept of a metric space, replacing the set of real numbers, by an ordered Banach space. After the work of Huang and Zhang Huang & Zhang (2007), Azam et al. Azam *et al.* (2009) introduced the notion of cone rectangular metric spaces and proved fixed point result for Banach type contraction in cone rectangular space. Malhotra et al. Malhotra *et al.* (2013*b*) generalized the result of Azam et al. Azam *et al.* (2009) in ordered cone rectangular metric spaces and proved some fixed point results for ordered Reich type contractions.

Recently, Jachymski Jachymski (2007) improved the Banach contraction principle for mappings on a metric space endowed with a graph. Jachymski Jachymski (2007) showed that the results of Ran and Reurings Ran & Reurings (2004) and Edelstein Edelstein (1961) can be derived by the results of Jachymski (2007). The results of Jachymski Jachymski (2007) was generalized by several authors (see, for example, Bojor (2012); Chifu & Petrusel (2012); Samreen & Kamran (2013); Asl *et al.* (2013); Abbas & Nazir (2013) and the references therein).

The fixed point results in cone rectangular metric spaces (also in rectangular metric spaces) endowed with a graph are not considered yet. In this paper, we prove some fixed point theorems for Reich type contractions on the cone rectangular metric spaces endowed with a graph. Our results extend the result of Jachymski Jachymski (2007) and the result of Malhotra et al. Malhotra *et al.* (2013*b*) into the cone rectangular metric spaces endowed with a graph. Some examples are provided which illustrate the results.

2. Preliminaries

First we recall some definitions about the cone rectangular metric spaces and graphs.

Definition 2.1. Huang & Zhang (2007) Let E be a real Banach space and P be a subset of E. The set P is called a cone if:
(i) P is closed, nonempty and $P \neq \{\theta\}$, here θ is the zero vector of E;
(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.
Given a cone $P \subset E$, we define a partial ordering " $\preceq$ " with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. While $x \ll y$ if and only if $y - x \in P^0$, where P^0 denotes the interior of P.
Let P be a cone in a real Banach space E, then P is called normal, if there exist a constant $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 2.2. Huang & Zhang (2007) Let X be a nonempty set, E be a real Banach space. Suppose that the mapping $d : X \times X \to E$ satisfies:

(i) $\theta \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. In the following we always suppose that E is a real Banach space, P is a solid cone in E, i.e., $P^0 \neq \phi$ and " $\preceq$ " is partial ordering with respect to P.

For examples and basic properties of normal and non-normal cones and cone metric spaces we refer Huang & Zhang (2007) and Rzepecki (1980).
The following remark will be useful in sequel.

Remark. Jungck *et al.* (2009) Let P be a cone in a real Banach space E, and $a, b, c \in P$, then:

(a) If $a \preceq b$ and $b \ll c$ then $a \ll c$.
(b) If $a \ll b$ and $b \ll c$ then $a \ll c$.
(c) If $\theta \preceq u \ll c$ for each $c \in P^0$ then $u = \theta$.
(d) If $c \in P^0$ and $a_n \to \theta$ then there exist $n_0 \in \mathbb{N}$ such that, for all $n > n_0$ we have $a_n \ll c$.
(e) If $\theta \preceq a_n \preceq b_n$ for each n and $a_n \to a,\ b_n \to b$ then $a \preceq b$.
(f) If $a \preceq \lambda a$ where $0 \leq \lambda < 1$ then $a = \theta$.

Definition 2.3. Azam *et al.* (2009) Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$, satisfies:

(i) $\theta \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [rectangular property].

Then d is called a cone rectangular metric on X, and (X, d) is called a cone rectangular metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in (X, d)$. If for every $c \in E$, with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_n x_n = x$ or $x_n \to x$, as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $m \in \mathbb{N}$ we have $d(x_n, x_{n+m}) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone rectangular metric space. If the underlying cone is normal then (X, d) is called normal cone rectangular metric space.

The concept of cone metric space is more general than that of a metric space, because each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$.

Example 2.1. Let $X = \mathbb{N}, E = \mathbb{R}^2$, $\alpha, \beta > 0$ and $P = \{(x, y) : x, y \geq 0\}$.
Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y, \\ 3(\alpha,\beta) & \text{if } x \text{ and } y \text{ are in } \{1,2\}, x \neq y, \\ (\alpha,\beta) & \text{otherwise.} \end{cases}$$

Now (X, d) is a cone rectangular metric space but (X, d) is not a cone metric space because it lacks the triangular property:

$$3(\alpha,\beta) = d(1,2) \succ d(1,3) + d(3,2) = (\alpha,\beta) + (\alpha,\beta) = 2(\alpha,\beta),$$

as $3(\alpha,\beta) - 2(\alpha,\beta) = (\alpha,\beta) \in P$.

Note that in above example (X, d) is a normal cone rectangular metric space. Following is an example of non-normal cone rectangular metric space.

Example 2.2. Let $X = \mathbb{N}, E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Then this cone is not normal (see Rezapour & Hamlbarani (2008)).
Define $d : X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} \theta & \text{if } x = y, \\ 3e^t & \text{if } x, y \in \{1,2\}, x \neq y, \\ e^t & \text{otherwise.} \end{cases}$$

Then (X, d) is non-normal cone rectangular metric space but (X, d) is not a cone metric space because it lacks the triangular property.

Now we recall some basic notions from graph theory which we need subsequently (see also Jachymski (2007)).

Let X be a nonempty set and Δ denote the diagonal of the cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the rectangular distance between its vertices.

By G^{-1} we denote the conversion of a graph G, that is, the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \in X \times X \colon (y, x) \in E(G)\}.$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}). \tag{2.1}$$

If x and y are vertices in a graph G, then a path in G from x to y of length l is a sequence $(x_i)_{i=0}^{l}$ of $l+1$ vertices such that $x_0 = x, x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, l$. A graph G is called connected if there is a path between any two vertices of G. G is weakly connected if $\widetilde{G}$ is connected.

Throughout this paper we assume that X is nonempty set, G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.

Now we define the G-Reich contractions in a cone rectangular metric space.

Definition 2.4. Let (X, d) be a cone rectangular metric space endowed with a graph G. A mapping $T: X \to X$ is said to be a G-Reich contraction if:

(GR1) T is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

(GR2) there exist nonnegative constants λ, μ, δ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty) \tag{2.2}$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

An obvious consequence of symmetry of $d(\cdot, \cdot)$ and (2.1) is the following remark.

Remark. If T is a G-Reich contraction then it is both a G^{-1}-Reich contraction and a $\widetilde{G}$-Reich contraction.

Example 2.3. Any constant function $T: X \to X$ defined by $Tx = c$, where $c \in X$ is fixed, is a G-Reich contraction since $E(G)$ contains all the loops.

Example 2.4. Any Reich contraction on a X is a G_0-Reich contraction, where $E(G_0) = X \times X$.

Example 2.5. Let (X, d) be a cone rectangular metric space, $\sqsubseteq$ a partial order on X and $T: X \to X$ be an ordered Reich contraction (see Malhotra *et al.* (2013*b*)), that is, there exist nonnegative constants λ, μ, δ such that $\lambda + \mu + \delta < 1$ and

$$d(Tx, Ty) \preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty)$$

for all $x, y \in X$ with $x \sqsubseteq y$,. Then T is a G_1-Reich contraction, where $E(G_1) = \{(x, y) \in X \times X : x \sqsubseteq y\}$.

Definition 2.5. Let (X, d) be a cone rectangular metric space and $T: X \to X$ be a mapping. Then for $x_0 \in X$, a Picard sequence with initial value x_0 is defined by $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. The mapping T is called a Picard operator on X if T has a unique fixed point in X and for all $x_0 \in X$ the Picard sequence $\{x_n\}$ with initial value x_0 converges to the fixed point of T. The mapping T is called weakly Picard operator, if for any $x_0 \in X$, the limit of Picard sequence $\{x_n\}$ with initial value x_0, that is, $\lim_{n\to\infty} x_n$ exits (it may depend on x_0) and it is a fixed point of T.

Now we can state our main results.

3. Main results

Let (X, d) be a cone rectangular metric space, and G be a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$. The set of all fixed points of a self mapping T of X is denoted by Fix$T$, that is, Fix$T = \{x \in X : Tx = x\}$ and the set of all periodic points of T is denoted by $P(T)$, that is, $P(T) = \{x \in X : T^n x = x, \text{ for some } n \in \mathbb{N}\}$. Also we use the notation $X_T = \{x \in X : (x, Tx), (x, T^2x) \in E(G)\}$. (X, d) is said to have the property (P) if:

$$\text{whenever a sequence } \{x_n\} \text{ in } X \text{ converges to } x \text{ with } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N}, \text{ then there is a subsequence } \{x_{k_n}\} \text{ with } (x_{k_n}, x) \in E(G) \text{ for all } n \in \mathbb{N}. \tag{P}$$

Proposition 3.1. *Let (X, d) be a cone rectangular metric space endowed with a graph G and $T : X \to X$ be a G-Reich contraction. Then, if $x, y \in FixT$ are such $(x, y) \in E(G)$ then $x = y$.*

Proof. Let $x, y \in$ Fix$T$ and $(x, y) \in E(G)$, then by (GR2) we have

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\preceq \lambda d(x, y) + \mu d(x, Tx) + \delta d(y, Ty) \\ &= \lambda d(x, y) + \mu d(x, x) + \delta d(y, y) = \lambda d(x, y). \end{aligned}$$

As $\lambda < 1$, by (f) of Remark 2, we have $d(x, y) = \theta$, that is, $x = y$. □

Theorem 3.1. *Let (X, d) be a cone rectangular metric space endowed with a graph G. Let $T : X \to X$ be a G-Reich contraction. Then for every $x_0 \in X_T$ the Picard sequence $\{x_n\}$, is a Cauchy sequence.*

Proof. Let $x_0 \in X_T$ and define the iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. Since $x_0 \in X_T$ we have $(x_0, Tx_0) \in E(G)$ and T is a G-Reich contraction, by (GR1) we have $(Tx_0, T^2x_0) = (x_1, x_2) \in E(G)$. By induction we obtain $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$.

Now since $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$ by (GR2) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\preceq \lambda d(x_{n-1}, x_n) + \mu d(x_{n-1}, Tx_{n-1}) + \delta d(x_n, Tx_n) \\ &= \lambda d(x_{n-1}, x_n) + \mu d(x_{n-1}, x_n) + \delta d(x_n, x_{n+1}), \end{aligned}$$

that is,

$$d(x_n, x_{n+1}) \preceq \frac{\lambda + \mu}{1 - \delta} d(x_{n-1}, x_n) = \alpha d(x_{n-1}, x_n),$$

where $\alpha = \dfrac{\lambda + \mu}{1 - \delta} < 1$ (as $\lambda + \mu + \delta < 1$). Setting $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$, we obtain by induction that

$$d_n \preceq \alpha^n d_0 \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Note that, if $x_0 \in P(T)$ then there exists $k \in \mathbb{N}$ such that $T^k x_0 = x_k = x_0$ and by (3.1) we have

$$d_0 = d(x_0, x_1) = d(x_0, Tx_0) = d(x_k, Tx_k) = d(x_k, x_{k+1}) \preceq \alpha^k d(x_0, x_1) = \alpha^k d_0.$$

Since $\lambda \in [0,1)$ the above inequality yields a contradiction. Thus, we can assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.
As $x_0 \in X_T$ we have $(x_0, T^2x_0) = (x_0, x_2) \in E(G)$ and by (GR1) we obtain $(Tx_0, Tx_2) = (x_1, x_3) \in E(G)$. By induction we obtain $(x_n, x_{n+2}) \in E(G)$ for all $n \geq 0$. Therefore it follows from (GR2) that

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq d(Tx_{n-1}, Tx_{n+1}) \\ &\preceq \lambda d(x_{n-1}, x_{n+1}) + \mu d(x_{n-1}, Tx_{n-1}) + \delta d(x_{n+1}, Tx_{n+1}) \\ &\preceq \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+1})] + \mu d(x_{n-1}, x_n) \\ &\quad +\delta d(x_{n+1}, x_{n+2}), \end{aligned}$$

that is,

$$d(x_n, x_{n+2}) \preceq \frac{\lambda+\mu}{1-\lambda} d_{n-1} + \frac{\lambda+\delta}{1-\lambda} d_{n+1}$$

which together with (3.1) yields

$$\begin{aligned} d(x_n, x_{n+2}) &\preceq \frac{\lambda+\mu+[\lambda+\delta]\alpha^2}{1-\lambda}\alpha^{n-1}d_0 \\ &\preceq \frac{2\lambda+\mu+\delta}{1-\lambda}\alpha^{n-1}d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2}) \preceq \beta\alpha^{n-1}d_0, \tag{3.2}$$

where $\beta = \frac{2\lambda+\mu+\delta}{1-\lambda} \succeq 0$. We shall show that the sequence $\{x_n\}$ is a Cauchy sequence.
We consider the value of $d(x_n, x_{n+p})$ in two cases.
If p is odd, say $2m+1$, then using rectangular inequality and (3.1) we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\preceq d(x_{n+2m}, x_{n+2m+1}) + d(x_{n+2m-1}, x_{n+2m}) + d(x_n, x_{n+2m-1}) \\ &= d_{n+2m} + d_{n+2m-1} + d(x_n, x_{n+2m-1}) \\ &\preceq d_{n+2m} + d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_n \\ &\preceq \alpha^{n+2m}d_0 + \alpha^{n+2m-1}d_0 + \alpha^{n+2m-2}d_0 + \cdots + \alpha^n d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2m+1}) \preceq \frac{\alpha^n}{1-\alpha} d_0. \tag{3.3}$$

If p is even, say $2m$, then using rectangular inequality, (3.1) and (3.2) we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) &\preceq d(x_{n+2m-1}, x_{n+2m}) + d(x_{n+2m-1}, x_{n+2m-2}) + d(x_n, x_{n+2m-2}) \\ &= d_{n+2m-1} + d_{n+2m-2} + d(x_n, x_{n+2m-2}) \\ &\preceq d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_{n+2} + d(x_n, x_{n+2}) \\ &\preceq \alpha^{n+2m-1}d_0 + \alpha^{n+2m-2}d_0 + \alpha^{n+2m-3}d_0 + \cdots + \alpha^{n+2}d_0 + \beta\alpha^{n-1}d_0, \end{aligned}$$

that is,

$$d(x_n, x_{n+2m}) \preceq \frac{\alpha^n}{1-\alpha} d_0 + \beta\alpha^{n-1}d_0. \tag{3.4}$$

Since $\beta \geq 0$ and $\alpha < 1$, we have $\frac{\alpha^n}{1-\alpha}d_0, \beta\alpha^{n-1}d_0 \to \theta$ as $n \to \infty$ so it follows from (3.3), (3.4) and (a), (d) of Remark 2 that: for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+p}) \ll c \quad \text{for all} \quad p \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. □

Theorem 3.2. *Let (X, d) be a complete cone rectangular metric space endowed with a graph G and has the property (P). Let $T : X \to X$ be a G-Reich contraction such that $X_T \neq \emptyset$, then T is a weakly Picard operator.*

Proof. If $X_T \neq \emptyset$ then let $x_0 \in X_T$. By Theorem 3.1, the Picard sequence $\{x_n\}$, where $x_n = T^{n-1}x_0$ for all $n \in \mathbb{N}$, is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that

$$x_n \to u \text{ as } n \to \infty. \tag{3.5}$$

We shall show that u is a fixed point of T. By Theorem 3.1 we have $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$, $d_n \preceq d(x_n, x_{n+1}) \preceq \alpha^n d_0$, where $\alpha = \frac{\lambda+\mu}{1-\delta} < 1$ and by the property (P) there exists a subsequence $\{x_{k_n}\}$ such that $(x_{k_n}, u) \in E(G)$ for all $n \in \mathbb{N}$. Also, we can assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. So, using (2.2) we have

$$\begin{aligned}
d(u, Tu) &\preceq d(u, x_{k_n}) + d(x_{k_n}, x_{k_n+1}) + d(x_{k_n+1}, Tu)\\
&= d(u, x_{k_n}) + d_{k_n} + d(Tx_{k_n}, Tu)\\
&\preceq d(u, x_{k_n}) + \alpha^{k_n}d_0 + \lambda d(x_{k_n}, u) + \mu d(x_{k_n}, Tx_{k_n}) + \delta d(u, Tu)\\
&\preceq (1+\lambda)d(u, x_{k_n}) + (1+\mu)\alpha^{k_n}d_0 + \delta d(u, Tu),
\end{aligned}$$

that is,

$$d(u, Tu) \preceq \frac{1+\lambda}{1-\delta}d(x_{k_n}, u) + \frac{1+\mu}{1-\delta}\alpha^{k_n}d_0 \tag{3.6}$$

Since $\alpha^{k_n}d_0 \to \theta$, $x_n \to u$ as $n \to \infty$ we can choose $n_0 \in \mathbb{N}$ such that, for every $c \in E$ with $\theta \ll c$ we have $d(x_{k_n}, u) \ll \dfrac{1-\delta}{2(1+\lambda)}c$ and $\alpha^{k_n}d_0 \ll \dfrac{1-\delta}{2(1+\mu)}c$ for all $n > n_0$. Therefore, it follows from (3.6) that: for every $c \in E$ with $\theta \ll c$ we have

$$d(u, Tu) \ll c \quad \text{for all} \quad n > n_0.$$

So, by (c) of Remark 2, we have $d(u, Tu) = \theta$, that is, $Tu = u$ theerefore $u \in \text{Fix}T$. Thus T is a weakly Picard operator. □

In the above theorem the mapping T is not necessarily a Picard operator. Indeed, such mapping T may has infinitely many fixed points. Following example verifies this fact.

Example 3.1. Let $X = \mathbb{N} = \bigcup\limits_{k \in \mathbb{N}_0} N_k$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $N_k = \left\{2^k(2n-1) : n \in \mathbb{N}\right\}$ for all $k \in \mathbb{N}_0$. Let $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Let $d : X \times X \to E$ be defined by

$$d(x, y) = \begin{cases} \theta & \text{if } x = y,\\ 3e^t & \text{if } x, y \in \{1, 2\}, x \neq y,\\ e^t & \text{otherwise.} \end{cases}$$

Then (X, d) is a cone rectangular metric space endowed with graph G, where

$$E(G) = \Delta \bigcup_{k\in\mathbb{N}_0\setminus\{1\}} (N_k \times N_k) \bigcup \{(1, x) \colon x \in N_1\}.$$

Note that (X, d) is not a cone metric space. Define a mapping $T \colon X \to X$ by

$$Tx = \begin{cases} 2^k, & \text{if } x \in N_k, k \in \mathbb{N}_0 \setminus \{1\}; \\ 6, & \text{if } x = 2; \\ 1, & \text{if } x \in N_1 \setminus \{2\}. \end{cases}$$

Then it is easy to see that T is a G-Reich contraction with $\lambda \in [1/3, 1), \mu = \delta = 0$. All the conditions of Theorem 3.2 are satisfied and T has infinitely many fixed points, precisely $\mathrm{Fix}T = \left\{2^k \colon k \in \mathbb{N}_0 \setminus \{1\}\right\}$, therefore T is not a Picard operator but weakly Picard operator. Note that, if a Reich contraction on a cone rectangular metric space has a fixed point then it is unique therefore T is not a Reich contraction in (X, d) since $\mathrm{Fix}T$ is not singleton.

Remark. Unlike from Reich contraction, the above example shows that there may be more than one fixed points of a G-Reich contraction in a cone rectangular metric space and therefore a G-Reich contraction in a cone rectangular space need not be a Picard operator.

In following theorem we give a necessary and sufficient condition for T to be a Picard operator.

Theorem 3.3. *Let (X, d) be a complete cone rectangular metric space endowed with a graph G and has the property (P). Let $T \colon X \to X$ be a G-Reich contraction such that $X_T \neq \emptyset$, then T is a weakly Picard operator. Furthermore, the subgraph G_{Fix} defined by $V(G_{\mathrm{Fix}}) = \mathrm{Fix}T$ is weakly connected if and only if T is a Picard operator.*

Proof. The existence of fixed point follows from Theorem 3.2. Let $u, v \in \mathrm{Fix}T$, then since G_{Fix} is weakly connected there exists a path $(x_i)_{i=0}^{l}$ in G_{Fix} from u to v, that is, $x_0 = u, x_l = v$ and $(x_{i-1}, x_i) \in E(G_{\mathrm{Fix}})$ for $i = 1, 2, \ldots, l$. Therefore by Proposition 3.1 and Remark 2 we obtain $u = v$. Thus, fixed point is unique and T is a Picard operator. □

Remark. In Jachymski (2007), for T to be a Picard operator Jachymski assumed that G must be weakly connected. From the above theorem it is clear that for T to be a Picard operator it is sufficient to take that $\mathrm{Fix}T$ is weakly connected. Next example will illustrate this fact.

Example 3.2. Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$, $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Let $d \colon X \times X \to E$ be defined by

$$d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{3}{10}e^t, \qquad d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{1}{5}e^t,$$

$$d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{3}\right) = \frac{3}{5}e^t, \qquad d(x, x) = \theta = 0 \quad \text{for all} \quad x \in X,$$

$$d\left(1, \frac{1}{n}\right) = \frac{n-1}{n}e^t \text{ for } n = 2, 3, 4, 5, \quad d(x, y) = d(y, x) \quad \text{for all} \quad x, y \in X,$$

Then (X, d) is a cone rectangular metric space endowed with graph G, where

$$E(G) = \Delta \cup \left\{ \left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{3}\right) \right\}.$$

Note that (X, d) is not a cone metric space. Define $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x = \frac{1}{2}, \frac{1}{5}; \\ \frac{1}{5}, & \text{if } x = \frac{1}{3}; \\ 1, & \text{if } x = \frac{1}{4}; \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Then T is a G-Reich contraction with $\lambda \in \left[\frac{2}{3}, 1\right), \mu = \delta = 0$. All the conditions of Theorem 3.3 are satisfied and T is a Picard operator and Fix$T$ $= \left\{\frac{1}{2}\right\}$. Note that the graph G is not weakly connected. Indeed, there is no path from 1 to $\frac{1}{n}$ or form $\frac{1}{n}$ to 1 for all $n = 2, 3, 4, 5$. Also, one can see that T is neither a Reich contraction in cone rectangular metric space (X, d) nor a G-Reich contraction with respect to the usual metric.

With suitable values of constants λ, μ and δ we obtain the following corollaries.

Corollary 3.1. *Let (X, d) be a complete cone rectangular metric space endowed with a graph G and has the property (P). Let $T : X \to X$ be a G-contraction, that is,*

(G1) *T is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
(G2) *there exists $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \preceq \lambda d(x, y) \text{ for all } x, y \in X \text{ with } (x, y) \in E(G).$$

Then, if $X_T \neq \emptyset$ then T is a weakly Picard operator. Furthermore, the subgraph G_{Fix} defined by $V(G_{Fix}) = FixT$ is weakly connected if and only if T is a Picard operator.

Corollary 3.2. *Let (X, d) be a complete cone rectangular metric space endowed with a graph G and has the property (P). Let $T : X \to X$ be a G-Kannan contraction, that is,*

(GK1) *T is edge preserving, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
(GK2) *there exists $\lambda \in [0, 1/2)$ such that*

$$d(Tx, Ty) \preceq \lambda[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X \text{ with } (x, y) \in E(G).$$

Then, if $X_T \neq \emptyset$ then T is a weakly Picard operator. Furthermore, the subgraph G_{Fix} defined by $V(G_{Fix}) = FixT$ is weakly connected if and only if T is a Picard operator.

Following corollary is a fixed point result for an ordered Reich contraction (see Malhotra *et al.* (2013*b*)) and a generalization of result of Ran and Reurings Ran & Reurings (2004) in cone rectangular metric spaces.

Corollary 3.3. *Let (X, d) be a complete cone rectangular metric space endowed with a partial order $\sqsubseteq$ and $T: X \to X$ be a mapping. Suppose the following conditions hold:*

(A) *T is an ordered Reich contraction;*
(B) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;*
(C) *T is nondecreasing with respect to $\sqsubseteq$;*
(D) *if $\{x_n\}$ is a nondecreasing sequence in X and converging to some z, then $x_n \sqsubseteq z$.*

Then T is a weakly Picard operator. Furthermore, $FixT$ is well ordered (that is, all the elements of $FixT$ are comparable) if and only if T is a Picard operator.

Proof. Let G be a graph defined by $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \sqsubseteq y\}$. Then by conditions (A) and (C), T is a G-Reich contraction and by condition (B) we have $X_T \neq \emptyset$. Also by condition (D) we see that property (P) is satisfied. Now proof follows from Theorem 3.3. □

Conclusion. In the present paper we have proved the existence and uniqueness of fixed point theorems for a G-Reich contraction in cone rectangular metric spaces endowed with a graph. We note that the results of this paper generalize the ordered version of theorem of Reich (see Reich (1971) and Malhotra *et al.* (2013*b*)). Note that, in usual metric spaces the fixed point theorem for G-contractions generalizes and unifies the ordered version as well as the cyclic version of corresponding fixed point theorems (see Kirk *et al.* (2003) and Kamran *et al.* (2013)). We conclude with an open problem that: is it possible to prove the cyclic version of the result of Reich in cone rectangular metric spaces or rectangular metric spaces?

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