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Mean-Variance Portfolio Selection with Inflation Hedging Strategy: a Case of a Defined Contributory Pension Scheme

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Abstract

In this paper, we consider a mean-variance portfolio selection problem with inflation hedging strategy for a defined contributory pension scheme. We establish the optimal wealth which involves a cash account and two risky assets for the pension plan member (PPM). The efficient frontier is obtained for the three asset classes which gives the PPM the opportunity to decide his or her own risk and wealth. It was found that inflation-linked bond is a suitable asset for hedging inflation risks in an investment portfolio.

Keywords: Mean-variance, inflation hedging, defined contribution, efficient frontier, optimal utility, expected wealth, inflation risks fighter.
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1. Introduction

A mean-variance optimization is a quantitative method that is adopted by fund managers, consultants and investment advisors to construct portfolios for the investors. When the market is less volatile, mean-variance model seems to be a better and more reasonable way of determining portfolio selection problem. One of the aims of mean-variance optimization is to find portfolio that optimally diversify risk without reducing the expected return and to enhance portfolio construction strategy. This method is based on the pioneering work of Markowitz (Markowitz, 1952, 1959). The optimal investment allocation strategy can be found by solving a mean and variance optimization problem.

There are extensive literature that exist on the area of accumulation phase of a DC pension plan and optimal investment strategies. For some of the literature, see for instance, (Cairns *et al.*,

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2006), (Di Giacinto *et al.*, 2011), (Haberman & Vigna, 2002), (Vigna, 2010), (Gao, 2008), (Nkeki, 2011), (Nkeki & Nwozo, 2012).

In the context of DC pension plans, the problem of finding the optimal investment strategy involving a riskiness asset and two distinct risky assets, and inflation hedging strategy under mean-variance efficient approach has not been reported in published articles. Bjarne Hϕjgaard and Elena Vigna (Hϕjgaard & Vigna, 2007) and Vigna (Vigna, 2010)assumed a constant flow of contributions into the pension scheme. This paper follows the same assumption.

In the literature, the problem of determining the minimum variance on trading strategy in continuous-time framework has been studied by Richardson (Richardson, 1989) via the Martingale approach. (Li & Ng, 2000) solved a mean-variance optimization problem in a discrete-time multi-period framework. (Zhou & Li, 2000) considered a mean-variance in a continuous-time framework. They shown the possibility of transforming the difficult problem of mean-variance optimization problem into a tractable one, by embedding the original problem into a stochastic linear-quadratic control problem, that can be solved using standard methods. These approaches have been extended and used by many in the financial literature, see for instance, Vigna (2010), (Bielecki *et al.*, 2005), (Hϕjgaard & Vigna, 2007), (Chiu & Li, 2006), (Josa-Fombellida & Rincn-Zapatero, 2008). In this paper, we study a mean-variance approach (MVA) to portfolio selection problem with inflation protection strategy in accumulation phase of a DC pension scheme. Our result shows that inflation-linked bond can be used to hedge inflation risk that is associated with the PPM's wealth. We found that our optimal portfolio is efficient in the mean-variance approach.

The remainder of this paper is organized as follows. In section 2, we present the financial market model problem. In section 3, we present the optimal portfolio and optimal expected terminal wealth of the PPM. The efficient frontier is presented in section 4. In section 5, some numerical examples were presented. Finally, section 6 concludes the paper.

2. The Problem

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Let $\mathbf{F}(\mathcal{F}) = \{\mathcal{F}_t : t \in [0,T]\}$, where $\mathcal{F}_t = \sigma(S(s), I(s) : s \leq t)$, where $S(t)$ is stock price process at time $s \leq t$, $I(t)$ is the inflation index at time $s \leq t$. The Brownian motions $W(t) = (W^I(t), W^S(t))'$, $0 \leq t \leq T$ is a 2-dimensional process, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}(\mathcal{F}), \mathbf{P})$, where $\mathbf{P}$ is the real world probability measure and σ^S and σ_I are the volatility vectors of stock and volatility of the inflation-linked bond with respect to changes in $W^S(t)$ and $W^I(t)$, respectively. μ is the appreciation rate for stock. Moreover, σ^S and σ_I are the volatilities for the stock and inflation-linked bond respectively, referred to as the coefficients of the market and are progressively measurable with respect to the filtration $\mathcal{F}$.

We assume that the investor faces a market that is characterized by a risk-free asset (cash account) and two risky assets, all of whom are tradeable. In this paper, we allow the stock price to be correlated to inflation. The dynamics of the underlying assets are given by (2.1) to (2.3)

$$dC(t) = rC(t)dt, C(0) = 1 \tag{2.1}$$

$$dS(t) = \mu S(t)dt + \sigma_1^S S(t)dW^I(t) + \sigma_2^S S(t)dW^S(t), S(0) = s_0 > 0 \tag{2.2}$$

$$dF(t, I(t)) = (r + \sigma_1 \theta^I)F(t, I(t))dt + \sigma_I F(t, I(t))dW(t), F(0) = F_0 > 0 \tag{2.3}$$

where, r is the nominal interest rate, θ^I is the price of inflation risk, $C(t)$ is the price process of the cash account at time t, $S(t)$ is stock price process at time t, $I(t)$ is the inflation index at time t and has the dynamics: $dI(t) = E(q)I(t)dt + \sigma_I I(t)dW(t)$, where $E(q)$ is the expected rate of inflation, which is the difference between nominal interest rate, r and real interest rate R (i.e. $E(q) = r - R$). $F(t, I(t))$ is the inflation-indexed bond price process at time t and $\sigma_I = (\sigma_1, 0)$.
Then, the volatility matrix

$$\Sigma := \begin{pmatrix} \sigma_1 & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix} \tag{2.4}$$

corresponding to the two risky assets and satisfies $det(\Sigma) = \sigma_1\sigma_2^S \neq 0$. Therefore, the market is complete and there exists a unique market price θ satisfying

$$\theta := \begin{pmatrix} \theta^I \\ \theta^S \end{pmatrix} = \begin{pmatrix} \theta^I \\ \frac{\mu - r - \theta^I\sigma_1^S}{\sigma_2^S} \end{pmatrix} \tag{2.5}$$

where θ^S is the market price of stock risks and θ^I is the market price of inflation risks (MPIR).

3. The Wealth Process

Let $X(t)$ be the wealth process at time t, where $\Delta(t) = (\Delta^I(t), \Delta^S(t))$ is the portfolio process at time t and $\Delta^I(t)$ is the proportion of wealth invested in the inflation-linked bond at time t and $\Delta^S(t)$ is the proportion of wealth invested in stock at time t. Then, $\Delta_0(t) = 1 - \Delta^I(t) - \Delta^S(t)$ is the proportion of wealth invested in cash account at time t. Let c be the contribution rate of PPM.

Definition 3.1. The portfolio process Δ is said to be self-financing if the corresponding wealth process $X(t)$, $t \in [0, T]$, satisfies

$$\begin{aligned} dX(t) &= \Delta^S(t)X(t)\frac{dS(t)}{S(t)} + \Delta^I(t)X(t)\frac{dF(t, I(t))}{F(t, I(t))} + (1 - \Delta^S(t) - \Delta^I(t))X(t)\frac{dC(t)}{C(t)} + cdt, \\ X(0) &= x_0. \end{aligned} \tag{3.1}$$

(3.1) can be re-written in compact form as follows:

$$\begin{aligned} dX(t) &= (X(t)(r + \Delta(t)A) + c)dt + X(t)(\Sigma\Delta'(t))'dW(t)], \\ X(0) &= x_0, \end{aligned} \tag{3.2}$$

where, $A = (\sigma_1\theta^I, \mu - r)'$ and $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ \sigma_1^S & \sigma_2^S \end{pmatrix}$. The amount x_0 is the initial fund paid in by the PPM. This amount can be null, if the PPM has just joined the pension scheme without any transfer fund. The PPM enters the plan at initial time, 0 and contributes for T years, thereafter he or she retires and withdraws all his or her entitlement (or converts it into annuity). The aim of the PPM is pursued the two conflicting objectives of maximum expected terminal wealth together with minimum variance of the terminal wealth. PPM seeks to minimize the vector

$$[-E(X(T)), Var(X(T))].$$

Definition 3.2. (Hϕjgaard & Vigna, 2007) The mean-variance optimization problem is defined as

$$\begin{aligned} &Minimize(\Psi_1(\Delta(\cdot)); \Psi_2(\Delta(\cdot))) \equiv (-E(X(T)), Var(X(T))) \\ &\text{subject to} \begin{cases} \Delta(\cdot) \;\; admissible \\ X(\cdot), \Delta(\cdot) \;\; satisfy \;\; (3.2). \end{cases} \end{aligned} \tag{3.3}$$

An admissible strategy $\Delta^*(\cdot)$ is called an efficient strategy if there exists no admissible strategy $\Delta(\cdot)$ such that

$$\Psi_1(\Delta(\cdot)) \leq \Psi_1(\Delta^*(\cdot)), \Psi_2(\Delta(\cdot)) \leq \Psi_2(\Delta^*(\cdot)) \tag{3.4}$$

and at least one of the inequalities holds strictly. In this case, the point $(\Psi_1(\Delta(\cdot)), \Psi 2(\Delta(\cdot))) \in \mathbf{R}^2$ is called an efficient point and the set of all efficient points is called the efficient frontier.

Hϕjgaard and Vigna (Hϕjgaard & Vigna, 2007) established that solving (3.3) will address the following problem

$$\min_{\Delta(\cdot)}[-E(X(T)) + \delta Var(X(T))], \tag{3.5}$$

where $\delta > 0$. (3.5) is not easy to tackle with standard stochastic control techniques, see (Hϕjgaard & Vigna, 2007). Zhou and Li (Zhou & Li, 2000) and Li and Ng (Li & Ng, 2000) shown that it is possible to transform (3.5) into a tractable one. They were able to show that (3.5) is equivalent to the following problem

$$\min_{\Delta(\cdot)} E[\delta X(T)^2 + \omega X(T)], \tag{3.6}$$

which is a linear-quadratic control problem. Zhou and Li (Zhou & Li, 2000) and Li and Ng (Li & Ng, 2000) further show that if $\Delta(\cdot)$ is a solution of (3.5), then it is a solution of (3.6) with

$$\omega^* = 1 + 2\delta E(X^*(T)). \tag{3.7}$$

Our aim now is to solve

$$\begin{aligned} &Minimize(\Psi(\Delta(\cdot)), \delta, \omega) \equiv E[\delta X(T)^2 + \omega X(T)] \\ &\text{subject to} \begin{cases} \Delta(\cdot) \;\; \text{admissible} \\ X(\cdot), \Delta(\cdot) \;\; \text{satisfy } (3.2) \end{cases} \end{aligned} \tag{3.8}$$

3.1. Optimal Portfolio Process

We now follow the approach presented by Zhou and Li (Zhou & Li, 2000) and Hϕjgaard and Vigna (Hϕjgaard & Vigna, 2007). Let $\eta = \frac{\omega^*}{2\delta}$ and $\Phi(t) = X(t) - \eta$. It therefore resulted that our problem is equivalent to solving

$$\min_{\Delta(\cdot)} E\left[\frac{1}{2}\delta\Phi(T)^2\right] = \min_{\Delta(\cdot)} \Psi(\Delta(\cdot); \delta), \tag{3.9}$$

where $\Phi(t)$ satisfies the stochastic differential equation

$$\begin{aligned} &d\Phi(t) = ((\Phi(t) + \eta)(\Delta(t)A + r) + c)dt + (\Phi(t) + \eta)(\Sigma\Delta'(t))' dW(t), \\ &\Phi(0) = x_0 - \eta. \end{aligned} \tag{3.10}$$

We now adopt the dynamic programming approach to solve the standard stochastic optimal control problem (3.9) and (3.10). Let define the value function

$$U(t,\Phi) = \inf_{\Delta(\cdot)} E_{t,\Phi}\left[\frac{1}{2}\delta\Phi(T)^2\right] = \min_{\Delta(\cdot)} \Psi(\Delta(\cdot);\delta). \tag{3.11}$$

Then U which is assume to be a convex utility function of Φ, satisfies the Hamilton-Jacobi-Bellmann (HJB) equation

$$\begin{array}{l} \inf_{\Delta\in\mathbf{R}}\left\{U_t + ((\Phi+\eta)(\Delta(t)A + r) + c)U_\Phi + \frac{1}{2}(\Phi+\eta)^2\Sigma\Delta(t)\Sigma'\Delta'(t)U_{\Phi\Phi}\right\} = 0, \\ U(T,\Phi) = \frac{1}{2}\delta\Phi^2. \end{array} \tag{3.12}$$

Let $\mathcal{H}$ be the Hamiltonian such that

$$\mathcal{H} = ((\Phi+\eta)(\Delta(t)A + r) + c)U_\Phi + \frac{1}{2}(\Phi+\eta)^2\Sigma\Delta(t)\Sigma'\Delta'(t)U_{\Phi\Phi}. \tag{3.13}$$

Then,

$$\frac{\partial\mathcal{H}}{\partial\Delta(t)} = (\Phi+\eta)AU_\Phi + (\Phi+\eta)^2\Sigma\Sigma'\Delta'(t)U_{\Phi\Phi} = 0$$

Therefore,

$$\Delta^{'*}(t) = -\frac{(\Sigma\Sigma')^{-1}AU_\Phi}{(\Phi+\eta)U_{\Phi\Phi}}. \tag{3.14}$$

Substituting (3.14) into (3.12), we obtain the following non-linear partial differential equation for the value function

$$U_t + (r(\Phi+\eta)+c)U_\Phi - \frac{1}{2}M\frac{U_\Phi^2}{U_{\Phi\Phi}} = 0, \tag{3.15}$$

where, $M = [(\Sigma\Sigma')^{-1}A]'A$. Let assume the solution of the form, see Hϕjgaard and Vigna (2007) and Vigna (2010),

$$U(t,\Phi) = P(t)\Phi^2 + Q(t)\Phi + R(t). \tag{3.16}$$

Finding the partial derivatives of U in (3.16) with respect to U_t, U_Φ and $U_{\Phi\Phi}$ and then substitute into (3.15), we have the following system of ordinary differential equations (ODEs):

$$\left.\begin{array}{l} P'(t) + (2r - M)P(t) = 0 \\ Q'(t) + 2(r\eta + c)P(t) + (r - M)Q(t) = 0 \\ R'(t) + (r + r\eta + c)Q(t) - \frac{1}{4}M\frac{Q(t)^2}{P(t)} = 0 \end{array}\right\} \tag{3.17}$$

with boundary conditions

$$P(T) = \frac{1}{2}\delta, Q(T) = 0, R(T) = 0.$$

Solving the system of ODEs (3.17) using the boundary conditions $P(T) = \frac{1}{2}\delta, Q(T) = 0, R(T) = 0,$ we have

$$\left.\begin{array}{l} P(t) = \frac{\delta}{2}\exp[(2r + M)(T - t) \\ Q(t) = \frac{\delta(c+r\eta)}{2M+r}\exp[-(M - r)(T - t)](\exp[(2M + r)(T - t)] - 1) \\ R(t) = \int_t^T \left((r + r\eta + c)Q(s) - \frac{1}{4}M\frac{Q(s)^2}{P(s)}\right)ds \end{array}\right\} \tag{3.18}$$

Hence, replacing the partial derivatives of U in (3.14), the optimal fraction of portfolio to be invested in the two risky assets at time time t, becomes

$$\Delta^{'*}(t) = -\frac{(\Sigma\Sigma')^{-1}A}{\Phi+\eta}G_{\Delta}(t), \tag{3.19}$$

where,

$$G_{\Delta}(t) = \Phi+\eta - \frac{\eta(2M+r)-(r\eta+c)(1-\exp[-(2M+r)(T-t)])}{2M+r}.$$

Now, replacing $\Phi+\eta$ with x in (3.19), we have

$$\Delta^{'*}(t) = -\frac{(\Sigma\Sigma')^{-1}A}{x}\left[x - \frac{\eta(2M+r)-(r\eta+c)(1-\exp[-(2M+r)(T-t)])}{2M+r}\right] \tag{3.20}$$

Simplifying (3.20), we have

$$\Delta^{'*}(t) = -\frac{(\Sigma\Sigma')^{-1}A}{x}\bar{G}(t), \tag{3.21}$$

where,

$$\bar{G}(t) = x - \frac{\eta(2M+\exp[-(2M+r)(T-t)])}{(2M+r)} + \frac{c(1-\exp[-(2M+r)(T-t)])}{2M+r}.$$

3.2. Expected Optimal Wealth

In this subsection, we determine the expected wealth that will accrued to the PPM at the final time horizon. We also consider in this subsection, the second moment of the expected final wealth of the PPM. These will enable us to established the efficient frontier in the next section.

Substituting (3.20) into (3.2), we have that the evolution of wealth of the PPM under optimal control $X^*(t)$ is obtained as follows:

$$\begin{aligned} dX^*(t) &= \{(r-M)X^*(t) + \frac{\eta M(2M-\exp[-(2M+r)(T-t)])}{2M+r} \\ &+ \frac{cM(1-\exp[-(2M+r)(T-t)])}{2M+r} + c\}dt - \Sigma^{-1}A\{X^*(t) \\ &- \frac{\eta(2M+\exp[-(2M+r)(T-t)])}{2M+r} + \frac{c(1-\exp[-(2M+r)(T-t)])}{2M+r}\}dW(t). \end{aligned} \tag{3.22}$$

Then, applying Itô lemma to (3.22), we obtain the SDE that satisfies the evolution of $X^{*2}(t)$:

$$\begin{aligned} dX^{*2}(t) &= \{(2r-M)X^{*2}(t) + 2cX^*(t) + M[\frac{\eta(2M+\exp[-(2M+r)(T-t)])}{2M+r} \\ &+ \frac{c(1-\exp[-(2M+r)(T-t)])}{2M+r}]^2\}dt - 2\Sigma^{-1}A\{X^{*2}(t) \\ &- \frac{\eta X^*(t)(2M-\exp[-(2M+r)(T-t)])}{2M+r} \\ &+ \frac{cX^*(t)(1-\exp[-(2M+r)(T-t)])}{2M+r}\}dW(t) \end{aligned} \tag{3.23}$$

Taking the mathematical expectation on both sides of (3.22) and (3.23), we have the following expected value of the optimal wealth and the expected value of its square:

$$\begin{aligned} dE(X^*(t)) &= E[(r-M)X^*(t) + \frac{\eta M(2M - \exp[-(2M+r)(T-t)])}{2M+r} \\ &+ \frac{cM(1-\exp[-(2M+r)(T-t)])}{2M+r} + c]dt, \\ E(X(0)) &= x_0. \end{aligned} \tag{3.24}$$

$$\begin{aligned} &dE(X^{*2}(t)) = E[(2r-M)X^{*2}(t) + 2cX^*(t) \\ &+ M\left(\frac{\eta(2M+\exp[-(2M+r)(T-t)])}{2M+r} + \frac{c(1-\exp[-(2M+r)(T-t)])}{2M+r}\right)^2]dt, \\ &E(X^{*2}(t)) = x_0^2. \end{aligned} \tag{3.25}$$

Solving (3.24) and (3.25), we have the following:

$$\begin{aligned} E(X^*(t)) &= x_0 \exp[-(M-r)t] + \frac{2M^2\eta}{(M-r)(2M+r)}(1+\exp[-(M-r)t]) \\ &+ \frac{c}{3(2M+r)}\exp[-2MT - r(T-t)](\exp[-Mt] - \exp[2Mt]) \\ &+ \frac{c(3M+r)}{(M-r)(2M+r)}(1-\exp[-(M-r)t]) \\ &- \frac{\eta \exp[-Mt - r(T-t)]}{3(2M+r)}(\exp[-3M(T-t)] - \exp[-2MT]), \end{aligned} \tag{3.26}$$

$$\begin{aligned} &E(X^{*2}(t)) = x_0^2 \exp[-(M-2r)t] + \frac{c^2\exp[-2r(T-t) - M(4T+t)](\exp[5Mt]-1)}{5(2M+r)^2} \\ &+ \frac{2c^2(3M+r)\exp[-(M-r)t]}{r(M-r)(2M+r)} - \frac{\eta^2\exp[-2r(T-t) - M(4T+t)]}{5(2M+r)^2} \\ &+ \frac{12cM^2\eta}{(2M+r)^2(M^2-3Mr+2r^2)} - \frac{c^2(4M+r)(3M+2r)\exp[-Mt+2rt]}{r(M-2r)(2M+r)^2} + D_1(t) \\ &+ \frac{c^2(13M^2+9Mr+2r^2)}{(2M+r)^2(M^2-3Mr+2r^2)} + D_2(t) - \frac{2c\eta}{5(2M+r)^2} \\ &(\exp[-2r(T-t) - M(4T+t)] - \exp[-2(2M+r)(T-t)]) \\ &- \frac{2c\eta\exp[-(2MT+Mt) - r(T-t)]}{r(6M+3r)} - \frac{2c\eta(M(5+6M)+r)\exp[-(2M+r)(T-t)]}{3(3M-r)(2M+r)^2} \\ &+ \frac{4c(M(M+r+Mr))\eta\exp[-(M-r)t - r(T-t) - 2MT]}{r(3M-r)(2M+r)^2} + \\ &\frac{2cx_0}{r}\exp[-(M-r)t](1-\exp[rt]) \\ &- \frac{2c\eta\exp[-(M-r)t]}{r(r^2+Mr-2M^2)} - \frac{4M^2\eta}{(M-2r)(2M+r)^2}\left(M\eta + \frac{2c(M+r)}{r}\right)\exp[-(M-2r)t] \end{aligned} \tag{3.27}$$

where,

$$D_1(t) = \frac{2c^2 \exp[-r(T-t) - M(2T+t)]}{3r(3M-r)(2M+r)^2} \times (6M^2 + Mr(1 + 5\exp[3Mt]) - r^2(1-\exp[3Mt]) - 6M(M-r)\exp[rt]), \tag{3.28}$$

$$D_2(t) = -\frac{4M^2\eta^2 \exp[-r(T-t) - M(2T+t)](\exp[3Mt] - \exp[rt])}{(3M-r)(2M+r)^2} + \frac{\eta^2(\exp[-2(2M+r)(T-t)] + \frac{20M^3}{M-2r})}{5(2M+r)^2} \tag{3.29}$$

At terminal time, that is, at $t = T$, we have:

$$E(X^*(T)) = x_0 \exp[-(M-r)T] + \frac{2M^2\eta}{(M-r)(2M+r)}(1 + \exp[-(M-r)T]) + \frac{c}{3(2M+r)}(\exp[-3MT] - 1) + \frac{c(3M+r)}{(M-r)(2M+r)}(1 - \exp[-(M-r)T]) - \frac{\eta \exp[-MT]}{3(2M+r)}(1 - \exp[-2MT]), \tag{3.30}$$

$$\begin{aligned} E(X^{*2}(T)) &= x_0^2 \exp[-(M-2r)T] + \frac{4M^3\eta^2}{(M-2r)(2M+r)^2}(1 - \exp[-(M-2r)T]) \\ &+ \frac{(\eta+c)^2}{5(2M+r)^2}(1 - \exp[-5MT]) - \frac{4M^2\eta^2}{(3M-r)(2M+r)^2}(1 - \exp[-3MT + rT]) \\ &- \left(\frac{c^2(4M+r)(3M+2r)}{r(M-2r)(2M+r)^2} + \frac{8cM^2\eta(M+r)}{r(M-2r)(2M+r)^2}\right)\exp[-(M-2r)T] \\ &- \frac{2c}{r}\left(x_0(1 + \exp[rT]) + \frac{\eta}{r^2 + Mr - 2M^2} - \frac{c(3M+r)}{r(M-r)(2M+r)}\right)\exp[-(M-r)T] \\ &+ \frac{4c^2M^2 \exp[-3MT](1 - (1 - \frac{r}{M} + \frac{r\eta}{c})\exp[rT])}{r(3M-r)(2M+r)^2} + \frac{12\eta M^2 c}{(2M+r)^2} \\ &\left(\frac{1}{(M-r)(M-2r)} - \frac{1}{3(3M-r)}\right) + \frac{c^2(13M^2 + 9Mr + 2r^2)}{(M-r)(M-2r)(2M+r)^2} \\ &+ \frac{2c(c(M-r)\exp[-3MT] + 5M(c-\eta) + r(c-\eta))}{3(3M-r)(2M+r)^2} \\ &- \frac{2c\eta \exp[-3MT]}{2M+r}\left(\frac{1}{3r} - \frac{2M}{(3M-r)(2M+r)}\left(\frac{M}{r} + 1\right)\exp[rT]\right). \end{aligned} \tag{3.31}$$

Since $\eta = \frac{\omega^*}{2\delta}$ and ω^* is as defined in (3.7), we have that

$$\eta = \frac{3(M-r)(2M+r)}{3(M-r)(2M+r) - 6M^2(1 + \exp[-(M-r)T]) + (M+r)\exp[-MT](1 - \exp[-2MT])} \times \left(\frac{1}{2\delta} + x_0 \exp[-(M-r)T] + \frac{c(\exp[-3MT] - 1)}{3(2M+r)} + \frac{c(3M+r)(1 - \exp[-(M-r)T])}{(M-r)(2M+r)}\right). \tag{3.32}$$

Observe that η is a decreasing function of δ. Therefore, the expected optimal terminal wealth of the PPM can be re-express in terms of δ as follows:

$$\begin{aligned} E(X^*(T)) &= \\ &\times \left(1 + \frac{6M^2(1+\exp[-(M-2r)T]) - (M-r)\exp[-MT](1-\exp[-2MT])}{3(M-r)(2M+r) - 6M^2(1+\exp[-(M-r)T]) + (M+r)\exp[-MT](1-\exp[-2MT])}\right) \\ &\times \left(x_0 \exp[-(M-r)T] + \frac{c(\exp[-3MT]-1)}{3(2M+r)} + \frac{c(3M+r)(1-\exp[-(M-r)T])}{(M-r)(2M+r)}\right) \\ &+ \frac{6M^2(1+\exp[-(M-2r)T]) - (M-r)\exp[-MT](1-\exp[-2MT])}{2\delta(3(M-r)(2M+r) - 6M^2(1+\exp[-(M-r)T]) + (M+r)\exp[-MT](1-\exp[-2MT]))} \end{aligned} \tag{3.33}$$

Observe that the expected optimal terminal wealth for the PPM is the sum of the wealth that invested would get for investing the whole portfolio always in both the riskless and the risky assets plus a term,

$$\frac{6M^2(1+\exp[-(M-2r)T]) - (M-r)\exp[-MT](1-\exp[-2MT])}{2\delta(3(M-r)(2M+r) - 6M^2(1+\exp[-(M-r)T]) + (M+r)\exp[-MT](1-\exp[-2MT]))}.$$

This term depends both on the goodness of the risky assets with respect to the riskless asset and on the weight given to the minimization of the variance. Hence, the higher the value of M (which is the Sharpe ratio of the risky assets, stock and inflation-linked bond), the higher the expected optimal terminal wealth, everything else being equal. The higher the parameter given to the minimization of the variance of the terminal wealth, δ, the lower its mean.

4. The Efficient Frontier

We now establish the efficient frontier for the three classes of assets in the investment portfolio. From (3.30), we have that

$$E(X^*(T)) = x_0 \exp[-(M-r)T] + \lambda, \tag{4.1}$$

where,

$$\begin{aligned} \lambda = &\frac{2M^2\eta}{(M-r)(2M+r)}(1+\exp[-(M-r)T]) \\ &+ \frac{c}{3(2M+r)}(\exp[-3MT]-1) + \frac{c(3M+r)}{(M-r)(2M+r)}(1-\exp[-(M-r)T]) \\ &- \frac{\eta\exp[-MT]}{3(2M+r)}(1-\exp[-2MT]). \end{aligned} \tag{4.2}$$

$$E(X^{*2}(T)) = x_0^2 \exp[-(M-2r)T] + \psi, \tag{4.3}$$

where,

$$\begin{aligned}
\psi = & \frac{4M^3\eta^2}{(M-2r)(2M+r)^2}(1-\exp[-(M-2r)T]) \\
+ & \frac{(\eta+c)^2}{5(2M+r)^2}(1-\exp[-5MT]) - \frac{4M^2\eta^2}{(3M-r)(2M+r)^2}(1-\exp[-3MT+rT]) \\
- & \left(\frac{c^2(4M+r)(3M+2r)}{r(M-2r)(2M+r)^2} + \frac{8cM^2\eta(M+r)}{r(M-2r)(2M+r)^2}\right)\exp[-(M-2r)T] \\
- & \frac{2c}{r}\left(x_0(1+\exp[rT]) + \frac{\eta}{r^2+Mr-2M^2} - \frac{c(3M+r)}{r(M-r)(2M+r)}\right)\exp[-(M-r)T] \\
+ & \frac{4c^2M^2\exp[-3MT](1-(1-\frac{r}{M}+\frac{r\eta}{c})\exp[rT])}{r(3M-r)(2M+r)^2} + \frac{12\eta M^2 c}{(2M+r)^2} \\
\times & \left(\frac{1}{(M-r)(M-2r)} - \frac{1}{3(3M-r)}\right) + \frac{c^2(13M^2+9Mr+2r^2)}{(M-r)(M-2r)(2M+r)^2} \\
+ & \frac{2c(c(M-r)\exp[-3MT]+5M(c-\eta)+r(c-\eta))}{3(3M-r)(2M+r)^2} \\
- & \frac{2c\eta\exp[-3MT]}{2M+r}\left(\frac{1}{3r} - \frac{2M}{(3M-r)(2M+r)}\left(\frac{M}{r}+1\right)\exp[rT]\right).
\end{aligned} \tag{4.4}$$

Therefore,

$$\begin{aligned}
Var(X^*(T)) &= x_0^2\exp[-(M-2r)T] + \psi - (E(X^*(T)))^2 \\
&= x_0^2\exp[rT]\exp[-(M-r)T] + \psi - (E(X^*(T)))^2 \\
&= x_0\exp[rT](E(X^*(T)) - \lambda) + \psi - (E(X^*(T)))^2 \\
&= x_0\exp[rT]E(X^*(T) - \lambda x_0\exp[rT] + \psi - (E(X^*(T)))^2 \\
&= E(X^*(T))(x_0\exp[rT] - x_0\exp[-(M-r)T] - \lambda) + \psi - \lambda x_0\exp[rT] \\
&= E(X^*(T))(x_0\exp[rT](1-\exp[-MT]) - \lambda) + \psi - \lambda x_0\exp[rT].
\end{aligned}$$

Therefore,

$$E(X^*(T)) = \frac{\lambda x_0\exp[rT] - \psi}{x_0\exp[rT](1-\exp[-MT]) - \lambda} + \frac{\sigma^2(X^*(T))}{x_0\exp[rT](1-\exp[-MT]) - \lambda}. \tag{4.5}$$

This show that the expected terminal wealth of the PPM is a function of its variance. The efficient frontier in the mean-variance diagram is a straight line with gradient $\frac{1}{x_0\exp[rT](1-\exp[-MT])-\lambda}$ which measures the rate at which the terminal wealth will increase or decrease as the variance increases by one unit. If $x_0\exp[rT](1-\exp[-MT]) < \lambda$, we have a negative gradient. If $x_0\exp[rT](1-\exp[-MT]) > \lambda$, we have a positive gradient. If $x_0\exp[rT](1-\exp[-MT]) = \lambda$, we have an infinite gradient. Note that if the gradient is negative, it implies that the mean will increase as the variance decreases. If the gradient is positive, it implies that the mean will increase as the variance increases. If the gradient is infinite, we have that the mean will tends to negative infinity. Observe that if the PPM entered the scheme with no initial endowment, then (4.5) will become

$$E(X^*(T)) = \frac{\bar{\psi}}{\lambda} - \frac{\sigma^2(X^*(T))}{\lambda}. \tag{4.6}$$

where,

$$
\begin{aligned}
\bar{\psi} &= \frac{4M^3\eta^2}{(M-2r)(2M+r)^2}(1-\exp[-(M-2r)T]) \\
&+ \frac{(\eta+c)^2}{5(2M+r)^2}(1-\exp[-5MT]) - \frac{4M^2\eta^2}{(3M-r)(2M+r)^2}(1-\exp[-3MT+rT]) \\
&- \left(\frac{c^2(4M+r)(3M+2r)}{r(M-2r)(2M+r)^2} + \frac{8cM^2\eta(M+r)}{r(M-2r)(2M+r)^2}\right)\exp[-(M-2r)T] \\
&- \frac{2c}{r}\left(\frac{\eta}{r^2+Mr-2M^2} - \frac{c(3M+r)}{r(M-r)(2M+r)}\right)\exp[-(M-r)T] \\
&+ \frac{4c^2M^2\exp[-3MT](1-(1-\frac{r}{M}+\frac{r\eta}{c})\exp[rT])}{r(3M-r)(2M+r)^2} + \frac{12\eta M^2 c}{(2M+r)^2} \\
&\times \left(\frac{1}{(M-r)(M-2r)} - \frac{1}{3(3M-r)}\right) + \frac{c^2(13M^2+9Mr+2r^2)}{(M-r)(M-2r)(2M+r)^2} \\
&+ \frac{2c(c(M-r)\exp[-3MT]+5M(c-\eta)+r(c-\eta))}{3(3M-r)(2M+r)^2} \\
&- \frac{2c\eta\exp[-3MT]}{2M+r}\left(\frac{1}{3r} - \frac{2M}{(3M-r)(2M+r)}\left(\frac{M}{r}+1\right)\exp[rT]\right). \qquad (4.7)
\end{aligned}
$$

In this case, the gradient of the mean-variance portfolio selection becomes $-\frac{1}{\lambda}$ and the intercept is $\frac{\bar{\psi}}{\lambda}$.

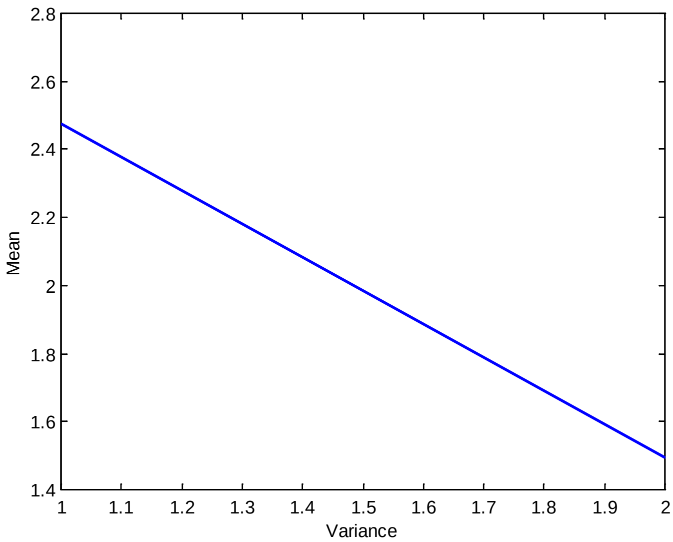

Figure 1: Efficient Frontier. We take $x_0 = 1$, $\delta = 0.05$, $T = 5$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, and $\alpha = 0.05$.

Figure 1 shows the efficient frontier of portfolios in the mean-variance plan and reports the points $(\sigma^2(X^*(T)), E(X^*(T)))$ for each strategy under consideration. Observe that figure 1 has negative gradient of −0.979322 and intercept of 3.45372. Observe that the higher the variance, the lower the mean and vice versa. But, with that presents of inflation-linked bond as one of the risky assets, the variance could be minimized.

5. Numerical Example

Suppose a market involve a cash account with nominal annual interest rate 4%, an inflation-linked bond with a nominal annual appreciation rate $r + \sigma_I \theta^I$, where $r = 4\%$ is the nominal annual interest rate, $\sigma_I = 35\%$ is the inflation volatility and $\theta^I = 30\%$ is the market price of inflation risks, and a stock with a nominal annual appreciation rate 9.2% and a standard deviations arising from inflation and stock market 38% and 45% respectively. Suppose also that the following parameters (which have been defined earlier) take the values as follows: $c = 0.07$ million, $x_0 = 1$ million, $\delta = 0.05$ and $T = 5$ (years), we have the following results.

A PPM who contributes a constant flow of 0.07 million and have initial wealth $x_0 = 1$ million in the pension scheme and wishes to obtain an expected wealth between 0 − 2.5 million has a portfolio value in inflation-linked bond as obtain in figure 2 and stock as obtain in figure 3 for 5 year period. Under the same strategy but for a period of 30 years, we have the results for inflation-linked bond and stock in figure 4 and 5, respectively.

In particular, at the initial time $t = 0$, $\Delta^S(0, x_0) = 0.280635$ million and $\Delta^I(0, x_0) = -1.60143$ million. These imply that the inflation-linked bond needs to be shorten for an amount 1.60143 million and then invest into cash account which is already having an amount 0.617935 million together with the initial endowment 1 million. It implies that at $t = 0$, a total of 3.219365 million should be invested in cash account.

We take $x_0 = 1$, $\delta = 0.05$, $T = 5$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

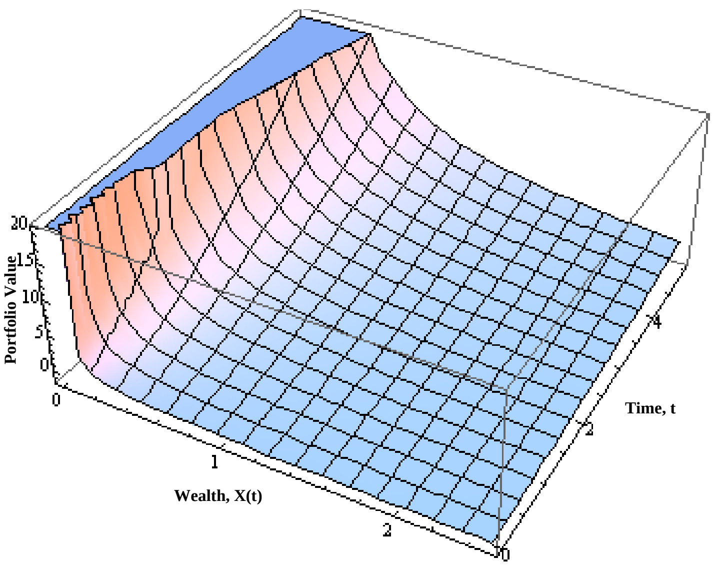

Figure 2: Portfolio Value in Inflation-linked Bond. We take $x_0 = 1$, $\delta = 0.05$, $T = 5$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

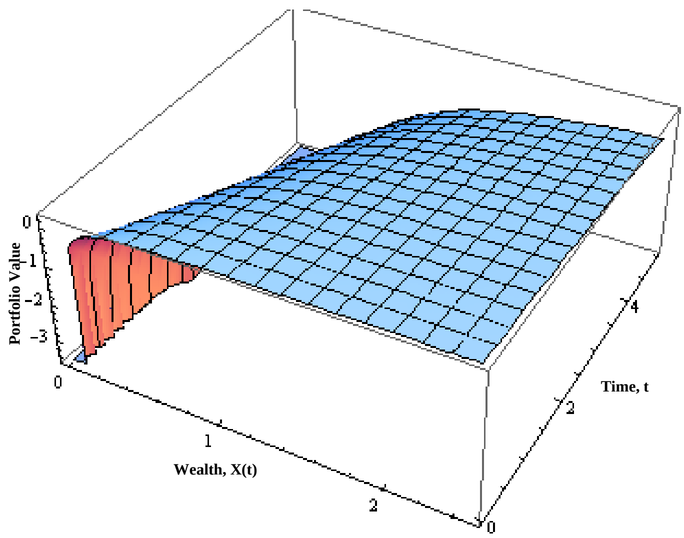

Figure 3: Portfolio Value in Stock. We take $x_0 = 1$, $\delta = 0.05$, $T = 5$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

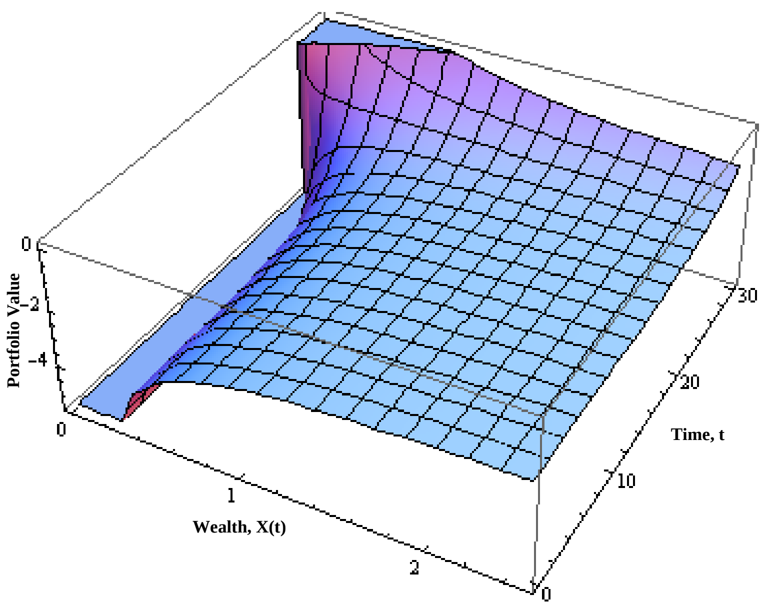

Figure 4: Portfolio Value in Inflation-linked Bond. We take $x_0 = 1$, $\delta = 0.05$, $T = 30$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

Table 1: EPMV at Different Value of c

c	$\Delta^{I*}(T)$	$\Delta^{S*}(T)$	$E(X^*(T))$	$Var(X^*(T))$
0.07	2.11382	-0.370427	2.24252	1.23677
0.10	2.16595	-0.379561	2.40778	1.17607
0.18	2.30494	-0.403918	2.84850	1.02772
0.50	2.86089	-0.501344	4.61135	0.63131
1.00	3.72958	-0.653573	7.36580	0.64283

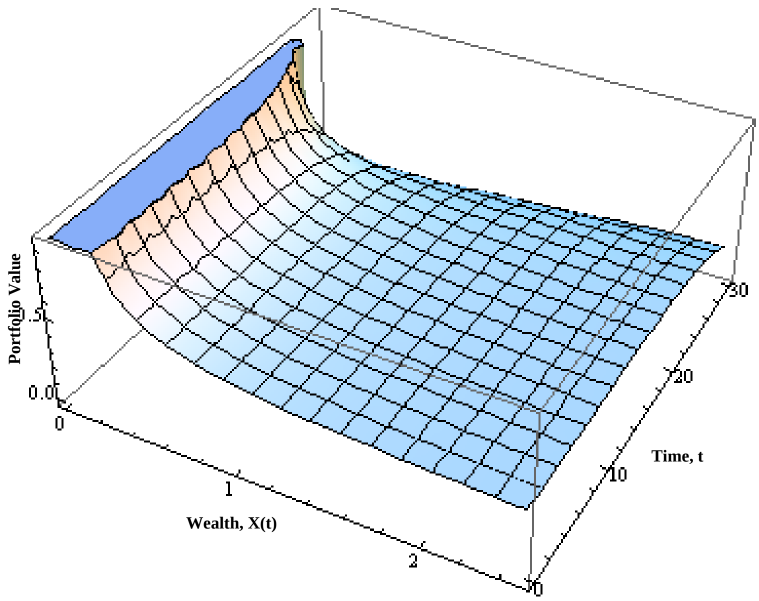

Figure 5: Portfolio Value in Stock. We take $x_0 = 1$, $\delta = 0.05$, $T = 30$, $\mu = 0.092$, $\sigma_1 = 0.35$, $\sigma_1^S = 0.38$, $\sigma_2^S = 0.45$, $\theta^I = 0.30$, $r = 0.04$, $c = 0.07$, $\alpha = 0.05$ and $X^* = 2.5$.

where, EPMV stands for Expected Portfolio, Mean and Variance

Table 2: EPMV at Different Value of θ^I

θ^I	$\Delta^{I*}(T)$	$\Delta^{S*}(T)$	$E(X^*(T))$	$Var(X^*(T))$
-0.40	1.04336	-0.62614	-0.73204	1.92241
-0.30	-0.0528	0.036369	0.63063	1.46643
-0.20	-0.7444	0.672109	2.08936	1.05585
-0.10	-0.1036	0.218662	1.95446	0.44016
0.12	-18.9559	-0.69169	-1.8832	161.417
0.20	-0.56135	0.052899	1.57551	0.00502
0.30	2.11382	-0.370427	2.24252	1.23677
0.40	0.36761	-0.081166	0.84008	1.92241

Table 3: EPMV at Different Value of x_0

x_0	$\Delta^{I*}(T)$	$\Delta^{S*}(T)$	$E(X^*(T))$	$Var(X^*(T))$
1.00	2.11382	-0.370427	2.24252	1.236770
2.00	2.36116	-0.413770	3.02677	0.839136
2.24	2.42114	-0.424282	3.21697	0.823284
2.25	2.42299	-0.424606	3.22284	0.823295
3.00	2.60849	-0.457113	3.81103	0.976339

Table 4: EPMV at Different Value of δ

δ	$\Delta^{I*}(T)$	$\Delta^{S*}(T)$	$E(X^*(T))$	$Var(X^*(T))$
5.00000	-1.34329	0.235398	1.1806	0.253598
0.50000	-1.02901	0.180323	1.27714	0.155583
0.05000	2.11382	-0.370427	2.24252	1.23677
0.00500	33.5421	-5.87793	11.8963	218.182
0.00050	347.825	-60.953	108.434	23001
0.00005	3490.66	-611.704	1073.81	2312170

5.1. Discussion of the Results in the Tables

From table 1, observe that as the contributions of the PPM increases, the portfolio value in inflation-linked bond increases while the portfolio value in stock decreases. Observe also that as the contributions increases, the expected terminal wealth increases and the variance decreases, which is an interesting result since the aim of an investor is to maximize wealth and minimize risks. The reason for this, is that the inflation risks in the investment profile have been hedged by the inflation-linked bond. This shows that inflation risks on the contributions of the PPM can be hedged by the inflation-linked bond. We conclude that the higher the contributions of the PPM, the higher the expected wealth and vice versa, which is an expected result. The expected optimal wealth (as in above) can be actualized only when the entire portfolio is invested in inflation-linked bond.

From table 2, we found that, when the market price of inflation risks, θ^I, is -0.40, the portfolio value in inflation-linked bond, $\Delta^I(T)$ at $T = 5$, is 1.04336 million and stock, $\Delta^S(T)$ is -0.62614 million. This means that the portfolio value in stock should be shorten by an amount 0.62614 million and invest it in inflation-linked bond. Observe also that when $\theta^I = -0.40$, the expected wealth is -0.73204 million and variance 1.92241 million. This shows negative expected wealth with high variance. Similar interpretation go to when $\theta^I = -0.30, -0.20$, and -0.10. Observe that at $\theta^I = 0.12$, $\Delta^I(T) = -18.9559$ million and $\Delta^S(T) = -0.69169$ million. This implies that that the entire portfolio values of the PPM should remain only in cash account. This is because the risks associated with the portfolio in stock and inflation-linked bond are very high. At $\theta^I = 0.20$, $\Delta^I(T) = -0.56135$ million and $\Delta^S(t) = 0.052899$ with expected wealth of 1.57551 million and variance of 0.00502. This means that the entire portfolio should remain in stock and cash account. Observe also that the PPM will have a higher expected wealth at $\theta^I = 0.30$. This occur when the entire portfolio is invested in inflation-linked bond.

From table 3, observe that the higher the initial endowment of the PPM, the higher the portfolio value in inflation-linked bond and the expected wealth, the lower the variance, which is an interesting result since the aim of an investor is to minimize risks and maximize wealth. The reason for the gradual reduction of the variance is because the inflation risks on the initial endowment has be hedged due to the presents of an inflation-linked bond in the investment profile. We therefore conclude that inflation-linked bond is an ”inflation risks fighter”.

From table 4, observe that the higher the weight given to the minimization of the variance, the lower the portfolio value in stock and inflation-linked bond, and vice versa, which is an expected result. Therefore, it is optimal to invest the entire portfolio into cash account when $\delta = +\infty$. We found that the lower the value of δ, higher the portfolio value in inflation-linked bond and expected wealth. This also lead to high variance.

6. Conclusion

In this paper, we have considered a mean-variance portfolio selection problem in the accumulation phase of a defined contribution pension scheme. The optimal portfolio and optimal expected terminal wealth for the pension plan member (PPM) were established. The efficient frontier was obtained for the three assets class. It was found that inflation-linked bond is an "inflation risks fighter".

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