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Distortion Estimate and the Radius of Starlikeness of Janowski Close-to-Star Functions

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Abstract

Let F be the class of all analytic functions in the open unit disc $\mathbb{D} = \{z \quad |z| < 1\}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Let $g(z)$ be an element of F such that $g(z)$ satisfies the condition

$$(z\frac{g'(z)}{g(z)}) = \frac{1 + A\phi(z)}{1 + B\phi(z)},$$

for all $z \in \mathbb{D}$, where $\phi(z)$ is analytic in $\mathbb{D}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$, and $-1 \leq B < A \leq 1$, then $g(z)$ is called Janowski Starlike function in $\mathbb{D}$. The class of such functions is denoted by $\mathcal{S}^*(A, B)$ (Janowski, 1973).
The aim of this paper is to give a distortion estimation and the radius of starlikeness of the class $\mathcal{S}^*K(A, B)$.

Keywords: Close-to-star function, the radius of starlikeness, distortion estimate.
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1. Introduction

Let Ω be the family of functions $\phi(z)$ which are regular in $\mathbb{D}$ and satisfy the condition $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. The family of functions $p(z) = 1 + p_1z + p_2z^2 + \cdots$ analytic in $\mathbb{D}$, and satisfying the conditions $p(0) = 1$, $Re\ p(z) > 0$ is denoted by $\mathcal{P}$ such that $p(z)$ in $\mathcal{P}$ if and only if

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}, \tag{1.1}$$

for some $\phi(z) \in \Omega$, and every $z \in \mathbb{D}$. The class $\mathcal{P}$ is the Caratheodory class (Nehari, 1952).

Next, for arbitrary fixed real numbers A and B which satisfy $-1 \leq B < A \leq 1$, we say $p(z)$ belongs to the class $P(A, B)$ if $p(z) = 1 + p_2z^2 + p_3z^3 + \cdots$ is analytic in $\mathbb{D}$ and $p(z)$ is given by

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, \tag{1.2}$$

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for some $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. The class $P(A, B)$ was introduced by (Janowski, 1973).

Let $f(z)$ be an element of F and $g(z)$ be an element of $\mathcal{S}^*(A, B)$, if the condition

$$Re(\frac{f(z)}{g(z)}) > 0 \tag{1.3}$$

is satisfied, then $f(z)$ is called Janowski close-to-star function in $\mathbb{D}$. The class of such functions is denoted by $\mathcal{S}^*K(A, B)$. The class of $\mathcal{S}^*K(A, B)$ is not empty because,

$$f(z) = \begin{cases} z(1 + Bz)^{\frac{A-B}{B}} . \frac{1+z}{1-z}; & B \neq 0, \\ ze^{Az} \frac{1+z}{1-z}; & B = 0. \end{cases} \tag{1.4}$$

Finally, let $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + \cdots$ be analytic functions in $\mathbb{D}$, if there exist a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for every $z \in \mathbb{D}$, then we say that $F(z)$ is subordinate to $G(z)$, and we write $F(z) \prec G(z)$. We also note that if $F(z) \prec G(z)$, then $F(\mathbb{D}) \subset G(\mathbb{D})$.

2. Main Results

Lemma 2.1. *Let $f(z)$ be an element of $\mathcal{S}^*K(A, B)$, then*

$$Re[(1 + Bz)^{-\frac{A-B}{B}} \frac{f(z)}{z}] > 0, B \neq 0, \tag{2.1}$$

$$Re[e^{-Az} \frac{f(z)}{z}] > 0, B = 0. \tag{2.2}$$

Proof. Since the function,

$$g(z) = \begin{cases} z(1 + Bz)^{\frac{A-B}{B}}; & B \neq 0, \\ ze^{Az}; & B = 0. \end{cases} \tag{2.3}$$

belongs to the class $S^*(A, B)$ (Janowski, 1973), then using the definition of the class $\mathcal{S}^*K(A, B)$,

$$Re(\frac{f(z)}{g(z)}) = Re[\frac{f(z)}{z(1 + Bz)^{\frac{A-B}{B}}}] = Re[(1 + Bz)^{-\frac{A-B}{B}} . \frac{f(z)}{z}] > 0, B \neq 0, \tag{2.4}$$

$$Re(\frac{f(z)}{g(z)}) = Re[\frac{f(z)}{ze^{Az}}] = Re[e^{-Az} \frac{f(z)}{z}] > 0, B = 0. \tag{2.5}$$

□

Theorem 2.2. *Let $f(z)$ be an element of $\mathcal{S}^*K(A, B)$, then for $r = |z|$*

$$F(-A, -B, r) \leq |f(z)| \leq F(A, B, r), B \neq 0, \tag{2.6}$$

$$G(-A, r) \leq |f(z)| \leq G(A, r), B = 0, \tag{2.7}$$

where

$$F(A,B,r) = \frac{r(1+r)(1+Br)^{\frac{A-B}{B}}}{(1-r)} \tag{2.8}$$

and

$$G(A,r) = \frac{r(1+r)e^{Ar}}{(1-r)}. \tag{2.9}$$

Proof. Let $g(z)$ be an element of $\mathcal{S}^*(A,B)$, then

$$F_1(-A,-B,r) \le |g(z)| \le F_1(A,B,r), \tag{2.10}$$

where

$$F_1(A,B,r) = \begin{cases} r(1+Br)^{\frac{A-B}{B}}; & B \ne 0, \\ re^{Ar}; & B = 0. \end{cases} \tag{2.11}$$

On the other hand, if $p(z) \in \mathcal{P}$ then we have,

$$\frac{1-r}{1+r} \le |p(z)| \le \frac{1+r}{1-r} \tag{2.12}$$

(Goodman, 1983).

Considering 2.10 and 2.12 together and after the straightforward calculations, we get 2.6 and 2.7. We also note that the inequalities 2.6 and 2.7 are sharp because the extremal functions are;

$$(1+Bz)^{-\frac{A-B}{B}}\frac{f(z)}{z} = p(z) = \frac{1+z}{1-z} \Rightarrow f(z) = \frac{z(1+z)(1+Bz)^{\frac{A-B}{B}}}{1-z}, B \ne 0, \tag{2.13}$$

$$e^{-Az}\frac{f(z)}{z} = p(z) = \frac{1+z}{1-z} \Rightarrow f(z) = \frac{z(1+z)e^{Az}}{1-z}, B = 0. \tag{2.14}$$

□

Remark. If we give the special values to A and B, we obtain that new inequalities and new growth theorems for the subclass of $S^*K(A,B)$. The special values of A and B can be ordered in the following manner:

i. $A = 1$, $B = -1$;
ii. $A = 1-2\alpha$, $B = -1$, $0 \le \alpha < 1$;
iii. $A = 1$, $B = 0$;
iv. $A = \alpha$, $B = 0$, $0 < \alpha < 1$;
v. $A = 1$, $B = -1 + \frac{1}{M}$, $M > \frac{1}{2}$;
vi. $A = \alpha$, $B = -\alpha$, $0 < \alpha < 1$.

Theorem 2.3. *The radius of starlikeness of the class $\mathcal{S}^*K(A,B)$ is the smallest positive roots r_0 of the equations,*

$$\begin{cases} Q_1(r) = -2r(1-Br) + (1-r^2)(1-Ar); & B \ne 0, \\ Q_2(r) = -2r + (1-Ar)(1-r^2); & B = 0. \end{cases} \tag{2.15}$$

Proof. Using Lemma 2.1, then we obtain

$$(1+Bz)^{-\frac{A-B}{B}}\frac{f(z)}{z} = p(z), B \neq 0, \tag{2.16}$$

$$e^{-Az}\frac{f(z)}{z} = p(z), B = 0, \tag{2.17}$$

$$\begin{cases} z\frac{f'(z)}{f(z)} = \frac{1+Az}{1+Bz} + z\frac{p'(z)}{p(z)}; & B \neq 0, \\ z\frac{f'(z)}{f(z)} = (1+Az) + z\frac{p'(z)}{p(z)}; & B = 0. \end{cases} \tag{2.18}$$

Thus,

$$\begin{cases} Re(z\frac{f'(z)}{f(z)}) \geq Min_{|z|=r}Re(\frac{1+Az}{1+Bz}) + Min_{|z|=r,p(z)\in\mathcal{P}}\, Re(z\frac{p'(z)}{p(z)}); & B \neq 0, \\ Re(z\frac{f'(z)}{f(z)}) \geq Min_{|z|=r}Re(1+Az) + Min_{|z|=r,p(z)\in\mathcal{P}}\, Re(z\frac{p'(z)}{p(z)}); & B = 0. \end{cases} \tag{2.19}$$

$$Re(z\frac{f'(z)}{f(z)}) \geq Min_{|z|=r}Rez\frac{p'(z)}{p(z)} + Min_{|z|=r}Rez\frac{(A-B)}{1+Bz} + Min_{|z|=r}Re(1). \tag{2.20}$$

Therefore we have:

$$\begin{cases} Re(z\frac{f'(z)}{f(z)}) \geq \frac{-2r(1-Br)+(1-r^2)(1-Ar)}{(1-r^2)(1-Br)}; & B \neq 0, \\ Re(z\frac{f'(z)}{f(z)}) \geq \frac{-2r+(1-Ar)(1-r^2)}{(1-r^2)}; & B = 0. \end{cases} \tag{2.21}$$

The denominator of the expression on the right hand sides of the inequalities 2.20 is positive for $0 \leq r < 1$, $Q_1(0) = 1$, $Q(1) = -2(1-B) < 0$, $Q_2(0) = 1$, $Q_2(1) = -2 < 0$.

Thus using intermediate value theorem and mean value theorem, the smallest positive roots of the equations 2.15 lies between 0 and 1.

Therefore the inequality

$$Re\left(z\frac{f'(z)}{f(z)}\right) > 0, \tag{2.22}$$

is valid for $r = |z| < r_0$. Hence the radius of starlikeness for $\mathcal{S}^*K(A,B)$ is not less than r_0. The theorem is proved. □

References

Goodman, A. W. (1983). *Univalent functions, Volume I and II*. Mariner publishing company, Tampa Florida, USA.

Janowski, W. (1973). Some extremal problems for certain families of analytic functions. I. *Annales Polonici Mathematici* **28**, 297–326.

Nehari, Z. (1952). *Conformal Mapping*. McGraw-Hill, New York.