Theory and Applications of
Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 4 (1) (2014) 65–80

Best Approximation in L^p-norm and Generalized (α,β)-growth of Analytic Functions

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Abstract

Let $0 < p \leq +\infty$ and $\Omega_R = \left\{ z \in \mathbf{C}^n; \exp V_E(z) < R \right\}$, for some $R > 1$, where $V_E = \sup \left\{ \frac{1}{d} \ln |P_d|, P_d \right.$ polynomial of degree $\left. \leq d, \|P_d\|_E \leq 1 \right\}$ is the Siciak extremal function of a L-regular compact E.

The aim of this paper is the characterization of the generalized growth of analytic functions of several complex variables in the open set by means of the best polynomial approximation in L_p-norm on a compact E with respect to the set $\Omega_r = \left\{ z \in \mathbf{C}^n; \exp V_E(z) \leq r \right\}$, $1 < r < R$.

Keywords: Extremal function, L-regular, generalized growth, best approximation of analytic function, L^p-norm.
2010 MSC: Primary 30E10; Secondary 41A21, 32E30.

1. Introduction

Let E be a compact L-regular of $\mathbb{C}^n$. For an entire function f in $\mathbb{C}^n$ developed according an extremal polynomial basis $(A_k)_k$ (see Zeriahi (1987)), M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) have generalized growth in term of coefficients with respect the sequence $(A_k)_k$. The growth used by M. Harfoui was defined according to the functions α and β (see Harfaoui (2010), pp. 5, eq. (2.14)), with respect to the set:

$$\Omega_r = \left\{ z \in \mathbb{C}^n,\ exp(V_E)(z) < r \right\},$$

where

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$$V_E = \sup\Big\{\frac{1}{d}\ \log|P_d|, P_d \quad \text{polynomial of degree} \leq d, \|P_d\|_E \leq 1\Big\}$$

is the Siciak's extremal function of E which is continuous in $\mathbb{C}^n$ (Because E is L-regular). The (α,β)-order and the (α,β)-type of f an entire function (or generalized order and generalized type) are defined respectively by:

$$\rho(\alpha,\beta) = \limsup_{r\to+\infty}\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(\log(r))} \quad \text{and} \quad \sigma(\alpha,\beta) = \limsup_{r\to+\infty}\frac{\alpha\Big(\|f\|_{\overline{\Omega}_r}\Big)}{\Big[\beta(r)\Big]^{\rho(\alpha,\beta)}},$$

where

$$\Big\|f\Big\|_{\overline{\Omega}_r} = \sup_{\overline{\Omega}_r}\Big|f(z)\Big|.$$

These results have been used to establish the generalized growth in terms of best approximation in L_p-norm for $p \geq 1$.

Let f be a function defined and bounded on E. For $k \in \mathbb{N}$ put

$$\pi_k^p(E,f) = \inf\Big\{\Big\|f - P\Big\|_{L^p(E,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\Big\},$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the family of all polynomials of degree $\leq k$ and μ the well-selected measure (The equilibrium measure $\mu = (dd^cV_E)^n$ associated to a L-regular compact E) (see Zeriahi (1983)) and $L^p(E,\mu)$, $p \geq 1$, is the class of all functions such that:

$$\Big\|f\Big\|_{L^p(E,\mu)} = \Big(\int_E |\ f\ |^p\ d\mu\Big)^{1/p} < \infty.$$

For an entire function $f \in \mathbb{C}^n$ M. Harfaoui established a precise relationship between the general growth with respect to the set (see (Harfaoui (2010)): $\Omega_r = \Big\{z \in \mathbb{C}^n\ :\ \exp(V_E)(z) < r\Big\}$, and the coefficients of the development of f with respect to the sequence $(A_k)_k$, called extremal polynomial (see Zeriahi (1987)). He used these results to give the relationship between the generalized growth of f and the sequence $(\pi_k^p(E,f))_k$. Note that M. Harfaoui did not study the case $0 < p < 1$ because the triangle inequality is not satisfied. A. Janik (see Janik (1991)) characterized the (α,β)-order of an analytic function g in Ω_R defned by

$$\Omega_R = \Big\{z \in \mathbb{C}^n,\ exp(V_E(z)) < R\Big\},\ \text{for some}\ R > 1,$$

by means of polynomial approximation and interpolation to g on on a L-regular compact E, with respect to the set

$$\Omega_r = \Big\{z \in \mathbb{C}^n,\ \exp(V_E(z)) < r,\quad 1 < r < R\Big\}.$$

In his work A. Janik used the best approximation defined, for a function defined and bounded on E, by:

$$\mathcal{E}_n^{(1)} = \mathcal{E}_n^{(1)}(f,E) = \|\ f - t_n\ \|,$$

$$\mathcal{E}_n^{(2)} = \mathcal{E}_n^{(2)}(f,E) = \|\ f - l_n\ \|,$$

$$\mathcal{E}_{n+1}^{(3)} = \mathcal{E}_{n+1}^{(3)}(f, E) = \| l_{n+1} - l_n \|,$$

where t_n denoted the nth Chebychev polynomial of the best approximation to f on E and l_n denoted the nth Lagrange interpolation polynomial for f with nodes at extremal points of E (see Siciak (1962)).

The (α, β)-order of an analytic function was defined as follows:

If E be a compact L-regular. If f is an analytic function in

$$\Omega_R = \Big\{ z \in \mathbb{C}^n : \ \exp(V_E(z)) < R \Big\}$$

for some $R > 1$. We define the (α, β)-order of f (or generalized order) by

$$\rho(\alpha, \beta) = \limsup_{r \to R} \frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r}) \Big)}{\beta(R/(R-r))}$$

where $\big\|f\big\|_{\overline{\Omega}_r} = \sup_{\overline{\Omega}_r} \big|f(z)\big| = \sup\Big\{ \mid f(z) \mid : \exp V_E(z) \le r, 1 < r < R \Big\}$.

In this work we study the generalized order and generalized type, which will be defined later, for an analytic function in the open set Ω_R, with respect to the set Ω_r in terms of coefficients of the analytic function in the development according to the sequence of extremal polynomials. So we obtain a generalization of the results of M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) and A. Janik (see Janik (1984), and Janik (1991)) replacing $\mathbb{C}^n$ by Ω_R and the entire function in $\mathbb{C}^n$ by analytic function in Ω_R .

After studying the generalized type of an analytic function in Ω_R, for some $R > 1$, we use this results to characterize the generalized type by means of best polynomial approximation on E in L_p-norm for $0 < p \le +\infty$.

Recall that the generalized growth used by M. Harfaoui (see Harfaoui (2010) and Harfaoui (2011)) called (α, β)-growth was defined with respect to functions α and β defined as:

Let α and β be two positive, strictly increasing to infinity differentiable functions $]0, +\infty[$ to $]0, +\infty[$such that for every $c > 0$:

such that

$$\begin{cases} \lim\limits_{x \to +\infty} \dfrac{\alpha(cx)}{\alpha(x)} = 1, \\ \lim\limits_{x \to +\infty} \dfrac{\beta\Big(1 + x\omega(x)\Big)}{\beta(x)} = 1, \quad \lim\limits_{x \to +\infty} \omega(x) = 0, \\ \lim\limits_{x \to +\infty} \dfrac{d\Big(\beta^{-1}(c\alpha(x))\Big)}{\alpha(\log(x))} \le b. \\ \alpha\Big(x/\beta^{-1}(c\alpha(x))\Big) = (1 + o(x))\alpha(x), \qquad \text{for } x \to +\infty, \end{cases}$$

where $d(u)$ means the differential of u.

2. Definitions and notations

Before we give some definitions and results which will be frequently used in this paper.

Definition 2.1. (Siciak (1977)) Let E be a compact set in $\mathbb{C}^n$ and let $\|.\|_E$ denote the maximum norm on E. The function

$$V_E = \sup\left\{\frac{1}{d}\ \log|P_d|, P_d \quad \text{polynomial of degree} \le d, \|P_d\|_E \le 1, d \in \mathbb{N}\right\}$$

is called the Siciak's extremal function of the compact E.

Definition 2.2. Zeriahi (1983) A compact E in $\mathbb{C}^n$ is said to be L-regular if the extremal function, V_E, associated to E is continuous on $\mathbb{C}^n$.

Regularity is equivalent to the following Bernstein-Markov inequality (see Siciak (1962)): For any $\epsilon > 0$, there exists an open $U \supset E$ such that for any polynomial P, $\|P\|_U \le e^{\epsilon.deg(P)}\|P\|_E$.

In this case we take $U = \left\{z \in \mathbb{C}^n; V_E(z) < \epsilon\right\}$.

Regularity also arises in polynomials approximation. For $f \in C(E)$, we let

$$\epsilon_d(E, f) = \inf\left\{\|f - P\|_E,\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the set of polynomials of degree at most d. Siciak (see Siciak (1977)) showed:

If E is L-regular, then $\limsup_{d\to+\infty}\Big(\varepsilon_d(E, f)\Big)^{1/d} = \dfrac{1}{r} < 1$ if and only if f has an analytic continuation to $\left\{z \in \mathbf{C}^n; V_E(z) < \log\left(\dfrac{1}{r}\right)\right\}$. It is known that if E is an compact L-regular of $\mathbb{C}^n$, there exists a measure μ, called extremal measure, having interesting properties (see Siciak (1962) and Siciak (1977)), in particular, we have:

(P_1) Bernstein-Markov inequality: $\forall\epsilon > 0$, there exists $C = C_\varepsilon$ is a constant such that

$$(BM) : \|P_d\|_E = C(1 + \varepsilon)^{s_k}\|P_d\|_{L^2(E,\mu)}, \tag{2.1}$$

for every polynomial of n complex variables of degree at most d.

(P_2) Bernstein-Waish (B.W) inequality:

For every set L-regular E and every real $r > 1$ we have:

$$\|f\|_E \le M.r^{deg(f)}\Big(\int_E |\ f\ |^p\ .d\mu\Big)^{1/p} \tag{2.2}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.

Let $s : \mathbb{N} \to \mathbb{N}^n, k \to s(k) = (s_1(k), ..., s_n(k))$ be a bijection such that

$$|s(k + 1)| \ge |s(k)| \ \text{ where } \ |s(k)| = s_1(k) + ... + s_n(k).$$

A. Zeriahi (see Zeriahi (1987)) has constructed according to the Hilbert Schmidt method a sequence of monic orthogonal polynomials according to a extremal measure (see Siciak (1962)), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)} \tag{2.3}$$

such that $\left\|A_k\right\|_{L^p(E,\mu)} = \left[\inf\left\{\left\|z^{s(k)} + \sum_{j=1}^{k-1} a_j z^{s(j)}\right\|_{L^2_{(E,\mu)}}, (a_1, a_2, ..., a_n) \in \mathbb{C}^n\right\}\right]^{1/s_k}$.

We need the following notations which will be used in the sequel: (N_1) $\nu_k = \nu_k(E) = \left\|A_k\right\|_{L^2(K,\mu)}$. (N_2) $a_k = a_k(E) = \left\|A_k\right\|_E = \max_{z\in K}\left|A_k(z)\right|$ and $\tau_k = \left(a_k\right)^{1/s_k}$, where $s_k = deg(A_k)$. With that notations and (B.W) inequality we have

$$\left\|A_k\right\|_{\overline{\Omega}_r} \le a_k.r^{s_k} \tag{2.4}$$

where $s_k = deg(A_k)$. For more details (see Zeriahi (1983)).

Definition 2.3. Zeriahi (1983) Let E be a compact L-regular. If f is an analytic function in

$$\Omega_R = \left\{z \in \mathbb{C}^n :\ \exp(V_E(z)) < R\right\}$$

for some $R > 1$. We define the (α,β)-growth ((α,β)-order and (α,β)-type) of f (or generalized order) by $\rho(\alpha,\beta) = \limsup_{r\to R}\dfrac{\alpha\left(\log(\|f\|_{\overline{\Omega}_r})\right)}{\beta(R/(R-r))}$, $\sigma(\alpha,\beta) = \limsup_{r\to R}\dfrac{\alpha\left(\log(\|f\|_{\overline{\Omega}_r})\right)}{\left[\beta(R/(R-r))\right]^{\rho(\alpha,\beta)}}$, where $\left\|f\right\|_{\overline{\Omega}_r} = \sup_{z\in\overline{\Omega}_r}\left|f(z)\right| = \sup\left\{\mid f(z)\mid : \exp V_E(z) \le r, 1 < r < R\right\}$.

Note that in the classical case $\alpha(x) = \beta(x) = \log(x)$. We need the following lemma (see Zeriahi (1987)).

Lemma 2.1. *(Zeriahi (1987)) If E is a compact L-regular subset of $\mathbb{C}^n$, then for every $\theta > 1$, there exists an integer $N_\theta \ge 1$ and a constant $C_\theta > 0$ such that:*

$$\pi_k^p(E,f) \le C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^k}. \tag{2.5}$$

for every $k \ge 1$, every $r > 1$ and every $f \in 0\left(\overline{\Omega}_{r\theta}\right)$. If $f = \sum_{k=0}^{+\infty} f_k.A_k$ be an entire function, then for every $\theta > 1$, there exists $N_\theta \in \mathbb{N}^*$ and $C_\theta > 0$ such that

$$\left|f_k\right|\nu_k \le C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^{s_k}}, \tag{2.6}$$

for every $k \ge 0$ and $r > 1$. C_θ and N_θ do not depend on r or k, or f.

Note that the second assertion of the lemma is a consequence of the first assertion and it replaces Cauchy inequality for complex function defined on the complex plane $\mathbb{C}$.

3. Generalized order and coefficient characterizations with respect to extremal polynomial

The purpose of this section is to establish the relationship of the generalized growth of an analytic function in Ω_R with respect to the set $\Omega_r = \left\{z \in \mathbb{C} : \exp(V_E(z)) < r\right\}$ and coefficients of an entire function $f \in \mathbb{C}^n$ in the development with respect to the sequence of extremal polynomials.

Let $(A_k)_k$ be a basis of extremal polynomial associated to the set E defined the relation (2.3). We recall that $(A_k)_k$ is a basis of $O(\mathbb{C}^n)$ (the set of entire functions on $\mathbb{C}^n$). So if f is an entire function then $f = \sum_{k\geq 1} f_k.A_k$.

Put

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \mu(\alpha,\beta). \tag{3.1}$$

To prove the aim result of this section we need the following lemmas:

Lemma 3.1. *(Zeriahi (1987)) Let E be a compact L-regular subset of $\mathbb{C}^n$. Then*

$$\lim_{k\to+\infty}\left[\frac{|A_k(z)|}{\nu_k}\right]^{1/s_k} = \exp(V_E(z)), \tag{3.2}$$

for every $z \in \mathbb{C}^n \setminus \widehat{E}$ the connected component of $\mathbb{C}^n \setminus E$,

$$\lim_{k\to+\infty}\left[\frac{\|A_k\|_E}{\nu_k}\right]^{1/s_k} = 1. \tag{3.3}$$

Lemma 3.2. *For every $r > 1$ and $\mu > 0$, the maximum of the function*

$$x \to \omega(x,r) = x.\log(r/R) + \frac{x}{\beta^{-1}\left(\alpha(x)/\mu\right)}$$

is reached for $x = x_r$ solution of the equation

$$x = \alpha^{-1}\left\{\mu\beta\left[\frac{1 - d\log\left(\beta^{-1}(\alpha(x)/\mu)\right)/d(\log(x))}{\log\left(R/r\right)}\right]\right\}. \tag{3.4}$$

Proof. Put $G(x,\mu) = \beta^{-1}\left(\alpha(x)/\mu\right)$, then $\omega(x,r) = x.\log(r/R) + \dfrac{x}{G(x,\mu)}$. The maximum of the function $x \to \omega(x,r)$ is reached for $x = x_r$ solution of the equation of $\dfrac{d\omega(x,r)}{dx} = 0$. We have $\dfrac{\omega(x,r)}{dx} = 0 \Leftrightarrow \log(\dfrac{r}{R}) + \dfrac{G(x,\mu) - x.\dfrac{dG(x,\mu)}{dx}}{\left(G(x,\mu)\right)^2} = 0$, or $G(x,\mu) = \dfrac{1 - \dfrac{x}{G(x,\mu)}.\dfrac{dG(x,\mu)}{dx}}{\log(R/r)}$.

Since $\dfrac{dG(x,\mu)}{dx} = \dfrac{dG(x,\mu)}{d\log(x)}.\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.\dfrac{dG(x,\mu)}{d\log(x)}$, we get

$$G(x,\mu) = \frac{1 - \dfrac{1}{G(x,\mu)}.\dfrac{dG(x,\mu)}{d\log(x)}}{\log(R/r)} = \frac{1 - \dfrac{d\log G(x,\mu)}{d\log(x)}}{\log(R/r)}.$$

We deduce $x = x_r = \alpha^{-1}\left\{\mu\alpha\left[\dfrac{1 - d\Big(\beta^{-1}(\alpha(x)/\mu)\Big)/d(\log(x))}{\log\big(R/r\big)}\right]\right\}$. □

Lemma 3.3. *Let* $f = \sum_{k\geq 0} f_k.A_k$ *and E a L-regular compact. For every* $r \in]1,R[$, *we put*

$$\begin{cases} \overline{M}(f,r) = \sup_{k\in\mathbb{N}}\Big\{ \parallel f_k.A_k \parallel_E .r^k,\ r > 0\Big\} \\ \overline{\rho}(\alpha,\beta) = \limsup_{r\to R}\dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\beta(R/(R-r))} \end{cases}$$

then $\overline{\rho}(\alpha,\beta) \leq \mu(\alpha,\beta)$ *and* $\rho(\alpha,\beta) \leq \overline{\rho}(\alpha,\beta)$.

Proof. By the definition of μ (3.1) we have, for r sufficiently close to R and $\overline{\mu} = \mu + \epsilon$,

$$\log\Big(\mid f_k \mid .\tau_k^{s_k}.R^{s_k}\Big) \leq \frac{\alpha(s_k)}{\beta^{-1}\Big(\dfrac{1}{\overline{\mu}}.\alpha(s_k)\Big)}.$$

Then $\log\Big(\mid f_k \mid .\tau_k^{s_k}.r^{s_k}\Big) \leq s_k \log(r/R) + \dfrac{\alpha(s_k)}{\beta^{-1}\Big(\dfrac{1}{\overline{\mu}}.\alpha(s_k)\Big)}$. By the proprieties of α and β,the function $t \to \log(t)$ and the Lemma 3.3 we get $x_r = (1 + o(1))\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big)$ as $r \to R$. Indeed this result is a consequence of $\lim_{x\to+\infty}\left|\dfrac{d\Big(\beta^{-1}(c\alpha(x))\Big)}{\alpha(\log(x))}\right| \leq b,\quad \log(1+t) = (1+o(t)).t,\quad t \to 0$. Therefore $\log\Big(\parallel f_k.A_k \parallel_E .r^{s_k}\Big) \leq C_0.\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big),\ \ k \in \mathbb{N}$. Passing to the maximum for the variable $k \in \mathbb{N}$ we obtain, for r sufficiently close to R $\log\Big(\overline{M}(f,r)\Big) \leq C_0.\alpha^{-1}\Big(\mu.\beta(R/(R-r))\Big),\ \ k \in \mathbb{N}$. Then, by the proprieties of α, we obtain $\dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\beta(R/(R-r))} \leq \mu$. Passing to upper limit for $r \to R$ we have

$$(*)\quad \overline{\rho}(\alpha,\beta) \leq \mu.$$

Moreover we have for $z \in \Omega_r$ and $k \in \mathbb{N}$, $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} \mid f_k \mid .\|A_k\|_{\Omega_r} \leq \sum_{k\geq 0} \mid f_k \mid .\|A_k\|_E.r^{s_k}$. Write $r = \sqrt{r.R}.\sqrt{r/R}$, then $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} \mid f_k \mid .\|A_k\|_E.(\sqrt{r.R})^{s_k}.(\sqrt{r/R})^{s_k}$. Because $\sqrt{r/R} < 1$

then $\|f\|_{\overline{\Omega}_r} \leq \sum_{k\geq 0} \sup_{k\in_N}\Big(\mid f_k \mid .\|A_k\|_E.(\sqrt{r.R})^{s_k}\Big).(\sqrt{r/R})^{s_k}$ thus $\|f\|_{\overline{\Omega}_r} \leq \overline{M}(f,r') \sum_{k\geq 0}(\sqrt{r/R})^{s_k} \leq \overline{M}(f,r').\dfrac{1}{1-\sqrt{r/R}}$. where $r' = \sqrt{r.R}$. Therefore $\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \log\Big(\overline{M}(f,r')\Big) - \log\Big(1-\sqrt{r/R}\Big)$.

We have $\dfrac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\beta(R/(R-r))} \leq \dfrac{\alpha\Big(\log(\overline{M}(f,\sqrt{r.R}) - \log(1-\sqrt{r/R}))\Big)}{\beta(R/(R-\sqrt{r.R}))}.\dfrac{\beta(R/(R-\sqrt{r.R}))}{\beta(R/(R-r.R))}$. Passing to the upper limit we get

$$(**)\quad \rho(\alpha,\beta) \leq \overline{\rho}(\alpha,\beta).$$

By the relations (*) and (**) we obtain $\rho(\alpha,\beta) \leq \mu(\alpha,\beta)$. □

Theorem 3.1. *Let E be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$ such that*

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\Big(|f_k|.\tau_k^{s_k}.R^{s_k}\Big)}\right]} = \mu(\alpha,\beta) < \infty. \tag{3.5}$$

Then f is analytic in Ω_R, for some $R > 1$ and its (α,β)-order $\rho(\alpha,\beta) = \mu(\alpha,\beta)$.

Proof. It is known that for every polynomial P(see Siciak (1977))

$$\mid P(z) \mid \leq \| P \|_E \Big(\exp(V_E(z))\Big)^{deg(P)}, \quad \text{for every } z \in \mathbb{C}^n. \tag{3.6}$$

So for every $r \in]1,R[$, and for $P = f_k.A_k$ we get

$$\mid f_k.A_k(z) \mid \leq \mid f_k \mid . \| A_k \|_E \Big(\exp(V_E(z))\Big)^{s_k}, \text{for every} z \in \mathbb{C}^n. \tag{3.7}$$

Then for every $z \in \Omega_r$, we have $\mid f_k.A_k(z) \mid \leq \mid f_k \mid . \| A_k \|_E .r^{s_k}$. So, for every $r \in]1,R[$ the series $\sum f_k.A_k$ is convergent in Ω_r, whence $\sum_{k\geq 1} f_k.A_k$ is analytic in Ω_R.

Now we shall show that μ is the (α,β)-order of f. By the Lemma 3.3, to complete the proof of the theorem it suffices to show that $\rho(\alpha,\beta) \geq \mu(\alpha,\beta)$. By definition of ρ, we have, for every $\epsilon > 0$ there exists $r_\epsilon \in]1,R[$ such that for every $r \in]r_\epsilon,R[$ $\log\Big(\|f\|_{\overline{\Omega}_r}\Big) \leq \alpha^{-1}\Big[(\rho(\alpha,\beta)+\epsilon).\beta(R/(R-r))\Big]$. Applying (2.6) and (3.3) we have, for every $k \in \mathbb{N}$ and $r > 1$ sufficiently close to R

$$\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq -s_k \log\Big(r/R\Big) + \log\Big(C_0.\frac{(r-1)^{N_\theta}}{(R-r)^{-(2N+1)}}\Big) + \log\Big(\|f\|_{\overline{\Omega}_r}\Big), \tag{3.8}$$

then $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq \varphi(r,s_k)$, where

$$\varphi(r,s_k) = -s_k \log\Big(r/R\Big) + \log\Big(C_0.\frac{(r-1)^{N_\theta}}{(R-r)^{-(2N+1)}} + \beta^{-1}\Big[(\rho(\alpha,\beta)+\epsilon).\beta(R/(R-r))\Big].$$

Put $\rho = \rho(\alpha,\beta)$ and $r_k = R.\left\{1 - \dfrac{1}{\beta^{-1}\left(\dfrac{1}{\rho+\epsilon}.\alpha\Big(\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}\Big)\right)}\right\}$. Replacing in the relation (3.8) r by r_k and applying the proprieties of the functions α and β:

$$\alpha\Big(x/\beta^{-1}(c\alpha(x))\Big) = (1+o(x))\alpha(x), \text{ for } c>0,\ x \to +\infty,$$

and the proprieties of the logarithm, we obtain $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq C_1.\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}$ where C_1 is a constant. Therefore $\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big) \leq C_1.\dfrac{s_k}{\beta^{-1}(\alpha(s_k)/(\rho+\epsilon))}$, thus

$$\beta\left(\frac{C_1.s_k}{\log\Big(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\Big)}\right) \geq \alpha(s_k)/(\rho+\epsilon).$$

Passing to the upper limit, after a simple calculus, we obtain $\mu(\alpha,\beta) \leq \rho(\alpha,\beta)$. □

4. Generalized type and coefficient characterizations with respect to extremal polynomial

The purpose of this section is to establish the relationship of the generalized type of an analytic function in Ω_R with respect to the set $\Omega_r = \Big\{z \in \mathbb{C} : \exp(V_E(z)) < r\Big\}$ and its coefficients in the development according to the sequence of extremal polynomials.

Let E be a compact L-regular and $f = \sum_{k\geq 1} f_k.A_k$ be an analytic function of (α,β)-order $\rho = \rho(\alpha,\beta)$, and put:

$$\tau_E(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\dfrac{s_k}{\log\big(|f_k|.\tau_k^{s_k}.R^{s_k}\big)}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{4.1}$$

We need the following proposition:

Proposition 4.1. *Let $f = \sum_{k\geq 0} f_k.A_k$ and E a L-regular compact. For every $r \in]1,R[$, we put*

$$\begin{cases} \overline{M}(f,r) = \sup_{k\in\mathbb{N}}\Big\{\mid f_k \mid . \parallel A_k \parallel_E .r^{s_k}\Big\} \\ \overline{\sigma_1}(\alpha,\beta) = \limsup_{r\to R}\dfrac{\alpha\Big(\log(\overline{M}(f,r))\Big)}{\Big(\beta(R/(R-r))\Big)^{\rho(\alpha,\beta)}} \end{cases}$$

then $\sigma(\alpha,\beta) \leq \overline{\sigma_1}(\alpha,\beta)$.

Proof. For $z \in \Omega_r$ and $k \in \mathbb{N}$, using the similar arguments and inequalities as in Lemma 2.3

$$\frac{\alpha\Big(\log(\|f\|_{\overline{\Omega}_r})\Big)}{\Big[\beta(R/(R-r))\Big]^{\rho(\alpha,\beta)}} \leq \frac{\alpha\Big(\log(\overline{M}(f,\sqrt{r.R}) - \log(1-\sqrt{r/R}))\Big)}{\Big[\alpha(R/(R-\sqrt{r.R}))\Big]^{\rho(\alpha,\beta)}}.\frac{\Big[\alpha(R/(R-\sqrt{r.R}))\Big]^{\rho(\alpha,\beta)}}{\Big[\alpha(R/(R-r))\Big]^{\rho(\alpha,\beta)}}.$$

We have $\limsup_{r\to R} \dfrac{\left[\alpha(R/(R-\sqrt{r.R}))\right]^{\rho(\alpha,\beta)}}{\left[\alpha(R/(R-r))\right]^{\rho(\alpha,\beta)}} = 1.$ □

Proceeding to the upper limit we get

$$(*) \quad \sigma(\alpha,\alpha) \le \overline{\sigma_1}(\alpha,\beta).$$

Theorem 4.1. *Let E be a compact L-regular and $f = \sum_{k\ge 1} f_k.A_k$. If f is of finite generalized (α,β)-order $\rho(\alpha,\beta)$, and*

$$\tau_E(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho(\alpha,\beta)}} < +\infty. \tag{4.2}$$

Then f is analytic in Ω_R, for some $R > 1$, and its (α,β)-type $\sigma(\alpha,\beta) = \tau_E(\alpha,\beta)$.

Proof. Put $\tau = \tau_E(\alpha,\beta)$, $\rho = \rho(\alpha,\beta)$, and $\sigma = \sigma(\alpha,\beta)$. The function is analytic by the definition $\tau_E(\alpha,\beta)$ and the arguments used in theorem 3.1.

1. Now we show that $\sigma(\alpha,\beta) \le \tau_E(\alpha,\beta)$. If $\tau < \infty$, by the definition of τ, for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k \ge k_\epsilon$ $\alpha(s_k) \le (\tau+\epsilon).\left\{\beta\left(\dfrac{s_k}{\log\left(\mid f_k\mid \tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho}$. A simple calculus gives for, $\overline{\tau} = \tau + \epsilon$.

$$\log\left(\mid f_k\mid \tau_k^{s_k}.R^{s_k}\right) \le \frac{s_k}{\beta^{-1}\left(\left(\dfrac{1}{\overline{\tau}}\alpha(s_k)\right)^{1/\rho}\right)}, \tag{4.3}$$

for every $k \ge k_\epsilon$for every $k \ge k_\epsilon$.
Since $\log\left(\mid f_k\mid .\tau_k^{s_k}.r^{s_k}\right) \le s_k\log(r/R) + \log\left(\mid f_k\mid \tau_k^{s_k}.R^{s_k}\right)$. By (4.3), we get

$$\log\left(\mid f_k\mid \tau_k^{s_k}.r^{s_k}\right) \le s_k\log(r/R) + \frac{s_k}{\beta^{-1}\left(\left(\dfrac{1}{\overline{\tau}}\alpha(s_k)\right)^{1/\rho}\right)}. \tag{4.4}$$

For every $r \in]1,R[$, and r and r sufficiently close to R, we put

$$\phi(x,r) = x\log(r/R) + \frac{x}{\beta^{-1}\left(\left(\dfrac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)}.$$

If we put $F = F(x,\overline{\tau},\dfrac{1}{\rho}) = \beta^{-1}\left(\left(\dfrac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)$ then $\phi(x,r) = x\log(r/R) + \dfrac{x}{F}$, and the maximum of the function $x \to \phi(x,r)$ is reached for $x = x_r$ solution of the equation of

$$\frac{d\phi(x,r)}{dx} = \frac{\partial\phi}{\partial x}(x,r) = \log(r/R) + \frac{d}{dx}\left\{\frac{x}{F}\right\} = 0.$$

We have $\dfrac{\phi(x,r)}{dx} = 0 \Leftrightarrow \log\left(\dfrac{r}{R}\right) + \dfrac{F - x.\dfrac{dF}{dx}}{(F)^2} = 0$, or $F = \dfrac{1 - \dfrac{x}{F}.\dfrac{dF}{dx}}{\log(R/r)}$. Since $\dfrac{dF}{dx} = \dfrac{dF}{d\log(x)}.\dfrac{d\log(x)}{dx} = \dfrac{1}{x}.\dfrac{dF}{d\log(x)}$, we get $F = \dfrac{1 - \dfrac{1}{F}.\dfrac{dF}{d\log(x)}}{\log(R/r)} = \dfrac{1 - \dfrac{d\log F}{d\log(x)}}{\log(R/r)}$, or

$$\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right) = \frac{1 - \dfrac{d\log\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)}{d\log(x)}}{\log(R/r)}.$$

We deduce $x = x_r = \alpha^{-1}\left\{\left[\overline{\tau}.\beta\left(\dfrac{1 - d\log\left(\beta^{-1}\left(\left(\frac{1}{\overline{\tau}}\alpha(x)\right)^{1/\rho}\right)\right)/d(\log(x))}{\log\left(R/r\right)}\right)\right]^{\rho}\right\}$. We have $\log\left(\dfrac{r}{R}\right) = \log\left(\dfrac{r-R}{R} + 1\right) \sim \dfrac{r-R}{R}$ $\left(\text{because } \dfrac{r-R}{R} \to 0\right)$ and $\left|\dfrac{d\left[\log\left(\beta^{-1}\left(\left(\alpha(x)\right)^{\rho}\right)\right)\right]}{d\log(x)}\right| \leq b$, where b is a positive constant. Then by the proprieties of α we get

$$x_r = (1 + o(1))\rho.\beta^{-1}\left(\overline{\tau}(\alpha(R/(R-r)))^{\rho}\right).$$

By (4.4), we have $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq \sup_{r\in\mathbb{N}} \phi(x,r) = \phi(x_r, r)$. Replacing s_k by x_r in this last relation we obtain $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq \dfrac{(1+o(1))\beta^{-1}\left(\overline{\tau}(\alpha(R/(R-r)))^{\rho}\right)}{R/(R-r)}$. Since $\dfrac{R}{R-r} > 1$ and $\dfrac{\rho-1}{\rho} < 1$, then $\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq C.\beta^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)$.

Then $\sup_{k\in\mathbb{N}} \log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k} \leq C.\alpha^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)\right.$ or $\log(\overline{M}(f,r)) \leq C.\beta^{-1}\left(\overline{\tau}.(\alpha(R/(R-r)))^{\rho}\right)$.

Therefore $\dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \overline{\tau}$.

Proceeding to the upper limit for $r \to R$, get $\overline{\sigma}_1(\alpha,\alpha) = \lim_{r\to R} \dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \tau$.

By the relations (*) of the proposition 4.1 we obtain $\sigma(\alpha,\alpha) = \lim_{r\to R} \dfrac{\alpha(\log(\overline{M}(f,r)))}{(\alpha(R/(R-r)))^{\rho}} \leq \tau$.

Thus $\sigma(\alpha,\beta) \leq \tau_E(\alpha,\beta)$. The result is obviously holds for $\tau = +\infty$.

2. Now we show that $\sigma(\alpha,\beta) \geq \tau_E(\alpha,\beta)$. Put $\overline{\sigma} = \sigma(\alpha,\beta) + \epsilon$, $\rho = \rho(\alpha,\beta)$. Suppose that $\sigma < \infty$. By definition of $\sigma(\alpha,\beta)$, we have for every $\epsilon > 0$, there exist $r_\epsilon \in]1, R[$, such that for every

$r > r_\epsilon$ ($R > r > r_\epsilon > 1$) $\log\left(\|f\|_{\overline{\Omega}_r}\right) \leq \alpha^{-1}\left[\overline{\sigma}.\left(\beta(R/(R-r))\right)^{\rho}\right]$. Applying (3.3) and (2.6) we get, for every $k \in \mathbb{N}$ and r sufficiently close to R:

$$\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right) \leq -s_k \log(r) + \log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \log\left(\|f\|_{\overline{\Omega}_r}\right).$$

As for every $r \in]1, R[$ $\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) = -s_k \log\left(r/R\right) + \log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right)$ then $\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) \leq -s_k \log\left(r/R\right) + \log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \log\left(\|f\|_{\overline{\Omega}_r}\right)$. or $\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) \leq -s_k \log\left(r/R\right) + \log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \alpha^{-1}\left[\overline{\sigma}.\left(\beta(R/(R-r))\right)^{\rho}\right]$.

Since $s_k \geq 1$, we obtain, for k sufficiently large, $\frac{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}{s_k} \leq \omega(r,k)$ where $\omega(r,k) = -\log\left(r/R\right) + \frac{1}{s_k}\log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \frac{1}{s_k}\alpha^{-1}\left[\overline{\sigma}.\left(\beta(R/(R-r))\right)^{\rho}\right]$.

Since $\lim\limits_{k\to+\infty} \frac{1}{s_k}\log\left(C_0.\frac{(r+1)^{N_\theta}}{(r-1)^{(2N+1)}}\right) + \frac{1}{s_k}\alpha^{-1}\left[\overline{\sigma}.\left(\beta(R/(R-r))\right)^{\rho}\right] = 0$ we get, for r sufficiently close to R, $\lim\limits_{k\to+\infty} \omega(r,k) = -\log\left(r/R\right) = \log\left(R/r\right)$.

Then for k sufficiently large and r sufficiently close to R, we have $\omega(r,k) = (1+o(1))\log\left(R/r\right)$, $k \to +\infty$, then

$$\frac{1}{s_k}\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right) \leq (1+o(1))\log\left(R/r\right). \tag{4.5}$$

Choose $r_k = R.\dfrac{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho}}{1+\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho}}$. Using the relation (4.5) and the proprieties of the function $t \to \log(t)$, we obtain, for r sufficiently close to R $\frac{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}{s_k} \leq (1+o(1))\left(\frac{R}{r}-1\right)$.

because $\log\left(\frac{R}{r}\right) = \log\left(\frac{R-r+r}{r}\right) = \log\left(1+\frac{R-r}{r}\right) \sim \frac{R-r}{r}$ $(r \to R)$.

Replacing r by the chosen r_k in this last relation we obtain $\frac{R-r_k}{r_k} = \dfrac{1}{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho}}$.

Then, for r sufficiently close to R and k r sufficiently large we get $\frac{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}{s_k} \leq \dfrac{1}{\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho}}$, thus $\beta^{-1}\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho} \leq \frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}$ or $\left(\frac{1}{\overline{\sigma}}\alpha(s_k)\right)^{1/\rho} \leq \beta\left(\frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.R^{s_k}\right)}\right)$.

Therefore $\frac{1}{\overline{\sigma}}\alpha(s_k) \leq \left\{\beta\left(\frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right)}\right)\right\}^{\rho}$ or $\dfrac{\alpha(s_k)}{\left\{\beta\left(\frac{s_k}{\log\left(\mid f_k \mid \tau_k^{s_k}.r^{s_k}\right)}\right)\right\}^{\rho}} \leq \overline{\sigma} = \sigma + \epsilon$.

Proceeding to the upper limit we obtain $\sigma(\alpha,\beta) \geq \tau_E(\alpha,\beta)$. The result is obviously holds for $\sigma(\alpha,\beta) = +\infty$.

□

5. Generalized (α,β)-growth and best polynomial approximation of analytic functions in L^p-norm.

Let E a L-regular compact of $\mathbb{C}^n$. The purpose of this paragraph is to give the relationship between the generalized order of an analytic function and speed of convergence to 0 in the best polynomial in L_p-norm on E. We need the following lemma.

Lemma 5.1. . *Let* $f = \sum_{k\geq 0} f_k.A_k$ *an element of* $L^p(E,\mu)$, *for* $p \geq 0$, *and*

$$\pi_k^p(E,f) = \inf\left\{\left\|f - P\right\|_{L^p(E,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}.$$

Then

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} \tag{5.1}$$

and

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\left\{\beta\left(\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right)\right\}^{\rho(\alpha,\beta)}} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\left\{\beta\left(\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{5.2}$$

Proof. Assume that $p \geq 2$. If $f \in L^p(E,\mu)$ where $p \geq 2$, then $f = \sum_{k=0}^{+\infty} f_k.A_k$ with convergence in $L^2(E,\mu)$, hence for $k \geq 0$, $f_k = \dfrac{1}{{\nu_k}^2}\displaystyle\int_E f.\overline{A}_k d\mu$ and therefore $f_k = \dfrac{1}{{\nu_k}^2}\displaystyle\int_E (f - P_{k-1}).\overline{A}_k d\mu$ (because $deg(A_k) = s_k$). Since the relation, $\left|f_k\right| \leq \dfrac{1}{{\nu_k}^2}\displaystyle\int_E \left|f - P_{k-1}\right|.\left|\overline{A}_k\right|\mu$ is satisfied, is easily verified by using inequalities Bernstein-walsh and Holder that we have for all $\varepsilon > 0$

$$\left|f_k\right|.\nu_k \leq C_\varepsilon.(1+\varepsilon)^{s_k}.\pi_{s_k-1}^p(E,f). \tag{5.3}$$

for all $k \geq 0$.

If $1 \leq p < 2$, let p' such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have $p' \geq 2$. According to the inequality of Hölder we have: $\left|f_k\right|.\nu_k^2 \leq \left\|f - P_{k-1}\right\|_{L^p(E,\mu)}.\left\||A_k|\right\|_{L^{p'}(E,\mu)}$. But $\left\|A_k\right\|_{L^{p'}(E,\mu)} \leq C.\left\|A_k\right\|_E = C.a_k(E)$. This shows, according to inequality (BM), that: $\left|f_k\right|.\nu_k^2 \leq C.C_\varepsilon.(1+\varepsilon)^{s_k}.\left\|f - P_{s_k-1}\right\|_{L^p(E,\mu)}$.

Hence the result $\left|f_k\right|.\nu_k^2 \leq C'_\varepsilon.(1+\varepsilon)^{s_k}.\pi_k^{s_k-1}(E,f)$. In both cases we have therefore

$$\left|f_k\right|.\nu_k^2 \leq A_\varepsilon.(1+\varepsilon)^{s_k}.\pi_{s_k-1}^p(E,f) \tag{5.4}$$

where A_ε is a constant which depends only on ε.
After passing to the upper limit in the relation (5.4) and applying the relation (3.3) we get

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} \leq \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}.$$

To prove the other inequality we consider the polynomial of degree s_k, $P_k(z) = \sum_{s_j=0}^{k} f_j.A_j$ then $\pi_{s_k-1}^p(E,f) \leq \sum_{s_j=s_k}^{+\infty} |f_j|.\left\|A_j\right\|_{L^p(E,\mu)} \leq C_0 \sum_{s_j=s_k}^{+\infty} |f_j|.\left\|A_j\right\|_E$. By Bernstein-Walsh inequality we have $\pi_k^p(E,f) \leq C_\epsilon \sum_{s_j=s_k}^{+\infty} (1+\epsilon)^{s_j}\left|f_j\right|.\nu_j$ for $k \geq 0$ and $p \geq 1$. If we take as a common factor $(1+\varepsilon)^{s_k}.\left|f_k\right|.\nu_k$ the other factor is convergent thus we have $\pi_k^p(K,f) \leq C(1+\epsilon)^{s_k}.\left|f_k\right|.\nu_k$ and by (3.3) we have, then

$$\pi_k^p(E,f) \leq C(1+\epsilon)^{2s_k}.\left|f_k\right|.\tau_k^{s_k}. \tag{5.5}$$

We deduce $\limsup_{k\to+\infty} \dfrac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} \geq \limsup_{k\to+\infty} \dfrac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}$. □

Applying this Lemma 5.1 we get the following main result:

Theorem 5.1. *Let $f \in L^p(E,\mu)$, then f is μ-almost-surely the restriction to E of an analytic function in $\mathbb{C}^n$ of finite generalized order $\rho(\alpha,\beta)$ if and only if*

$$\rho(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} + \infty. \tag{5.6}$$

Theorem 5.2. *Let $f \in L^p(E,\mu)$, then f is μ-almost-surely the restriction to E of an analytic function in $\mathbb{C}^n$ of finite generalized order $\rho(\alpha,\beta)$ and finite generalized type $\sigma(\alpha,\beta)$ if and only if*

$$\sigma(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\left\{\beta\left(\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right)\right\}^{\rho(\alpha,\beta)}}. \tag{5.7}$$

Proof. We prove only the first Theorem 5.1, the second is proved by the same arguments.

Suppose that f is μ-almost-surely the restriction to E of an entire function g of general order ρ $(0 < \rho < +\infty)$ and show that $\rho = \rho(\alpha, \beta)$.

We have $g \in L^p(E,\mu)$, $p \geq 2$ and $g = \sum_{k\geq 0} g_k.A_k$ in $L^2(E,\mu)$ Since g is an element of $L^2(E,\mu)$ then $g = \sum_{k=0}^{+\infty} g_k.A_k$ and according to the Theorem 3.1 $\rho(g,\alpha,\beta) = \limsup_{k\to+\infty} \dfrac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]}$ and with the Lemma 5.1 (relation(5.1)) we have $\limsup_{k\to+\infty} \dfrac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \dfrac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]}$.

But $g = f$ on E hence $\rho = \limsup_{k\to+\infty} \dfrac{\alpha(s_k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$.

Now suppose that f is a function of $L^p(E,\mu)$ such that the relation (5.6) is verified. The proof is done in three steps $p \geq 2$, $1 \leq p < 2$ and $0 < p < 1$.

Step.1. Let $p \geq 2$, then $f = \sum_{k=0}^{+\infty} f_k.A_k$, because f is an element of $L^2(E,\mu)$ $\left((L^p(E,\mu))_{p\geq 1}\right.$ is decreasing sequence$\left.\right)$. Consider in $\mathbb{C}^n$ the series $\sum f_k.A_k$, $k \geq 0$. By the relation (5.6) and the inequality (BW) we have the inequality on coefficients $|A_k|$ (2.4), it can be seen that this series converges normally on all compact of $\mathbb{C}^n$, to an analytic function denoted f_1. We have $f_1 = f$, obviously, μ-almost surly on E.

We verify easily that this series converges normally on all compact of $\mathbb{C}^n$ to an analytic function denoted f_1. We have $f_1 = f$, obviously, μ-almost surly on E, and by Theorem 3.1 we have

$$\limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[\dfrac{s_k}{\log\left(|f_k|.\tau_k^{s_k}.R^{s_k}\right)}\right]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty.$$

According to the Lemma 5.1 we get $\rho(f_1) = \limsup_{k\to+\infty} \dfrac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$.

Let $f_1 = \sum_{k\geq 0} f_k.A_k$, then $f_1(z) = f(z)$ μ-almost surely for every z in E. Therefore the (α,β)-order of f_1 is: $\rho(f_1,\alpha,\beta) = \limsup_{k\to+\infty} \dfrac{\alpha(k)}{\beta\left[\dfrac{k}{\log\left(\pi_k^p(E,f).R^k\right)}\right]} < +\infty$ (see Theorem 3.1). By Lemma 5.1 we check $\rho(f_1) = \rho$ so the proof is completed.

Step.2. Now let $p \in [1,2[$ and $f \in L^p(E,\mu)$. By (BM) inequality and Hölder inequality we have again the inequality the relation (5.4) and by the previous arguments we obtain the result.

Step.3. Let $0 < p < 1$, of course, for $0 < p < 1$ the L_p-norm does not satisfy the triangle inequality. But our relations (5.3) and relation (5.4) are also satisfied for $0 < p < 1$ (see Kumar (2011)), because using Holder's inequality we have, for some $M > 0$ and all $r > p$ (p fixed)

$$\|\|\| f \|_{L^p(E,\mu)} \leq M. \| f \|_{L^r(E,\mu)} .$$

Using the inequality $\int_E \mid f \mid^p \, d\mu \leq \| f \|_E^{p-r} . \int_E \mid f \mid^r \, d\mu$ we get $\| f \|_{L^p(E,\mu)} \leq \| f \|_E^{1-(r/p)} . \| f \|_{L^r(E,\mu)}^{r/p}$. We deduce that (E, μ) satisfies the Bernstein.Markov inequality. For $\epsilon > 0$ there is a constant $C = C(\epsilon, p) > 0$ such that, for all (analytic) polynomials P we have

$$\|\|\| P \|_E \leq C(1+\epsilon)_{deg(P)}. \ \| P \|_{L^p(E,\mu)} .$$

Thus if (E, μ) satisfies the Bernstein-Markov inequality for one $p > 0$ then (5.4) and (5.5) are satisfied for all$p > 0$.

The rest of proof is easily deduced using the same reasoning as in step 1 and step 2. □

References

Harfaoui, M. (2010). Generalized order and best approximation of entire function in $L^p - norm$. *Int. J. Math. Mathematical Sciences* **2010**, 15 pages.

Harfaoui, M. (2011). Generalized growth of entire function by means best polynomial approximation in L^p-norm. *JP Journal of Mathematical Sciences* **1**(2), 111–126.

Janik, A. (1984). A characterisation of the growth of analytic functions by means of polynomial approximation. *Univ. Iagel. Acta Math.* **24**, 295–319.

Janik, A. (1991). On approximation of analytic functions and generalized ordersf polynomial. *Annales Polinici Mathematici* **55**, 163–167.

Kumar, D. (2011). Generalized growth and best approximation of entire functions in L^p-norm in several complex variables. *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **57**(2), 353–372.

Siciak, J. (1962). On some extremal functions and their applications in the theory of analytic functions of several complex variables. *Trans. Am. Math. Soc.* **105**, 322–357.

Siciak, J. (1977). Extremal plurisubharmonic functions in $\mathbb{C}^n$. In: *Proceedings of the first Fininish-Polish Summer School in Complex Analysis at Prodlesice (Lodz 1977), University of Lodz*. pp. 115–152.

Zeriahi, A. (1983). Families of almost everywhere bounded polynomials. *Bulletin des Sciences Mathématiques* **107**(1), 81–91.

Zeriahi, A. (1987). Meilleure approximation polynomiale et croissance des fonctions entières sur certaines variétés algébriques affines. (Best polynomial approximation and growth of entire functions on certain affine algebraic varieties). *Ann. Inst. Fourier* **37**(2), 79–104.