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Satellite Formation Control Using the Approximating Sequence Riccati Equations

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Abstract

In this study we develop a reliable algorithm to control the satellite formation using the Approximating Sequence of Riccati Equations(ASRE) minimizing the fuel consumption and the deviation of the orbit from the nominal orbit. The nonlinear Clohessy -Wiltshire(CW) equations of motions are used to describe the motion of the satellite formation about a virtual reference position maintained at the formation center. The nonlinear dynamics of the system will be factorized in such a way that the new factorized system is accessible. The problem is tackled using the Approximating Sequence Riccati Equations(ASRE) method. The technique is based on Linear Quadratic Regulator(LQR) with fixed terminal state, which guarantees closed loop solution.

Keywords: Nonlinear Feedback, Linear Quadratic Regulator, Approximation Sequence Riccati Equation, Satellite Formation.

1. Introduction

Satellite formation flying is one of the space dynamics branches which gained much consideration in recent years. Despite the topic evolved two decades ago, the implementation of formation flying is not yet mature.

A satellite formation consists of two or more satellite flying together in close proximity, cooperating together to achieve some space mission such as terrestrial or deep space one. This system of distributed satellites has several advantages over the single satellite system such as, larger capability, reliability, flexibility, and more importantly less cost. *Satellite formation in contrast to satellite constellation in which the satellites are moving independently, the satellites affecting each other in co orbital motion about a virtual reference position maintained at the formation center.* The

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nonlinear Clohessy -Wiltshire equations of motions are used to describe the motion of the satellite formation. CW equations are developed for rendezvous(Clohessy & Wiltshire, 1960). Later on the linear inhomogeneous CW are studied(Meirovitch, 1970). CW equations have been solved by simplifying the nonlinear equations of motion via coordinate transformation of the central gravity field dynamics in presence of quadratic drag force(Thomas Carter, 2002).

The nonlinear dynamics of the system will be factorized in such a way that the new factorized system is accessible. The problem is tackled using the Approximating Sequence Riccati Equations method. The most common way of solving the orbit rendezvous of a satellite is the low thrust orbit rendezvous approach, which is a nonlinear optimal control problem. In the open loop context the problem can be solved via indirect and then direct method. The indirect method was developed through Pontryagin Maximum Principle(PMP) (A. J. Bryson, 1975),(L. Pontryagin & Mishchenko, 1952) . The direct method was developed using the Karush-Kuhn-Tucker(KKT) algebric equation(Enright & Conway, 1992).

one of the most common methods for solving the nonlinear feedback optimal control problem in the is the State Dependent Riccati Equations(SDRE) (Cimen, 2006). The Approximating Sequence of Riccati Equations (Cimen, 2004) technique is an iterative approach to solve the nonlinear optimal control problem. The ASRE is developed(Topputo & Bernelli-Zazzera, 2012) using the state transition matrix. By the virtue of the closed-loop nature of this control law, a trajectory designed in this way has the property to respond to perturbations acting during the transfer that continuously alter the state of the spacecraft. The optimal feedback control for linear systems with quadratic objective functions is addressed through the matrix Riccati equation: this is a matrix differential equation that can be integrated backward in time to yield the initial value of the Lagrange multipliers (A. J. Bryson, 1975). Recently, the nonlinear feedback control of circular coplanar low-thrust orbital transfers has been faced using continuous orbital elements feedback and Lyapunov functions(Chang & Marsden, 2002) and proved optimal by(Alizadah & Villac, 2011). Later on the problem has been solved using the primer vector approximation method(Haung, 2012). The problem is tackled using the Approximating Sequence Riccati Equation(ASRE) method based on Linear Quadratic Regulator(LQR) with fixed terminal state and the method is applied to GNSS circular constellation (Owis, 2013). In this work the control of the satellite formation described in the Earth Centered Earth Fixed Frame Fig. 1 is developed.

Linear Quadratic Regulator(LQR) with Fixed Terminal State

Consider the following system with linear dynamics and quadratic performance index as follows:

$$\dot{X} = AX + BU, \qquad X(t_0) = X_0 \in \mathbb{R}^n, \tag{1.1}$$

the following performance index

$$J = X_f^T Q_f X_f + \frac{1}{2}\int_{t_0}^{t_f} [X^T QX + U^T RU]dt, \tag{1.2}$$

Where A , B , Q , and R are constant coefficients matrices of the suitable dimen-

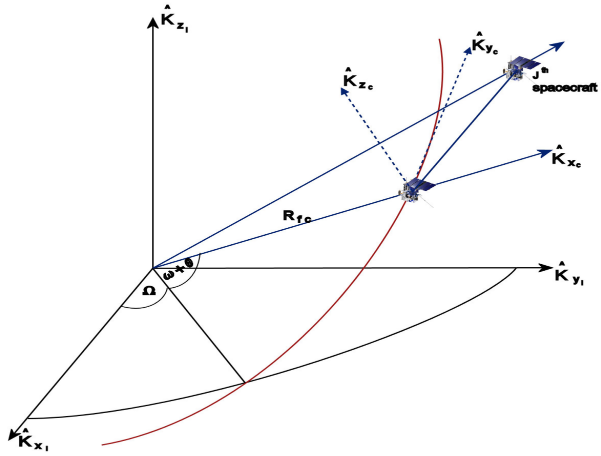

Figure 1. Satellite Formation Flying in the Earth Centered Earth Fixed Frame

sions. we have to find the m-dimensional control functions $U(t)$, $t \in [t_0\ t_f]$ which minimizes the J, which is an open loop (with t_0 fixed) optimal control. We optimize the performance index J, by adjoining the dynamics and the performance index (integrand) to form the Hamiltonian:

$$H(X, \lambda, U, t) = \frac{1}{2}(X^T QX + U^T RU) + \lambda^T (A(t)X + B(t)U),$$

where the Lagrange multiplier λ is called the adjoint variable or the costate. The necessary conditions for optimality are:

1. $\dot{X} = H_\lambda = A(t)X + B(t)U, \qquad X(t_0) = X_0,$
2. $\dot{\lambda} = -H_x = -QX - A^T\lambda, \qquad \lambda(t_f) = Q_f X_f,$
3. $H_u = 0 \Longrightarrow RU + B^T\lambda = 0 \Longrightarrow U^\star = -R^{-1}B^T\lambda.$

To find the minimum solution we have to check for $H_{uu} = \dfrac{\partial^2 H}{\partial \lambda^2} > 0$ or equivalently $R > 0$. Now we have that

$$\dot{X} = AX + BU^\star = AX - BR^{-1}B^T\lambda,$$

which can be combined to the the equation of the costate as follows

$$\begin{bmatrix} \dot{X} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ \lambda \end{bmatrix}, \tag{1.3}$$

which is called the Hamiltonian matrix, it represents a $2n$ boundary value problem with $X(t_0) = X_0$ and, $\lambda(t_f) = Q_f X_f$.

We can solve this $2n$ boundary value problem using the transition matrix method as follows. Let's define a transition matrix

$$\phi_(t_1, t_0) = \begin{bmatrix} \phi_{11}(t_1, t_0) & \phi_{12}(t_1, t_0) \\ \phi_{21}(t_1, t_0) & \phi_{22}(t_1, t_0) \end{bmatrix},$$

we use this matrix to relate the current values of X and λ to the final values X_f and λ_f as follows

$$\begin{bmatrix} X \\ \lambda \end{bmatrix} = \begin{bmatrix} \phi_{11}(t, t_f) & \phi_{12}(t, t_f) \\ \phi_{21}(t, t_f) & \phi_{22}(t, t_f) \end{bmatrix} \begin{bmatrix} X(t_f) \\ \lambda(t_f) \end{bmatrix},$$

so we have

$$\begin{aligned} X &= \phi_{11}(t, t_f)X(t_f) + \phi_{12}(t, t_f)\lambda(t_f) \\ &= [\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f]X(t_f), \end{aligned}$$

we can eliminate $X(t_f)$ to get

$$\begin{aligned} X &= [\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f][\phi_{11}(t_0, t_f) + \phi_{12}(t_0, t_f)Q_f]^{-1}X(t_0) \\ &= X(t, X_0, t_0), \end{aligned}$$

now we can find $\lambda(t)$ in terms of $X(t_f)$ as

$$\lambda(t) = [\phi_{21}(t, t_f) + \phi_{22}(t, t_f)Q_f]X(t_f),$$

then we can eliminate $X(t_f)$ to get

$$\begin{aligned} \lambda(t) &= [\phi_{21}(t, t_f) + \phi_{22}(t, t_f)Q_f][\phi_{11}(t, t_f) + \phi_{12}(t, t_f)Q_f]^{-1}X(t), \\ &= \phi_{\lambda x}X(t). \end{aligned}$$

Now we search a solution for $\phi_{\lambda x}$. By differentiating $\lambda(t)$ we get

$$\dot{\lambda}(t) = \dot{\phi}_{\lambda x}X(t) + \phi_{\lambda x}\dot{X}(t).$$

Comparing the last equation with the Hamiltonian matrix we get

$$-QX(t) - A^T\lambda(t) = \dot{\phi}_{\lambda x}X(t) + \phi_{\lambda x}\dot{X}(t),$$

then we have

$$\begin{aligned} -\dot{\phi_{\lambda x}}(t)X(t) &= QX(t) + A^T\lambda(t) + \phi_{\lambda x}\dot{X}(t) \\ &= QX(t) + A^T\lambda(t) + \phi_{\lambda x}(AX - BR^{-1}B^T\lambda(t)) \\ &= (Q + \phi_{\lambda x}A)X(t) + (A^T - \phi_{\lambda x}BR^{-1}B^T)\lambda(t) \\ &= (Q + \phi_{\lambda x}A)X(t) + (A^T - \phi_{\lambda x}BR^{-1}B^T)\phi_{\lambda x}X(t) \\ &= [Q + \phi_{\lambda x}A + A^T\phi_{\lambda x} - \phi_{\lambda x}BR^{-1}B^T\phi_{\lambda x}]X(t). \end{aligned}$$

Since this is true for arbitrary $X(t)$, $\phi_{\lambda x}$ must satisfy

$$-\dot{\phi}_{\lambda x}(t) = Q + \phi_{\lambda x}A + A^T\phi_{\lambda x} - \phi_{\lambda x}BR^{-1}B^T\phi_{\lambda x}, \tag{1.4}$$

which is the matrix differential Riccati Equation . We can solve for $\phi_{\lambda x}$ by solving Riccati Equation backwards in time from t_f with $\phi_{\lambda x}(t_f) = Q_f$. The optimal control is then given by

$$U^\star = -R^{-1}B^T\lambda(t) = -R^{-1}B^T\phi_{\lambda x}X = -K(t)X(t, X_0, t_0). \tag{1.5}$$

From 1.5 we notice that the optimal control is a linear full-state feedback control, therefore the linear quadratic terminal controller is feedback by default.

2. The Approximating Sequence of Riccati Equations(ASRE)

Assume that we have the following nonlinear system

$$\dot{X} = f(X, U, t) \tag{2.1}$$

$$X(t_0) = X_0, \quad X(t_f) = X_f \in R^n \tag{2.2}$$

with performance index

$$J = \phi(X_f, t_f) + \int_{t_0}^{t_f} L(X, U, t)dt \tag{2.3}$$

This system can be rewritten in the state dependent quasi-linear system as follows

$$\dot{X}^i = A(X^{i-1})X^i + B(X^{i-1})U^i \tag{2.4}$$

$$X(t_0) = X_0^0, \quad X(t_f) = X_f^n \in R^n \tag{2.5}$$

$$J = X_f^{i\,T} Q(X_f^{i-1}) X_f^i + \frac{1}{2}\int_{t_0}^{t_f} [X^{i^T} Q(X^{i-1}) X^i \; + U^{i^T} R(X^{i-1}) U^i] dt, \qquad (2.6)$$

where i represents the iteration step over the time interval $[t_i - 1, t_i]$ Fig. the technique is based of the previously introduced Linear Quadratic Regulator with fixed terminal state, which is a full state feedback and therefore the obtained solution will be a closed loop one, I.e. able to respond to the unexpected change in the inputs. The technique works as follows: the initial state is used to compute A_0, and B_0 and we solve for the first LQR iteration and compute X^1 and then used to compute new value of A_1, and B_1 for the second iteration until the final state error reaches a value below a set threshold.

3. Satellite formation control

Consider a satellite in the central gravity field. The eqaution of motion can be written in the cartesian frame as follows

$$\ddot{\mathbf{r}}) = -\frac{\mu}{r^3}\mathbf{r} + \frac{\mathbf{f}}{m} \qquad (3.1)$$

Where μ is the gravitational constant of the Earth($3.986005 \times 10^{14} m^3/s^2$). In the rotating coordinate frame along a circular orbit at a constant angular velocity, the position, velocity, and the acceleration become

$$\begin{aligned} \mathbf{r} &= \mathbf{R} + \delta\mathbf{r} = (R + x)i + yj + zk \\ \dot{\mathbf{r}} &= (\dot{x} - \omega y)i + [(\dot{y} + \omega(R + x)]j + \dot{z}k \\ \ddot{\mathbf{r}} &= [\ddot{x} - 2\omega\dot{y} - \omega^2(R + x)]i + [(\ddot{y} + 2\omega\dot{x}) - \omega^2 y]j + \ddot{z}k \end{aligned} \qquad (3.2)$$

Plugging third equation of (3.2) into equ. (3.1) and substituting $r = \sqrt{[(R + x)^2 + y^2 + z^2]}$ we get

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - \omega^2(R + x) &= -\frac{\mu}{r^3}(R + x) + U_x \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y &= -\frac{\mu}{r^3}y + U_y \\ \ddot{z} &= -\frac{\mu}{r^3}z + U_z \end{aligned} \qquad (3.3)$$

If we nondimensionalize the problem by setting the radius of the reference orbit $R = 1$ and reference time $\frac{1}{\omega}$ and in this system of units the gravitational constant μ is unity the nondimensionalized equation of motion can be written as

$$\begin{aligned} \ddot{x} - 2\dot{y} - (1 + x)(\frac{1}{r^3} - 1) &= U_x \\ \ddot{y} + 2\dot{x} + y(\frac{1}{r^3} - 1) &= U_y \\ \ddot{z} + \frac{1}{r^3} z &= U_z \end{aligned} \tag{3.4}$$

where $r = \sqrt{[(1 + x)^2 + y^2 + z^2]}$, for simplicity we consider the in plan motion. We define the state vector of the system

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} \tag{3.5}$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \end{bmatrix} \tag{3.6}$$

Then Equation (3.4) can be written in the form :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{3.7}$$

Choosing a suitable factorization equation (3.7) is rewritten in the factored state variable form :

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{3.8}$$

where :

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Gamma + \frac{\Gamma}{x_1} & 0 & 0 & 2 \\ 0 & \Gamma & 2 & 0 \end{bmatrix} \tag{3.9}$$

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.10}$$

where $\Gamma = \frac{1}{r^3} - 1$

4. Factored Controllability

For the factored system (3.8) the controllability is established by verifying that the controllability matrix

$$\mathbf{M}_{cl} = [\mathbf{B}\ \mathbf{AB}\ \mathbf{A}^2\mathbf{B}\ \mathbf{A}^3\mathbf{B}]$$

has a rank equals to $n = 4\ \forall x$ in the domain.
Since $\mathbf{A}$ and $\mathbf{B}$ have nonvanishing rows the controllability matrix $\mathbf{M}_{cl}$ for the System (3.8) is of rank 4.

Nondimensionalization of the problem In order to simplify the calculation we dimensionalize the system by removing the units from the equations of motion via multiplying or dividing some constants. The two constant we divid by are the radial distance of the initial orbit and the gravitational constant μ in this case the radius of the initial orbit is unity and velocity is divided by the circular velocity of the initial orbit $\sqrt{\frac{\mu}{r_0^2}}$ and the time is multiplied by $\sqrt{\frac{\mu}{r_0^3}}$ In application we would like to make an optimal orbit transfer(i.e. from $(r = 1)$ to $(r = 1.2)$ in time $t_f = 4.469, 5.2231$ (time unit) Fig. 2 with optimal velocity Fig. 3 and optimal control function of both radial and tangential components Figs. 4, 5. The initial angle is $(\theta_0 = \frac{\pi}{2})$ and the final angle is $(\theta_f = \frac{3\pi}{2})$. $\dot{r}_0 = 0$ and $\dot{r}_f = 0$ for the initial and final orbits. $\dot{\theta}_0 = \sqrt{\frac{1}{r_0^3}} = 1$ and $\dot{\theta}_f = \sqrt{\frac{1}{r_f^3}} = 0.54433105395$. In the second $\theta_f = \frac{5\pi}{2}$ with $t_f = 6.866$.
in example the matrices $\mathbf{Q}$ and $\mathbf{R}$ are the identity matrices.

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

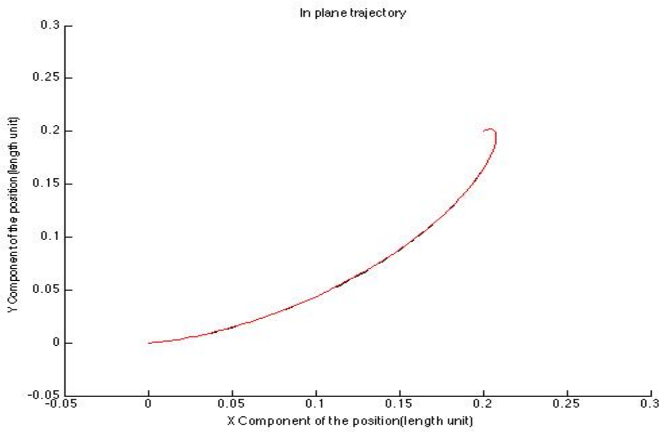

Figure 2. Trajectory of orbit rendezvous manoeuvre in the non dimensional coordinates

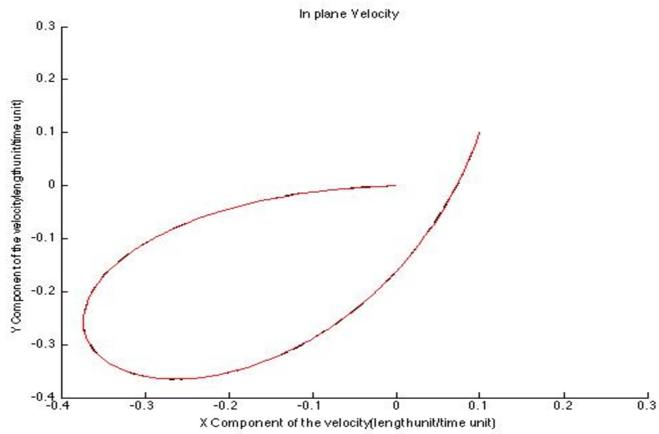

Figure 3. Velocity in the non dimensional coordinates

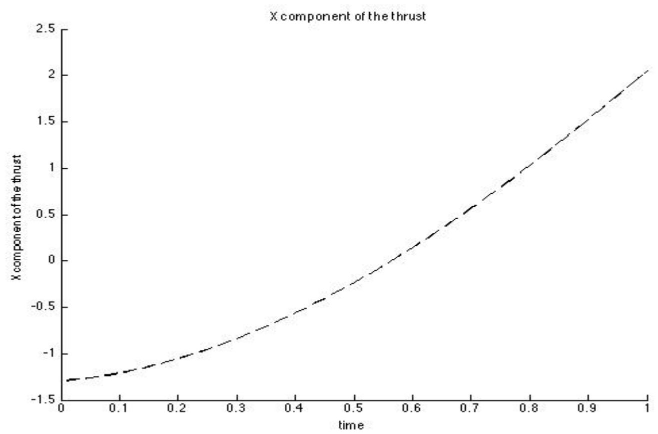

Figure 4. Control X component in the non dimensional coordinates

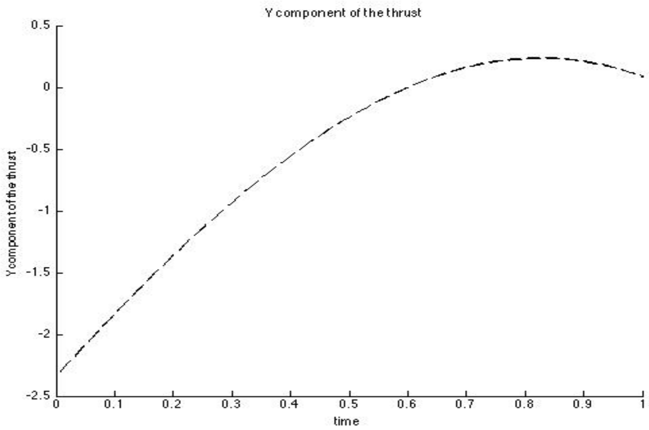

Figure 5. Control Y component in the non dimensional coordinates

5. Conclusion

The nonlinear orbital dynamics of the satellite formation with respect to the Earth Center Earth Fixed Coordinates are developed. The feedback optimal control of the satellite formation can be solved by factorizing the original nonlinear dynamics into accessible (weakly controllable) linear dynamics of state dependent factors. The factorized problem has been solved using the the Approximating Sequence Riccati Equations(ASRE) method. The technique is based on Linear Quadratic Regulator(LQR) with fixed terminal state, which guarantees closed loop solution. A computer simulation verified that the adopted technique is relaible.

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