Theory and Applications of Mathematics & Computer Science

(ISSN 2067-2764, EISSN 2247-6202)
http://www.uav.ro/applications/se/journal/index.php/tamcs

Theory and Applications of Mathematics & Computer Science 3 (2) (2013) 93–102

Starlikeness and Convexity of Certain Classes of Meromorphically Multivalent Functions

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Abstract

The purpose of this paper is to investigate the problems of finding the order of starlikeness and the order of convexity of the products of certain meromorphically p-valent functions belonging to some interesting classes of β-uniformly p-valent starlike functions and β-uniformly p-valent convex functions in the open unit disk $\mathbb{U}$. The main results presented in the paper are capable of being specialized suitably in order to deduce the solutions of the corresponding problems for relatively more familiar subclasses of meromorphically p-valent functions in $\mathbb{U}$.

Keywords: Analytic functions, Meromorphically p-valent starlike functions, Meromorphically p-valent convex functions, Products of meromorphic functions, Uniformly starlike functions, Uniformly convex functions.
2010 MSC: Primary 30C45.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of all functions $f(z)$ which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1.$$

A function $f(z) \in \mathcal{A}$ is said to be *uniformly convex* (or *uniformly starlike*) in $\mathbb{U}$ if, for every circular arc Γ contained in $\mathbb{U}$, with center at ω_0 also in $\mathbb{U}$, the arc $f(\Gamma)$ is convex (or starlike) with respect to the point $f(\omega_0)$. The classes of all uniformly convex function in $\mathbb{U}$ and all uniformly starlike

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functions in $\mathbb{U}$ are denoted by UCV and UST, respectively. These analytic function classes UCV and UST were introduced and studied by Goodman (Goodman, 1991*a*,*b*) who showed, among other things, that

$$f \in UCV \Longleftrightarrow \Re\left(1 + (z - \zeta)\,\frac{f''(z)}{f'(z)}\right) \geqq 0 \qquad (z, \zeta \in \mathbb{U})$$

and

$$f \in UST \Longleftrightarrow \Re\left(\frac{(z - \zeta)\, f'(z)}{f(z) - f(\zeta)}\right) \geqq 0 \qquad (z, \zeta \in \mathbb{U}).$$

Rønning (Rønning, 1993, 1994) and Ma and Minda (Ma & Minda, 1992) gave the following one-variable characterization of the class UCV of uniformly convex functions in $\mathbb{U}$.

Theorem A. *A function $f(z) \in \mathcal{A}$ is said to be in the class UCV of uniformly convex functions in $\mathbb{U}$ if it satisfies the following condition*:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqq \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathbb{U}).$$

Since the Alexander type result that

$$f \in UCV \Longleftrightarrow zf'(z) \in UST$$

does not hold true (Rønning, 1994), the class $\mathcal{S}_p$ defined by

$$\mathcal{S}_p := \{f : zf'(z) \in UCV\}$$

was introduced by Rønning (Rønning, 1993). On the other hand, Shams *et al.* (Shams *et al.*, 2004) initiated a study of the class $SD(\alpha,\beta)$ of β-uniformly starlike functions of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ consisting of functions $f(z) \in \mathcal{A}$ which satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (\beta \geqq 0;\ 0 \leqq \alpha < 1;\ z \in \mathbb{U}).$$

The class $KD(\alpha,\beta)$ of β-uniformly convex of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ is defined as follows:

$$f \in KD(\alpha,\beta) \Longleftrightarrow zf'(z) \in SD(\alpha,\beta).$$

Motivated by the above-defined function classes $SD(\alpha,\beta)$ and $KD(\alpha,\beta)$, Nishiwaki and Owa (Nishiwaki & Owa, 2007) introduced the class $MD(\alpha,\beta)$ consisting of all functions $f(z) \in \mathcal{A}$ which satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) < \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (\beta \leqq 0;\ \alpha > 1;\ z \in \mathbb{U}).$$

The function class $ND(\alpha,\beta)$ may also be considered as a subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in MD(\alpha,\beta)$.

The class of uniformly convex functions and various other related function classes have been studied by several authors (see, for example, (Ali & Ravichandran, 2010; Frasin, 2011; Kanas & Srivastava, 2000; Kanas & Wisniowska, 1999, 2000; Murugusundaramoorthy & Magesh, 2004; Rønning, 1991); see also (Srivastava & Owa (Editors), 1992)).

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic and p-valent in the *punctured* unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}.$$

A function $f \in \Sigma_p$ is said to be in the class $\Sigma S_p^*(\alpha)$ of meromorphically p-valent starlike functions of order α in $\mathbb{U}$ if and only if

$$\Re\left[\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)\right] < -\alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.2}$$

Also a function $f \in \Sigma_p$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphically p-valent convex functions of order α in $\mathbb{U}$ if and only if

$$\Re\left[\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < -\alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1). \tag{1.3}$$

It is easy to observe from (1.2) and (1.3) that

$$f(z) \in \Sigma C_p(\alpha) \Longleftrightarrow -\frac{zf'(z)}{p} \in \Sigma S_p^*(\alpha). \tag{1.4}$$

We note that the meromorphically p-valent function classes $\Sigma S_p^*(\alpha)$ and $\Sigma C_p(\alpha)$ were introduced by Kumar and Shukla (Kumar & Shukla, 1982).

We next denote by $\Sigma M_p(\alpha)$ and $\Sigma N_p(\alpha)$ the subclasses of the meromorphically p-valent function class Σ_p which satisfy the following inequalities:

$$\Sigma M_p(\alpha) := \left\{f : f \in \Sigma_p \quad \text{and} \quad \Re\left[-\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)\right] < \alpha \quad (z \in \mathbb{U};\ \alpha > 1)\right\}$$

and

$$\Sigma N_p(\alpha) := \left\{f : f \in \Sigma_p \quad \text{and} \quad \Re\left[-\frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < \alpha \quad (z \in \mathbb{U};\ \alpha > 1)\right\},$$

respectively. The meromorphically p-valent function classes $\Sigma M_p(\alpha)$ and $\Sigma N_p(\alpha)$ are analogous, respectively, to the subclasses $M(\alpha)$ and $N(\alpha)$ of the analytic function class $\mathcal{A}$ which were introduced by Owa and Nishiwaki (Owa & Nishiwaki, 2002).

Recently, Kumar *et al.* (Kumar *et al.*, 2005) introduced the following subclass $\Sigma\mathcal{S}_p^*(\alpha,\beta)$ of meromorphically p-valent starlike functions $f \in \Sigma_p$ in $\mathbb{U}$, which is similar to the class $SD(\alpha,\beta)$, by means of the following inequality:

$$\Re\left[-\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)\right] > \alpha\left|\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)+1\right|+\beta \tag{1.5}$$

$$(z \in \mathbb{U};\ \alpha \geqq 0;\ 0 \leqq \beta < 1).$$

Analogously, we define here the subclass $\Sigma\mathcal{C}_p(\alpha,\beta)$ of meromorphically p-valent convex functions in $\mathbb{U}$, which is similar to the class $KD(\alpha,\beta)$, consisting of all functions $f \in \Sigma_p$ which satisfy the following inequality:

$$\Re\left[-\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \alpha\left|\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)+1\right|+\beta \tag{1.6}$$

$$(z \in \mathbb{U};\ \alpha \geqq 0;\ 0 \leqq \beta < 1).$$

Similarly, for $-1 < \alpha \leqq 0$ and $\beta > 1$, we let $\Sigma\mathcal{M}_p(\alpha,\beta)$ be the subclass consisting of all functions $f \in \Sigma_p$ which satisfy the following inequality:

$$\Re\left[-\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)\right] < \alpha\left|\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)+1\right|+\beta \tag{1.7}$$

$$(z \in \mathbb{U};\ -1 < \alpha \leqq 0;\ \beta > 1).$$

We also let $\Sigma\mathcal{N}_p(\alpha,\beta)$ be the subclass consisting of all functions $f \in \Sigma_p$ which satisfy the following inequality:

$$\Re\left[-\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] < \alpha\left|\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)+1\right|+\beta \tag{1.8}$$

$$(z \in \mathbb{U};\ -1 < \alpha \leqq 0;\ \beta > 1).$$

The main purpose of this paper is to investigate the problems of finding the order of starlikeness and the order of convexity of certain products of meromorphically p-valent functions belonging to some of the above-defined classes of β-uniformly p-valent starlike functions in $\mathbb{U}$ and β-uniformaly p-valent convex functions in $\mathbb{U}$. Our main results in Section 2 (stated as Theorems 1 to 4 *and* Corollaries 1 to 5) can indeed be specialized suitably in order to deduce the solutions of the corresponding problems for relatively more familiar subclasses of meromorphically p-valent functions in $\mathbb{U}$.

2. The main results and their consequences

Our first main result is asserted by Theorem 1 below.

Theorem 1. *Let* $f_j \in \Sigma S_p^*(\gamma_j) \quad (j = 1, \cdots, n)$, *where*

$$\gamma_j := 1 - \alpha_j \geqq 0 \qquad \textit{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\kappa := 1 - \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then the product $F_p(z)$ *defined by*

$$F_p(z) := z^{-p} \prod_{j=1}^{n} \left\{ z^p f_j(z) \right\} \tag{2.1}$$

is in the class $\Sigma S_p^*(\kappa)$ *of meromorphically p-valent starlike functions of order* κ *in* $\mathbb{U}$.

Proof. Clearly, $F_p(z) \in \Sigma_p$. By differentiating (2.1) logarithmically with respect to z, we obtain

$$\frac{1}{p}\left(\frac{zF_p'(z)}{F_p}\right) = -1 + \sum_{j=1}^{n} \left[\frac{1}{p}\left(\frac{zf_j'(z)}{f_j(z)}\right) + 1\right], \tag{2.2}$$

which readily yields

$$\frac{1}{p}\left(\frac{zF_p'(z)}{F_p}\right) = -1 + \left(1 - \gamma_j\right) + \sum_{j=1}^{n} \left[\frac{1}{p}\left(\frac{zf_j'(z)}{f_j(z)}\right) + \gamma_j\right]. \tag{2.3}$$

We thus find that

$$\Re\left[\frac{1}{p}\left(\frac{zF_p'(z)}{F_p}\right)\right] = -1 + \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \Re\left[\frac{1}{p}\left(\frac{zf_j'(z)}{f_j(z)}\right) + \gamma_j\right]. \tag{2.4}$$

Since, by hypothesis, $f_j \in \Sigma S_p^*(\gamma_j) \quad (j = 1, \cdots, n)$, we have

$$\Re\left[\frac{1}{p}\left(\frac{zF_p'(z)}{F_p}\right)\right] < -\left(1 - \sum_{j=1}^{n} \alpha_j\right) =: \kappa, \tag{2.5}$$

which evidently completes the proof of Theorem 1. □

Upon setting

$$f_j(z) = f(z), \quad \gamma_j = \gamma \quad \text{and} \quad \alpha_j = \alpha \qquad (j = 1, \cdots, n)$$

in Theorem 1, we have the following corollary.

Corollary 1. *Let* $f \in \Sigma S_p^*(\gamma) \quad (\gamma := 1 - \alpha \geqq 0)$, *where* $\alpha \geqq 0$. *Also let* $1 - n\alpha \geqq 0$. *Then the product* $\Theta_p(z)$ *defined by*

$$\Theta_p(z) := z^{-p} \left[z^p f(z)\right]^n$$

is in the class $\Sigma S_p^*(1 - n\alpha)$ *of meromorphically p-valent starlike functions of order* $1 - n\alpha$ *in* $\mathbb{U}$.

Corollary 2. *Let* $f_j \in \Sigma S_p^*(\gamma_j) \quad (j = 1, \cdots, n)$, *where*

$$\gamma_j := 1 - \alpha_j \geqq 0 \qquad \text{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\kappa := 1 - \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then the function $\Phi_p(z)$ *defined by*

$$\Phi_p(z) := -p \int_0^z t^{-p-1} \prod_{j=1}^{n} \left\{ t^p f_j(t) \right\} \mathrm{d}t \tag{2.6}$$

is in the class $\Sigma C_p(\kappa)$ *of meromorphically p-valent convex functions of order* κ *in* $\mathbb{U}$.

Proof. The result asserted by Corollary 2 follows immediately from Theorem 1, since

$$\Phi_p(z) \in \Sigma C_p(\kappa) \Longleftrightarrow -\frac{z\Phi'(z)}{p} =: F_p(z) \in \Sigma S_p^*(\kappa).$$

□

Corollary 3. *Let* $f_j \in \Sigma C_p(\gamma_j) \quad (j = 1, \cdots, n)$, *where*

$$\gamma_j := 1 - \alpha_j \geqq 0 \qquad \text{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\kappa := 1 - \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then the product $G_p(z)$ *defined by*

$$G_p(z) = z^{-p} \prod_{j=1}^{n} \left\{ -\left(\frac{z^{p+1} f_j'(z)}{p} \right) \right\} \tag{2.7}$$

is in the class $\Sigma S_p^*(\kappa)$ *of meromorphically p-valent starlike functions of order* κ *in* $\mathbb{U}$.

Proof. From the fact that

$$f_j(z) \in \Sigma C_p(\gamma_j) \Longleftrightarrow -\frac{z f_j'(z)}{p} \in \Sigma S_p^*(\gamma_j) \qquad (j = 1, \cdots, n),$$

by replacing $f_j(z)$ by $-\frac{z f_j'(z)}{p}$ in Theorem 1, we are led easily to Corollary 3. □

Corollary 4. *Let* $f_j \in \Sigma C_p(\gamma_j) \quad (j = 1, \cdots, n)$, *where*

$$\gamma_j := 1 - \alpha_j \geqq 0 \qquad \text{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\kappa := 1 - \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then the function $\Psi_p(z)$ *defined by*

$$\Psi_p(z) = -p \int_0^z t^{-p-1} \prod_{j=1}^{n} \left\{ -\left(\frac{t^{p+1} f_j'(t)}{p} \right) \right\} \mathrm{d}t \tag{2.8}$$

is in the class $\Sigma C_p(\kappa)$ *of meromorphically p-valent convex functions of order* κ.

Proof. The result asserted by Corollary 4 follows immediately from Corollary 3, since

$$\Psi_p(z) \in \Sigma C_p(\kappa) \Longleftrightarrow -\frac{z\Psi_p'(z)}{p} =: G_p(z) \in \Sigma S_p^*(\kappa).$$

□

By applying the same method and technique as in our proofs of Theorem 1 as well as of Corollaries 2, 3 and 4, we can establish Theorem 2 below.

Theorem 2. *Let* $f_j \in \Sigma_p \quad (j = 1, \cdots, n)$. *Suppose that*

$$\gamma_j := 1 + \alpha_j \geqq 0 \qquad \text{and} \qquad \alpha_j \geqq 0 \qquad (j = 1, \cdots, n).$$

Also let

$$\sigma := 1 + \sum_{j=1}^{n} \alpha_j \geqq 0.$$

Then each of the following assertions holds true:

(i) *If* $f_j \in \Sigma M_p(\gamma_j) \quad (j = 1, \cdots, n)$, *then the product* $F_p(z)$ *defined by* (2.1) *is in the class* $\Sigma M_p(\sigma)$.

(ii) *If* $f_j \in \Sigma M_p(\gamma_j) \quad (j = 1, \cdots, n)$, *then the integral operator* Φ_p *defined by* (2.6) *is in the class* $\Sigma N_p(\sigma)$.

(iii) *If* $f_j \in \Sigma N_p(\gamma_j) \quad (j = 1, \cdots, n)$, *then the product* $G_p(z)$ *defined by* (2.7) *is in the class* $\Sigma M_p(\sigma)$.

(iv) *If* $f_j \in \Sigma N_p(\gamma_j) \quad (j = 1, \cdots, n)$, *then the integral operator* Ψ_p *defined by* (2.8) *is in the class* $\Sigma N_p(\sigma)$.

Theorem 3. *Let*

$$\alpha_j \geqq 0 \qquad \text{and} \qquad 0 \leqq \beta_j < 1 \qquad (j = 1, \cdots, n)$$

and suppose that

$$\delta := 1 - \sum_{j=1}^{n} \left(\frac{1 - \beta_j}{1 + \alpha_j} \right).$$

Also let the products $F_p(z)$ *and* $G_p(z)$ *be defined by* (2.1) *and* (2.7), *respectively. Then each of the following assertions holds true*:

(i) *If* $f_j \in \Sigma\mathcal{S}_p^*(\alpha_j,\beta_j) \quad (j=1,\cdots,n)$, *then* $F_p(z) \in \Sigma S_p^*(\delta)$.

(ii) *If* $f_i \in \Sigma\mathcal{C}_p(\alpha_j,\beta_j) \quad (j=1,\cdots,n)$, *then* $G_p(z) \in \Sigma S_p^*(\delta)$.

Proof. By following the lines as in (Kumar *et al.*, 2005), we first prove that

$$\Sigma\mathcal{S}_p^*(\lambda,\mu) \subset \Sigma S_p^*\left(\frac{\lambda+\mu}{1+\lambda}\right).$$

Indeed, if we let $f \in \Sigma\mathcal{S}_p^*(\lambda,\mu)$, then the quantity w defined by

$$w := \frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)$$

satisfies the following inequality:

$$-\Re(w) - \mu \geqq \lambda\,|w+1| \geqq \lambda\Re\,(w+1)\,,$$

which immediately yields

$$-\Re(w) \geqq \frac{\lambda+\mu}{1+\lambda}.$$

We thus have

$$f \in \Sigma\mathcal{S}_p^*(\lambda,\mu) \Longrightarrow f \in \Sigma S_p^*\left(\frac{\lambda+\mu}{1+\lambda}\right).$$

Next, since

$$f_j \in \Sigma\mathcal{S}_p^*(\alpha_j,\beta_j) \quad (j=1,\cdots,n),$$

we have

$$f_j \in \Sigma S_p^*\left(\frac{\alpha_j+\beta_j}{1+\alpha_j}\right) \quad (j=1,\cdots,n),$$

The assertion (i) of Theorem 3 now follows readily from an application of Theorem 1.

The proof of the assertion (ii) of Theorem 3 follows similarly by using Corollary 3. □

Corollary 5. *Let*

$$\alpha_j \geqq 0 \qquad \text{and} \qquad 0 \leqq \beta_j < 1 \qquad (j=1,\cdots,n)$$

and suppose that

$$\delta := 1 - \sum_{j=1}^{n}\left(\frac{1-\beta_j}{1+\alpha_j}\right).$$

Also let the functions $\Phi_p(z)$ *and* $\Psi_p(z)$ *be defined by* (2.6) *and* (2.8), *respectively. Then each of the following assertions holds true*:

(i) *If* $f_j \in \Sigma\mathcal{S}_p^*(\alpha_j,\beta_j) \quad (j=1,\cdots,n)$, *then* $\Phi_p(z) \in \Sigma C_p(\delta)$.

(ii) *If* $f_j \in \Sigma\mathcal{C}_p(\alpha_j,\beta_j) \quad (j=1,\cdots,n)$, *then* $\Psi_p(z) \in \Sigma C_p(\delta)$.

Proof. The results asserted by Corollary 5 would follow immediately from Theorem 3, since

$$\Phi_p(z) \in \Sigma C_p(\delta) \Longleftrightarrow -\frac{z\Phi_p'(z)}{p} =: F_p(z) \in \Sigma S_p^*(\delta)$$

and

$$\Psi_p(z) \in \Sigma C_p(\delta) \Longleftrightarrow -\frac{z\Psi'(z)}{p} =: G_p(z) \in \Sigma S_p^*(\delta).$$

□

Finally, if we make use of the same method and technique as in our proofs of Theorem 3 and Corollary 5, we are led easily to Theorem 4 below.

Theorem 4. *Let*

$$-1 < \alpha_j \leqq 0 \qquad \text{and} \qquad \beta_j > 1 \qquad (j = 1, \cdots, n)$$

and suppose that

$$\nu := 1 + \sum_{j=1}^{n} \left(\frac{\beta_j - 1}{1 + \alpha_j} \right).$$

Also let the products $F_p(z)$ *and* $G_p(z)$ *be defined by* (2.1) *and* (2.7), *respectively, and the functions* $\Phi_p(z)$ *and* $\Psi_p(z)$ *be defined by* (2.6) *and* (2.8), *respectively. Then each of the following assertions holds true*:

(i) *If* $f_j \in \Sigma\mathcal{M}_p(\alpha_j,\beta_j) \quad (j = 1, \cdots, n)$, *then* $F_p(z) \in \Sigma M_p(\nu)$.
(ii) *If* $f_j \in \Sigma\mathcal{N}_p(\alpha_j,\beta_j) \quad (j = 1, \cdots, n)$, *then* $G_p(z) \in \Sigma M_p(\nu)$.
(iii) *If* $f_j \in \Sigma\mathcal{M}_p(\alpha_j,\beta_j) \quad (j = 1, \cdots, n)$, *then* $\Phi_p(z) \in \Sigma N_p(\nu)$.
(iv) *If* $f_j \in \Sigma\mathcal{N}_p(\alpha_j,\beta_j) \quad (j = 1, \cdots, n)$, *then* $\Psi_p(z) \in \Sigma N_p(\nu)$.

3. Concluding remarks and observations

In our present investigation, we have considered several interesting subclasses of the familiar class of meromorphically p-valent functions in the open unit disk $\mathbb{U}$. Our main purpose has been to successfully address the problems of finding the order of starlikeness and the order of convexity of the products of functions belonging to each of the various classes of β-uniformly p-valent starlike functions and β-uniformly p-valent convex functions in $\mathbb{U}$, which we have introduced here. The main results (stated as Theorems 1 to 4 *and* Corollaries 1 to 5) can indeed be specialized suitably in order to deduce the solutions of the corresponding problems for relatively more familiar subclasses of meromorphically p-valent functions in $\mathbb{U}$.

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