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On Ideal Convergent Difference Double Sequence Spaces in n-Normed Spaces Defined by Orlicz Function

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Abstract

The main aim of this paper is to define the generalized difference double sequence spaces ${}_2W^I(M, \|., ..., .\|, \Delta_m^n, p)$, ${}_2W_0^I(M, \|., ..., .\|, \Delta_m^n, p)$ and ${}_2W_\infty^I(M, \|., ..., .\|, \Delta_m^n, p)$ defined over a n-normed space $(X, \|., ..., .\|)$. Here we also study their properties and establish some inclusion relations.

Keywords: Double sequence spaces, n-norm, Orlicz Function, difference sequence spaces, I convergence.
2010 MSC: 46E30, 46E40, 46B20.

1. Introduction

The notion of ideal convergence was introduced first by Kostyrko et-al- (Kostyrko *et al.*, 2000) as an interesting generalization of statistical convergence (Khan & Tabassum, 2012) which was further studied in topological spaces. A family $I \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if

1. $\emptyset \in I$;
2. $A, B \in I$ imply $A \cup B \in I$;
3. $A \in I, B \subset A$ imply $B \in I$,

while an admissible ideal I further satisfies $\{x\} \in I$ for each $x \in Y$ (Kostyrko *et al.*, 2000, 2005; Savas, 2010).
Given $I \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. Let X be a normed space. The sequence (x_j) in X is said to be $I-$*convergent* to $\xi \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{j \in \mathbb{N} : \|x_j - \xi\| \geq \varepsilon\}$ belongs to I (Khan & Tabassum, 2010).

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The concept of 2-normed spaces was initially introduced by Ğahler (Gähler, 1963) in the mid of 1960's as an interesting nonlinear generalization of a normed linear space. Since then, many researchers have studied this concept and obtained various results, see for instance (Gunawan & Mashadi, 2001; Khan & Tabassum, 2010; Savas, 2010).

Recall (Khan & Tabassum, 2012) that an *Orlicz Function* is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$. If convexity of M is replaced by $M(x+y) \leq M(x)+M(y)$, then it is called a *Modulus funtion* (Maddox, 1986).

Let w be the space of all sequences. Lindenstrauss and Tzafriri (Lindenstrauss & Tzafiri, 1971) used the idea of Orlicz sequence space. Let

$$l_M := \left\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\right\}$$

is Banach space with respect to the norm

$$\|x\|_M := \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\right\}.$$

Orlicz function has been studied by V. A. Khan (Khan, 2008*a*,*b*) and many others.

Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. An n-norm on X is a function $\|., ..., .\| : X \times X \times ... \times X \to \mathbb{R}$ which satisfies the following four conditions:

1. $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
2. $\|x_1, x_2, ..., x_n\|$ is invariant under permutation,
3. $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$, for any $\alpha \in \mathbb{R}$,
4. $\|x + x', x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|x', x_2, ..., x_n\|$.

The pair $(X, \|., ., ..., .\|)$ is called an n-normed space (Savas, 2011).

Example 1.1. (see (Savas, 2011)). As a standard example of a n-normed space we may take R^n being equipped with the n-norm $\|x_1, x_2, ..., x_n\|_E$ = the volume of the n-dimensional parallelopiped spaned by the vectors $x_1, x_2, ..., x_{n-1}, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, ..., x_n\|_E = \begin{vmatrix} \langle x_1, x_2\rangle & ... & \langle x_1, x_n.\rangle \\ . & ... & \\ . & ... & \\ \langle x_n, x_1\rangle & ... & \langle x_n, x_n\rangle \end{vmatrix}.$$

where $\langle ., .\rangle$ denotes inner product.

Example 1.2. (see (Savas, 2011)). Let $(X, \|., ., ..., .\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, ..., a_n\}$ be a linearly independent set in X. Then the following function $\|., ., ..., .\|_\infty$ defined by

$$\|x_1, x_2, ..., x_{n-1}, x_n\|_\infty = \max\{\|x_1, x_2, ..., x_{n-1}, a_i\| : i = 1, 2, ..., n\}$$

defines an $(n-1)-$norm on X with respect to $\{a_1, a_2, ..., a_n\}$.

Definition 1.1. (see (Savas, 2011)). A sequence (x_j) in an n-normed space $(X, \|., ., ..., .\|)$ is said to be converge to some $L \in X$ in the n-norm if

$$\lim_{j\to\infty} \|x_j - L, x_1, ..., x_{n-1}\| = 0, \quad \text{for every } x_1, ..., x_{n-1} \in X.$$

Example 1.3. (see (Khan & Tabassum, 2010)). A sequence (x_j) in an n-normed space $(X, \|., ., ..., .\|)$ is said to be Cauchy with respect to the n-norm if

$$\lim_{j,k\to\infty} \|x_j - x_k, x_1, ..., x_{n-1}\| = 0, \quad \text{for every } x_1, ..., x_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let w, l_∞, c and c_0 denote the spaces of all, bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\| = \sup_k |x_k|.$$

Kizmaz (Kizmaz, 1981), defined the difference sequences $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

for $Z = l_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$.

The above spaces are Banach spaces, normed by

$$\|x\|_\Delta = |x_1| + \sup_k |\Delta x_k|.$$

The notion of difference sequence spaces was generalized by Et. and Colak (Et & Colak, 1995) as follows:

$$Z(\Delta^n) = \{x = (x_k) : (\Delta^n x_k) \in Z\},$$

for $Z = l_\infty, c$ and c_0, where $n \in \mathbb{N}, (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})$ and so that

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}.$$

In 2005, Tripathy and Esi (Tripathy & Esi, 2006), introduced the following new type of difference sequence spaces:

$$Z(\Delta_m) = \{x = (x_k) \in w : \Delta_m x \in Z\}, \text{ for } Z = l_\infty, c \text{ and } c_0$$

where $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$, for all $k \in \mathbb{N}$.

Later on, Tripathy, Esi and Tripathy (B. C. Tripathy & Tripathy, 2005), generalized the above notions and unified these as follows:
Let m, n be non negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}$$

where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^{\circ} x_k = x_k$ for all $k \in \mathbb{N}$. The difference operator is equivalent to the binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

A *paranorm* is a function $g : X \to \mathbb{R}$ which satisfies the following axioms:
For any $x, y, x_0 \in X$, $\lambda, \lambda_0 \in \mathbb{C}$:

(i) $g(\theta) = 0$;
(ii) $g(x) = g(-x)$;
(iii) $g(x + y) \le g(x) + g(y)$
(iv) the scalar multiplication is continuous, that is $\lambda \to \lambda_0$, $x \to x_0$ imply $\lambda x \to \lambda_0 x_0$.

Throughout, a double sequence $x = (x_{jk})$ is a double infinite array of elements x_{jk}. for $j, k \in \mathbb{N}$. Double sequences have been studied by V. A. Khan and S. Tabassum (Khan & Tabassum, 2012; V. & Tabassum, 2011; Khan & Tabassum, 2011, 2010), Moricz and Rhoades (Moricz & Rhoades, 1952) and many others.

Definition 1.2. (see (Khan & Tabassum, 2010)). A double sequence space X is said to be *Solid (Normal)*, if $(\alpha_{jk} x_{jk}) \in X$ whenever $(x_{jk}) \in X$ and for all double sequence (α_{jk}) of scalars with $|\alpha_{jk}| \le 1$ for all $j, k \in \mathbb{N}$.

2. Main Results

In 2010 E. Savas (Savas, 2010) introduced certain new sequence spaces using ideal convergence in 2-normed spaces. Later on V. A. Khan and S. Tabassum (Khan & Tabassum, 2010) introduced similar kind of double sequence spaces using difference operator in 2-normed spaces. In this paper we generalized these sequence spaces in n-normed spaces.

Let $p = (p_{jk})$ be any bounded sequence of positive numbers, m, n be non-negative integers and let I be an admissible ideal of $\mathbb{N}$. Let ${}_2W(n - X)$ be the space of X-valued double sequence spaces defined over a n-normed space $(X, ||., ..., .||)$. Then for an Orlicz function M we define the following sequence spaces:

$$ {}_2W^I(M, ||., ..., .||, \Delta_m^n, p) = \Big\{ x = (x_{jk}) \in {}_2W(n - X) : \forall \varepsilon > 0 \text{ the set } \Big\{ (j, k) \in N \times N :$$

$$\lim_{j,k\to\infty} \left(M\left(\left\| \frac{\Delta_m^n x_{jk} - L}{\rho}, z_1, z_2, ..., z_{n-1} \right\| \right) \right)^{p_{jk}} \ge \varepsilon \Big\} \in I, \text{ for some } \rho > 0, L \in X, z_1, z_2, ..., z_{n-1} \in X \Big\}.$$

$$ {}_2W_0^I(M, ||., ..., .||, \Delta_m^n, p) = \Big\{ x = (x_{jk}) \in {}_2W(n - X) : \forall \varepsilon > 0 \text{ the set} \Big\{ (j, k) \in N \times N :$$

$$\lim_{j,k\to\infty}\left(M\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \geq \varepsilon\right\} \in I, \text{ for some } \rho > 0, z_1, z_2, ..., z_{n-1} \in X\right\}.$$

$${}_2W_\infty^I(M, \|., ..., .\|, \Delta_m^n, p) = \left\{x = (x_{jk}) \in {}_2W(n-X) : \exists K > 0 \text{ s.t. } \left\{(j,k) \in N \times N :\right.\right.$$

$$\sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \geq K\right\} \in I, \text{ for some } \rho > 0, z_1, z_2, ..., z_{n-1} \in X\right\}$$

where

$$(\Delta_m^n x_{jk}) = (\Delta_m^{n-1} x_{jk} - \Delta_m^{n-1} x_{j+1,k} - \Delta_m^{n-1} x_{j,k+1} + \Delta_m^{n-1} x_{j+1,k+1})$$

and

$$(\Delta_m^0 x_{jk}) = x_{jk} \quad \text{for all } j,k \in \mathbb{N},$$

which is equivalent to the following binomial representation:

$$\Delta_m^n x_{jk} = \sum_{u=0}^{n}\sum_{v=0}^{n}(-1)^{u+v}\binom{n}{u}\binom{n}{v} x_{j+mu,k+mv}.$$

and $\Delta x_{j,k} = x_{j,k} - x_{j+1,k} - x_{j,k+1} + x_{j+1,k+1}$.
The following inequality will be used throughout the paper. Let $p_{j,k}$ be a double sequence of positive real numbers with $0 < p_{jk} \leq \sup_{j,k} p_{jk} = H$, and let $D = \max\{1, 2^{H-1}\}$. Then for the factorable sequences (a_{jk}) and (b_{jk}) in the complex plane, we have

$$|a_{jk} + b_{jk}|^{q_{jk}} \leq D(|a_{jk}|^{q_{jk}} + |b_{jk}|^{q_{jk}})$$

Theorem 2.1. *If* $\{\Delta_m^n x_{jk}, z_1, z_2, ..., z_{n-1}\}$ *is a linearly independent set in* $(X.\|., ..., .\|)$ *for all but finite* j,k *where* $x = (x_{jk}) \in {}_2W(n-X)$ *and* $\inf_{j,k} p_{jk} > 0$, *then*

(i) $\lim_{j,k\to\infty}\left[M\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} = 0, \text{ for every } \rho > 0,$

(ii) $\lim_{j,k\to\infty}\left[M\left(\left\|\frac{\Delta_m^n x_{jk} - L}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} < \infty, \text{ for every } \rho > 0.$

Proof. **(i).** Assume that $\{\Delta_m^n x_{jk}, z_1, z_2, ..., z_{n-1}\}$ is a linearly independent set in $(X.\|., ..., .\|)$ for all but finite j,k. Then we have $\|\Delta_m^n x_{jk}, z_1, z_2, ..., z_{n-1}\| \to 0$ as $j,k \to \infty$.
Since M is continuous and $0 < p_{jk} \leq \sup p_{jk} < \infty$, for each j,k, we have

$$\lim_{j,k\to\infty}\left[M\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} = 0, \text{ for every } \rho > 0.$$

(ii). Proof of this part is similar to part **(i)**. □

Theorem 2.2. ${}_2W^I(M, \|., ..., .\|, \Delta_m^n, p)$, ${}_2W_0^I(M, \|., ..., .\|, \Delta_m^n, p)$ *and* ${}_2W_\infty^I(M, \|., ..., .\|, \Delta_m^n, p)$ *are linear spaces.*

Proof. We prove the assertion for ${}_2W_0^I(M, \|., ..., .\|, \Delta_m^n, p)$ the others can be proved similarly. Assume that $x = (x_{jk})$ and $y = (y_{jk}) \in {}_2W_0^I(M, \|., ..., .\|, \Delta_m^n, p)$ and $\alpha, \beta \in \mathbb{R}$, so

$$\left\{(j,k) \in N \times N : \lim_{j,k\to\infty} \left(M\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \geq \varepsilon\right\} \in I, \text{ for some } \rho_1 > 0, \tag{2.1}$$

$$\left\{(j,k) \in N \times N : \lim_{j,k\to\infty} \left(M\left(\left\|\frac{\Delta_m^n y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \geq \varepsilon\right\} \in I, \text{ for some } \rho_2 > 0, \tag{2.2}$$

Since $\|., ..., .\|$ is a n-norm, and M is an Orlicz function the following inequality holds:

$$\begin{aligned} &\lim_{j,k\to\infty} \left(M\left(\left\|\tfrac{\Delta_m^n(\alpha x_{jk}+\beta y_{jk})}{|\alpha|\rho_1+|\beta|\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \\ &\leq D \lim_{j,k\to\infty} \left[\tfrac{|\alpha|\rho_1}{|\alpha|\rho_1+|\beta|\rho_2} M\left(\left\|\tfrac{\Delta_m^n x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \\ &+D \lim_{j,k\to\infty} \left[\tfrac{|\beta|\rho_2}{|\alpha|\rho_1+|\beta|\rho_2} M\left(\left\|\tfrac{\Delta_m^n y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \\ &\leq DF \lim_{j,k\to\infty} \left[M\left(\left\|\tfrac{\Delta_m^n x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \\ &+DF \lim_{j,k\to\infty} \left[M\left(\left\|\tfrac{\Delta_m^n y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \end{aligned} \tag{2.3}$$

where

$$F = \max\left[1, \left(\frac{|\alpha|}{\alpha\rho_1 + |\beta|\rho_2}\right)^H, \left(\frac{|\beta|}{\alpha\rho_1 + |\beta|\rho_2}\right)^H\right] \tag{2.4}$$

From the above inequality, we get

$$\begin{aligned} &\left\{(j,k) \in N \times N : \lim_{j,k\to\infty} \left(M\left(\left\|\tfrac{\Delta_m^n \alpha x_{jk}+\Delta_m^n \beta y_{jk}}{|\alpha|\rho_1+|\beta|\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \geq \varepsilon\right\} \\ &\subseteq \left\{(j,k) \in N \times N : DF \lim_{j,k\to\infty} \left(M\left(\left\|\tfrac{\Delta_m^n x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \geq \tfrac{\varepsilon}{2}\right\} \\ &\cup\left\{(j,k) \in N \times N : DF \lim_{j,k\to\infty} \left(M\left(\left\|\tfrac{\Delta_m^n y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \geq \tfrac{\varepsilon}{2}\right\}. \end{aligned} \tag{2.5}$$

The sets on the right hand side belong to I and this completes the proof. □

Theorem 2.3. *For any fixed $(j,k) \in N \times N$, ${}_2W_\infty^I(M, \|., ..., .\|, \Delta_m^n, p)$ is paranomed space with respect to the paranorm defined by:*

$$g(x) = \inf_{j,k} \left\{\rho^{\frac{p_{jk}}{H}} : \left(\sup_{j,k\geq 1} \left(M\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}}\right)^{\frac{1}{H}} \leq 1, \forall z_1, z_2, ..., z_{n-1} \in X\right\}. \tag{2.6}$$

Proof. (**i**) $x = \theta$ implies that then $\|0, z_1, z_2..., z_{n-1}\| = 0$ since the set containing 0 is linearly dependent. Also $M(0) = 0$ implies that $g(\theta) = 0$.

(ii) $g(x) = g(-x)$

(iii) Let $x = (x_{jk}), y = (y_{jk}) \in {}_2W^I_\infty(M, \|., ..., .\|, \Delta^n_m, p)$.
Then there exists $\rho_1, \rho_2 > 0$ such that: $\sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta^n_m x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \leq 1$ and

$$\sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta^n_m y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \leq 1 \tag{2.7}$$

for each $z_1, z_2, ..., z_{n-1} \in X$.

Let $\rho = \rho_1 + \rho_2$. Then by convexity of Orlicz function we have:

$$\begin{aligned} &\sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta^n_m x_{jk}+\Delta^n_m y_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right) \leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{j,k\geq 1} M\left(\left\|\frac{\Delta^n_m x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right) \\ &+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{j,k\geq 1} M\left(\left\|\frac{\Delta^n_m y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right). \end{aligned} \tag{2.8}$$

Thus $\sup_{j,k\geq 1} M\left(\left\|\frac{\Delta^n_m x_{jk}+\Delta^n_m y_{jk}}{\rho_1+\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)^{p_{jk}} \leq 1$ and hence

$$\begin{aligned} g(x+y) &\leq \inf_{j,k}\left\{\rho_1^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta^n_m x_{jk}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \leq 1\right\} \\ &+ \inf_{j,k}\left\{\rho_2^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta^n_m y_{jk}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \leq 1\right\}. \end{aligned} \tag{2.9}$$

The arbitrary ρ_1 and ρ_2 implies that $g(x + y) \leq g(x) + g(y)$.

(iv) Let $\alpha \to 0$ and $g(x^n - x) \to 0 \quad (n \to \infty)$

$$g(\alpha x) = \inf\left\{\left(\frac{\rho}{|\alpha|}\right)^{\frac{p_{jk}}{H}} : \sup_{j,k\geq 1}\left(M\left(\left\|\frac{\Delta^n_m \alpha x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \leq 1\right\}. \tag{2.10}$$

□

Theorem 2.4. *Let M, M_1, M_2, be Orlicz functions. Then we have*

(i) ${}_2W^I_0(M_1, \|., ..., .\|, \Delta^n_m, p) \subseteq {}_2W^I_0(M \circ M_1, \|., ..., .\|, \Delta^n_m, p)$
provided (p_{jk}) is such that $H_0 = \inf p_{jk} > 0$.

(ii) ${}_2W^I_0(M_1, \|., ..., .\|, \Delta^n_m, p) \cap {}_2W^I_0(M_2, \|., ..., .\|, \Delta^n_m, p) \subseteq {}_2W^I_0(M_1 + M_2, \|., ..., .\|, \Delta^n_m, p)$.

Proof. **(i).** For given $\varepsilon > 0$, first choose $\varepsilon_0 > 0$ such that $\max\{\varepsilon^H_0, \varepsilon^{H_0}_0\} < \varepsilon$. Now using the continuity of M choose $0 < \delta < 1$ such that $0 < t < \delta$, implies that $M(t) < \varepsilon_0$, Let $(x_{jk}) \in {}_2W^I_0(M_1, \|., .\|, \Delta^n_m, p)$. Now by definition:

$$A(\delta) = \left\{(j,k) \in N \times N : \lim_{j,k\to\infty}\left(M_1\left(\left\|\frac{\Delta^n_m x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \geq \delta^H\right\} \in I. \tag{2.11}$$

Thus if $(j, k) \notin A(\delta)$ then

$$\left(M_1\left(\left\|\frac{\Delta^n_m x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} \leq \delta^H, \qquad \forall j, k \in \mathbb{N}. \tag{2.12}$$

That is

$$\left(M_1\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} < \delta, \qquad \forall j,k \in \mathbb{N}. \tag{2.13}$$

Hence from above using continuity of M we must have

$$M\left(M_1\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)^{p_{jk}} < \varepsilon_0, \qquad \forall j,k \in \mathbb{N} \tag{2.14}$$

Which consequently implies that

$$\lim_{j,k\to\infty}\left[M\left(M_1\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)\right]^{p_{jk}} < \max\{\varepsilon_0^H, \varepsilon_0^{H_0}\} < \varepsilon. \tag{2.15}$$

This shows that

$$\left\{(j,k) \in N \times N : \lim_{j,k\to\infty}\left[M\left(M_1\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right)\right]^{p_{jk}} \geq \varepsilon\right\} \subset A(\delta) \tag{2.16}$$

and so belongs to I. This completes the result.
(ii). Let $x_{jk} \in {}_2W_0^I(M_1, \|., ..., .\|, \Delta_m^n, p) \cap {}_2W_0^I(M_2, \|., ..., .\|, \Delta_m^n, p)$
Then the fact that

$$\begin{aligned}\lim_{j,k\to\infty}\left[(M_1+M_2)\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} &\leq D\lim_{j,k\to\infty}\left[M_1\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}\\ +D\lim_{j,k\to\infty}\left[M_2\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}}.\end{aligned} \tag{2.17}$$

This gives the result. □

Theorem 2.5. *The sequence space* ${}_2W_0^I(M, \|., ..., .\|, \Delta_m^n, p), {}_2W_\infty^I(M, \|., ..., .\|, \Delta_m^n, p)$ *are Solid.*

Proof. We give the proof for ${}_2W_0^I(M, \|., ..., .\|, \Delta_m^n, p)$ only.

Let $(x_{jk}) \in {}_2W_0^I(M, \|., ..., .\|, \Delta_m^n, p)$ and let (α_{jk}) be a double sequence of scalars such that $|\alpha_{jk}| \leq 1$ for all $j,k \in \mathbb{N}$. Then we have

$$\begin{aligned}&\left\{(j,k) \in N \times N : \lim_{j,k\to\infty}\left[M\left(\left\|\frac{\Delta_m^n(\alpha_{jk}x_{jk})}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \geq \varepsilon\right\}\\ &\subseteq \left\{(j,k) \in N \times N : E\lim_{j,k\to\infty}\left[M\left(\left\|\frac{\Delta_m^n x_{jk}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{jk}} \geq \varepsilon\right\} \in I.\end{aligned} \tag{2.18}$$

Where $E = \max_{j,k}\{1, |\alpha_{jk}|^H\}$. Hence $(\alpha_{jk}x_{jk}) \in {}_2W_0^I(M, \|., .\|, \Delta_m^n, p)$ for all double sequence of scalars (α_{jk}) with $|\alpha_{jk}| \leq 1$ for all $j,k \in \mathbb{N}$ whenever $(x_{jk}) \in {}_2W_0^I(M, \|., ..., .\|, \Delta_m^n, p)$. □

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