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The $(\mathcal{G}, \beta, \phi, h(\cdot, \cdot), \rho, \theta)$-Univexities of Higher-Orders with Applications to Parametric Duality Models in Minimax Fractional Programming

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Abstract

Based on the recently introduced (see (Verma, 2012)) major higher order generalizations ($\mathcal{G}$, β, ϕ, $h(\cdot,\cdot)$, ρ, θ) - univexities, several second-order parametric duality models for a semiinfinite minimax fractional programming problem are developed with appropriate duality results under various generalized second-order ($\mathcal{G}$, β, ϕ, $h(\cdot,\cdot)$, ρ, θ) - univexity assumptions. The obtained results encompass a large variety of investigations on generalized univexities and their extensions in the literature.

Keywords: Semiinfinite programming, minimax fractional programming, generalized second-order univex functions, infinitely many equality and inequality constraints, dual problems, duality theorems.
2010 MSC: 49N15, 90C26, 90C30, 90C32, 90C45, 90C47.

1. Introduction

In this paper, we intend to establish some results on second-order duality under various generalized $(\mathcal{G}, \beta, \phi, h(\cdot, \cdot), \rho, \theta)$-univexity assumptions for the semiinfinite discrete minimax fractional programming problem of the form:

$$(P) \qquad \text{Minimize} \max_{1\le i\le p} \frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x,t) \le 0 \ \text{ for all } t \in T_j, \quad j \in \underline{q} = \{1, 2, \cdots, q\},$$
$$H_k(x,s) = 0 \ \text{ for all } s \in S_k, \quad k \in \underline{r} = \{1, 2, \cdots, r\},$$
$$x \in X,$$

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where $p,\ q$, and r are positive integers, X is a nonempty open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), for each $j \in \underline{q} = \{1, 2, \ldots, q\}$ and $k \in \underline{r} = \{1, 2, \ldots, r\}$, T_j and S_k are compact subsets of complete metric spaces, for each $i \in \underline{p}$, f_i and g_i are twice continuously differentiable real-valued functions defined on X, for each $j \in \underline{q}$, $z \to G_j(z, t)$ is a twice continuously differentiable real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r}$, $z \to H_k(z, s)$ is a twice continuously differentiable real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r}$, $t \to G_j(x, t)$ and $s \to H_k(x, s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$, and for each $i \in \underline{p}$, $g_i(x) > 0$ for all x satisfying the constraints of (P). The present communication is concerned with the major generalization $(\mathcal{G}, \beta, \phi, h(\cdot, \cdot), \rho, \theta)$- univexity of the second order introduced by Verma (see (Verma, 2012)) that generalizes $(\mathcal{F}, \beta, \phi, \rho, \theta)$-univexity introduced by Zalmai (see (Zalmai, 2012)) and the first order univexity studied by Zalmai and Zhang (see (Zalmai & Zhang, 2007)) with its applications to parametric duality models in minimax fractional programming. The obtained results not only generalize the work of Zalmai on second order univexities, but also generalize other investigations on general invexities, including the valued-contributions of Jeyakumar (see (Jeyakumar *b*, 1985)), Liu (see (Liu, 1999)), Mangasarian (see (Mangasarian, 1975)), Mishra (see (Mishra, 1997), (Mishra, 2000)), Mishra and Rueda (see (Mishra & Rueda, 2000), (Mishra & Rueda, 2006)), Mond (see (Mond, 1974)) and others. Based on Mangasarian's second-order dual problem, Mond (see (Mond, 1974)) established some duality results under relatively simpler conditions involving a certain second-order generalization of the concept of convexity, while observed some possible computational advantages of second-order duality results, and also studied a pair of second-order symmetric dual problems. Mond's original notion of second-order convexity was followed by generalizations by other authors in different ways and applied establishing several second-order duality results for several classes of nonlinear programming problems. Although there exist various second-order duality results in the related literature for several classes of mathematical programming problems with a finite number of constraints, we feel our second-order duality results established in this paper are new and general in nature to the context of semiinfinite programming. For more details on second order duality results, we refer the reader (see (Aghezzaf, 2003) - (Zalmai & Zhang, 2007)), but more importantly, (see (Aghezzaf, 2003) - (Jeyakumar *b*, 1985), (Mond & Weir, 1981-1983), (Mond & Zhang, 1995) - (Zalmai & Zhang, 2007)).

Note that second-order duality for a conventional nonlinear programming problem is of the form

$$(P_0) \qquad \text{Minimize } \ f(x) \ \text{ subject to } \ g_i(x) \le 0, \quad i \in \underline{m}, \quad x \in \mathbb{R}^n,$$

where f and g_i, $i \in \underline{m}$, are twice differentiable real-valued functions defined on $\mathbb{R}^n$, was initially considered and studied by Mangasarian (see (Mangasarian, 1975)). The idea underlying his approach to constructing a second-order dual problem was based on taking linear and quadratic approximations of the objective and constraint functions about an arbitrary but fixed point, leading to the Wolfe dual of the approximated problem, and then allowing the fixed point to vary. Mangasarian (see (Mangasarian, 1975)), more specifically, formulated the following second-order dual problem for (P_0):

$$(D_0) \qquad \text{Maximize } \ f(y) + \sum_{i=1}^m u_i g_i(y) - \tfrac{1}{2}\Big\langle z, \Big[\nabla^2 f(y) + \sum_{i=1}^m u_i \nabla^2 g_i(y)\Big] z \Big\rangle$$

subject to

$$\nabla f(y) + \sum_{i=1}^{m} u_i \nabla g_i(y) + \Big[\nabla^2 f(y) + \sum_{i=1}^{m} u_i \nabla^2 g_i(y)\Big] z = 0,$$

$$y \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad u \geq 0, \quad z \in \mathbb{R}^n,$$

where $\nabla F(y)$ and $\nabla^2 F(y)$ denote, respectively, the gradient and Hessian of the function $F : \mathbb{R}^n \to \mathbb{R}$ evaluated at y and $\langle a, b\rangle$ denotes the inner product of the vectors a and b. Then, by imposing somewhat complicated conditions on f, g_i, $i \in \underline{m}$, and z, he proved weak, strong, and converse duality theorems for (P_0) and (D_0).

We observe that all the duality results established in this paper can easily be modified and restated for each one of the following classes of nonlinear programming problems, that are special cases of (P):

$(P1)$ Minimize $\dfrac{f_1(x)}{g_1(x)}$;

$(P2)$ Minimize $\max\limits_{1\leq i\leq p} f_i(x)$;

$(P3)$ $\underset{x\in\mathbb{F}}{\text{Minimize}}\ f_1(x)$,

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \{x \in \mathbb{R}^n : G_j(x,t) \leq 0 \text{ for all } t \in T_j,\ j \in \underline{q},\quad H_k(x,s) = 0 \text{ for all } s \in S_k,\ k \in \underline{r}\};$$

$(P4)$ Minimize $\max\limits_{1\leq i\leq p} \dfrac{f_i(x)}{g_i(x)}$

subject to

$$\tilde{G}_j(x) \leq 0, \quad j \in \underline{q}, \quad \tilde{H}_k(x) = 0, \quad k \in \underline{r}, \quad x \in \mathbb{R}^n,$$

where f_i and g_i, $i \in \underline{p}$, are as defined in the description of (P), and $\tilde{G}_j$, $j \in \underline{q}$, and $\tilde{H}_k$, $k \in \underline{r}$, are real-valued functions defined on X;

$(P5)$ $\underset{x\in\mathbb{G}}{\text{Minimize}}\ \dfrac{f_1(x)}{g_1(x)}$;

$(P6)$ $\underset{x\in\mathbb{G}}{\text{Minimize}}\ \max\limits_{1\leq i\leq p} f_i(x)$;

$(P7)$ $\underset{x\in\mathbb{G}}{\text{Minimize}}\ f_1(x)$,

where $\mathbb{G}$ is the feasible set of $(P4)$, that is,

$$\mathbb{G} = \{x \in \mathbb{R}^n : \tilde{G}_j(x) \leq 0,\ j \in \underline{q},\quad \tilde{H}_k(x) = 0,\ k \in \underline{r}\}.$$

2. Preliminaries

In this section we recall, the recently introduced major generalization $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)-$univexiity by Verma (see (Verma, 2012)) to the notion of the Zalmai type $(\mathcal{F}, \beta, \phi, \rho, \theta)$-univexity of higher order (See (Zalmai, 2012)) to the context of parametric duality models in semiinfinite discrete minimax fractional programming. The obtained notion, in fact, reduces to most of the existing notions of invexities and univexities in the literature.

Recall that a function $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}$ is said to be *sublinear*(*superlinear*) if

$$\mathcal{G}(x+y) \leq (\geq) \mathcal{G}(x) + \mathcal{G}(y) \, \forall \, x, y \in \mathbb{R}^n,$$

and $\mathcal{G}(ax) = a\mathcal{G}(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}_+ = [0, \infty)$.

Let $x^* \in X$ and let us assume that the function $f : X \to \mathbb{R}$ is twice continuously differentiable at x^*.

Definition 2.1. The function f is said to be *(strictly)* $(\mathcal{G}, \beta, \phi, h(x^*, z), \rho, \theta)$*-univex* at x^* of higher order if there exist functions $\beta : X \times X \to \mathbb{R}_+ \backslash \{0\} = (0, \infty)$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{G}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z))(>) \geq \mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) + \rho(x, x^*)\|\theta(x, x^*)\|^2,$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with respect to the second component.

Definition 2.2. The function f is said to be *(strictly)* $(\mathcal{G}, \beta, \phi, h(x^*, z), \rho, \theta)$*-pseudounivex* at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{G}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) \geq -\rho(x, x^*)\|\theta(x, x^*)\|^2 \Rightarrow \phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z))(>) \geq 0,$$

equivalently,

$$\phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z))(\leq) < 0 \Rightarrow \mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) < -\rho(x, x^*)\|\theta(x, x^*)\|^2,$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with respect to the second component.

Definition 2.3. The function f is said to be *prestrictly* $(\mathcal{G}, \beta, \phi, h(x^*, z), \rho, \theta)$*-pseudounivex* at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{G}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X (x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) > -\rho(x, x^*)\|\theta(x, x^*)\|^2 \Rightarrow \phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z)) \geq 0,$$

equivalently,

$$\phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z)) < 0 \Rightarrow \\ \mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) \le -\rho(x, x^*)\|\theta(x, x^*)\|^2,$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with respect to the second component.

Definition 2.4. The function f is said to be *(prestrictly)*$(\mathcal{G}, \beta, \phi, h(x^*, z), \rho, \theta)$*-quasiunivex* at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{G}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z))(<) \le 0 \\ \Rightarrow \quad \mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) \le -\rho(x, x^*)\|\theta(x, x^*)\|^2,$$

equivalently,

$$\mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) > -\rho(x, x^*)\|\theta(x, x^*)\|^2 \Rightarrow \\ \phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z))(\ge) > 0,$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with respect to the second component.

Definition 2.5. The function f is said to be *strictly* $(\mathcal{G}, \beta, \phi, h(x^*, z), \rho, \theta)$*-quasiunivex* at x^* if there exist functions $\beta : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $\phi : \mathbb{R} \to \mathbb{R}$, $\rho : X \times X \to \mathbb{R}$, $\theta : X \times X \to \mathbb{R}^n$, and a sublinear function $\mathcal{G}(x, x^*; \cdot) : \mathbb{R}^n \to \mathbb{R}$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z)) \le 0 \\ \Rightarrow \quad \mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) < -\rho(x, x^*)\|\theta(x, x^*)\|^2,$$

equivalently,

$$\mathcal{G}(x, x^*; \beta(x, x^*)[\nabla_z h(x^*, z)]) \ge -\rho(x, x^*)\|\theta(x, x^*)\|^2 \Rightarrow \\ \phi(f(x) - f(x^*) + \langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z)) > 0,$$

where $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with respect to the second component.

We note that the generalized $(\mathcal{G}, \beta, \phi, h(\cdot, \cdot), \rho, \theta)$-univexities (see (Verma, 2012)) at x^* of higher order reduce to the Zalmai type $(\mathcal{F}, \beta, \phi, \rho, \theta)$-univexities (see (Zalmai, 2012)) of higher-order if we set

$$h(x^*, z) = \langle z, \nabla f(x^*)\rangle + \frac{1}{2}\langle z, \nabla^2 f(x^*) z\rangle.$$

Then, we have

$$\nabla_z h(x^*, z) = \nabla f(x^*) + \nabla^2 f(x^*) z$$

and

$$\langle z, \nabla_z h(x^*, z)\rangle - h(x^*, z) = \frac{1}{2}\langle z, \nabla^2 f(x^*)z\rangle.$$

We observe some of the implications from the above definitions as follows: if f is $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-univex at x^*, then it is both $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-pseudounivex and $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-quasiunivex at x^*, if f is $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-quasiunivex at x^*, then it is prestrictly $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-quasiunivex at x^*, and if f is strictly $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-pseudounivex at x^*, then it is $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-quasiunivex at x^*.

Note that during the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. We conclude this section by recalling a set of parametric necessary optimality conditions for (P) based on the following result.

Theorem 2.1. *(See (Verma, 2013)) Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1\le i\le p} f_i(x^*)/g_i(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z,t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z,s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an optimal solution of (P), if the second order generalized Abadie constraint qualification holds at x^*, and if for any critical direction y, the set cone*

$$\{\left(\nabla G_j(x^*,t), \langle y, \nabla^2 G_j(x^*,t)y\rangle\right) : t \in \hat{T}_j(x^*), j \in \underline{q}\}$$
$$+ \quad span\{\left(\nabla H_k(x^*,s), \langle y, \nabla^2 H_k(x^*,s)y\rangle\right) : s \in S_k, k \in \underline{r}\},$$

where $\hat{T}_j(x^*) = \{t \in T_j : G_j(x^*,t) = 0\}$, is closed, then there exist $u^* \in U = \{u \in \mathbb{R}^p : u \ge 0, \sum_{i=1}^p u_i = 1\}$ and integers ν_0^* and ν^*, with $0 \le \nu_0^* \le \nu^* \le n+1$, such that there exist ν_0^* indices j_m, with $1 \le j_m \le q$, together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0^*}$, $\nu^* - \nu_0^*$ indices k_m, with $1 \le k_m \le r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*}\backslash\underline{\nu_0^*}$, and ν^* real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^*(\nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^*[\nabla G_{j_m}(x^*,t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_k(x^*,s^m) = 0, \tag{2.1}$$

$$\langle y, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^*\nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^*\nabla^2 G_{j_m}(x^*,t^m) + \sum_{m=\nu_0^*+1}^{\nu^*} v_m^*\nabla^2 H_k(x^*,s^m)\Big]y\rangle \ge 0. \tag{2.2}$$

We shall call x a *normal* feasible solution of (P) if x satisfies all the constraints of (P), if the generalized Abadie constraint qualification holds at x, and if the set $cone\{\nabla G_j(x,t) : t \in \hat{T}_j(x), j \in \underline{q}\} + span\{\nabla H_k(x,s) : s \in S_k, k \in \underline{r}\}$ is closed.

The above theorem on the necessary optimality conditions provides us with clear guidelines for formulating numerous Wolfe-type duality models for (P). From now on, the functions f_i, g_i, $i \in \underline{p}$, $z \to G_j(z,t)$, and $z \to H_k(z,s)$ are twice continuously differentiable on X for all $t \in T_j$, $j \in \underline{q}$, and all $s \in S_k$, $k \in \underline{r}$.

3. Duality Models

In this section, we consider two duality models with special constraint structures that allow for a greater variety of generalized $(\mathcal{G}, \beta, \phi, h(x,z), \rho, \theta)$-univexity conditions under which duality can be established based on the following set:

$$\mathbb{H} = \Big\{(y, z, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s}) : \; y \in X; \; z \in \mathbb{R}^n; \; u \in U; \; 0 \le \nu_0 \le \nu \le n+1;$$
$$v \in \mathbb{R}^\nu, v_i > 0, 1 \le i \le \nu_0; \; \lambda \in \mathbb{R}_+; \; J_{\nu_0} = (j_1, j_2, \ldots, j_{\nu_0}), \; 1 \le j_i \le q; \; K_{\nu\backslash\nu_0} =$$
$$(k_{\nu_{0+1}}, \ldots, k_\nu), 1 \le k_i \le r; \; \bar{t} = (t^1, t^2, \ldots, t^{\nu_0}), \; t^i \in T_{j_i}; \; \bar{s} = (s^{\nu_0+1}, \ldots, s^\nu), \; s^i \in S_{k_i}\Big\}.$$

Consider the following two problems:

(DI) $$\sup_{(y,z,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$$

subject to

$$\sum_{i=1}^{p} u_i[\nabla_z h_i(y,z) - \lambda\nabla_z \kappa_i(y,z)] + \sum_{m=1}^{\nu_0} v_m[\nabla_z \mu_{j_m}(y, t^m, z)]$$
$$+ \sum_{m=\nu_0+1}^{\nu} v_m[\nabla_z \psi_{k_m}(y, s^m, z)] = 0, \tag{3.1}$$

$$f_i(y) - \lambda g_i(y) + \sum_{i=1}^{p} u_i[h_i(y,z) - \lambda\kappa_i(y,z) - \langle z, \nabla_z h_i(y,z) - \lambda\nabla_z \kappa_i(y,z)]\rangle \ge 0, \quad i \in \underline{p}, \tag{3.2}$$

$$G_{j_m}(y, t^m) + \mu_{j_m}(y, t^m, z) - \langle z, \nabla_z \mu_{j_m}(y, t^m, z)\rangle \ge 0, \quad m \in \underline{\nu_0}, \tag{3.3}$$

$$v_m H_{k_m}(y, s^m) + v_m\psi_{k_m}(y, s^m, z) - \langle z, v_m\nabla_z \psi_{k_m}(y, s^m, z)\rangle \ge 0, \quad m \in \underline{\nu}\backslash\underline{\nu_0}; \tag{3.4}$$

$(\tilde{D}I)$ $$\sup_{(y,z,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$$

subject to (3.2)-(3.4) and

$$\mathcal{G}\Big(x, y; \sum_{i=1}^{p} u_i[\nabla_z h_i(y,z)] - \sum_{i=1}^{p} u_i\lambda[\nabla_z \kappa_i(y,z)] + \sum_{m=1}^{\nu_0} v_m[\nabla_z \mu_{j_m}(y, z, t^m)]$$
$$+ \sum_{m=\nu_0+1}^{\nu} v_m[\nabla_z \psi_{k_m}(y, z, s^m)] \ge 0 \;\text{ for all } x \in \mathbb{F}, \tag{3.5}$$

where $\mathcal{G}(x, y; \cdot)$ is a sublinear function from $\mathbb{R}^n$ to $\mathbb{R}$.

Note that if we Compare (DI) and $(\tilde{D}I)$, we see that $(\tilde{D}I)$ is relatively more general than (DI) in the sense that any feasible solution of (DI) is also feasible for $(\tilde{D}I)$, but the converse may not be necessarily true.

Lemma 3.1. *(See (Zalmai, 2012)) For each $x \in X$,*

$$\varphi(x) \equiv \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)} = \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

The next theorem shows that (DI) is a dual problem for primal (P).

Theorem 3.1. *(Weak Duality) Let x and $w = (y, z, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu \backslash \nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DI), respectively, and let us assume that any one of the following five sets of hypotheses is satisfied:*

(a) *(i) for each $i \in \underline{p}$, f_i is $(\mathcal{G}, \beta, \bar{\phi}, h_i(\cdot,\cdot), \bar{\rho}_i, \theta)$-univex and $-g_i$ is $(\mathcal{G}, \beta, \bar{\phi}, \kappa_i(\cdot,\cdot), \tilde{\rho}_i, \theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \ge 0 \Rightarrow a \ge 0$;*

(ii) the function $\xi \to G_{j_m}(\xi, t^m)$ is $(\mathcal{G}, \beta, \hat{\phi}_m, \mu_m(\cdot,\cdot), \hat{\rho}_m, \theta)$-quasiunivex at y, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;

(iii) the function $\xi \to v_m H_{k_m}(\xi, s^m)$ is $(\mathcal{G}, \beta, \breve{\phi}_m, \psi_m(\cdot,\cdot), \breve{\rho}_m, \theta)$- quasiunivex at y, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \backslash \underline{\nu_0}$;

(iv) $\rho^*(x, y) + \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x, y) \ge 0$ where $\rho^*(x, y) = \sum_{i=1}^{p} u_i[\bar{\rho}_i(x, y) + \lambda \tilde{\rho}_i(x, y)]$;

(b) *(i) for each $i \in \underline{p}$, f_i is $(\mathcal{G}, \beta, \bar{\phi}, h_i(\cdot,\cdot), \bar{\rho}_i, \theta)$-univex and $-g_i$ is $(\mathcal{G}, \beta, \bar{\phi}, \kappa_i(\cdot,\cdot), \tilde{\rho}_i, \theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \ge 0 \Rightarrow a \ge 0$;*

(ii) the function $\xi \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi, t^m)$ is $(\mathcal{G}, \beta, \hat{\phi}, \mu_m(\cdot,\cdot), \hat{\rho}, \theta)$-quasiunivex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iii) the function $\xi \to v_m H_{k_m}(\xi, s^m)$ is $(\mathcal{G}, \beta, \breve{\phi}_m, \psi_m(\cdot,\cdot), \breve{\rho}_m, \theta)$-quasiunivex at y, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \backslash \underline{\nu_0}$;

(iv) $\rho^*(x, y) + \hat{\rho}(x, y) + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x, y) \ge 0$;

(c) *(i) for each $i \in \underline{p}$, f_i is $(\mathcal{G}, \beta, \bar{\phi}, h_i(\cdot,\cdot), \bar{\rho}_i, \theta)$-univex and $-g_i$ is $(\mathcal{G}, \beta, \bar{\phi}, \tilde{\rho}_i, \kappa_i(\cdot,\cdot), \theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \ge 0 \Rightarrow a \ge 0$;*

(ii) the function $\xi \to G_{j_m}(\xi, t^m)$ is $(\mathcal{G}, \beta, \hat{\phi}_m, \mu_m(\cdot,\cdot), \hat{\rho}_m, \theta)$-quasiunivex at y, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;

(iii) the function $\xi \to \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi, s^m)$ is $(\mathcal{G}, \beta, \breve{\phi}, \psi_m(\cdot,\cdot), \breve{\rho}, \theta)$-quasiunivex at y, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$;

(iv) $\rho^*(x, y) + \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) + \breve{\rho}(x, y) \ge 0$;

(d) *(i) for each $i \in \underline{p}$, f_i is $(\mathcal{G}, \beta, \bar{\phi}, h_i(\cdot,\cdot), \bar{\rho}_i, \theta)$-univex and $-g_i$ is $(\mathcal{G}, \beta, \bar{\phi}, \kappa_i(\cdot,\cdot), \tilde{\rho}_i, \theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \ge 0 \Rightarrow a \ge 0$;*

(ii) *the function* $\xi \rightarrow \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi, t^m)$ *is* $(\mathcal{G}, \beta, \hat{\phi}, \mu_m(\cdot,\cdot), \hat{\rho}, \theta)$*-quasiunivex at* y*,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) *the function* $\xi \rightarrow \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi, s^m)$ *is* $(\mathcal{G}, \beta, \breve{\phi}, \psi_m(\cdot,\cdot), \breve{\rho}, \theta)$*-quasiunivex at* y*,* $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$*;*

(iv) $\rho^*(x, y) + \hat{\rho}(x, y) + \breve{\rho}(x, y) \geq 0$*;*

(e) *(i)* *for each* $i \in \underline{p}$*,* f_i *is* $(\mathcal{G}, \beta, \bar{\phi}, h_i(\cdot,\cdot), \bar{\rho}_i, \theta)$*-univex and* $-g_i$ *is* $(\mathcal{G}, \beta, \bar{\phi}, \kappa_i(\cdot,\cdot), \tilde{\rho}_i, \theta)$*-univex at* y*,* $\bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) *the function* $\xi \rightarrow \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi, s^m)$ *is* $(\mathcal{G}, \beta, \hat{\phi}, \tau_m, \hat{\rho}, \theta)$*-quasiunivex at* y*,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\rho^*(x, y) + \hat{\rho}(x, y) \geq 0$*.*

Then $\varphi(x) \geq \lambda$*.*

Proof. (a): Applying (i), we have the following inequality:

$$\bar{\phi}\Big(\sum_{i=1}^{p} u_i[f_i(x) - f_i(y)] + \Big\langle z, \sum_{i=1}^{p} u_i \nabla_z h_i(y, z) \Big\rangle - \sum_{i=1}^{p} u_i h_i(y, z)$$
$$+ \lambda[\sum_{i=1}^{p} u_i[-g_i(x) + g_i(y)] - \Big\langle z, \sum_{i=1}^{p} u_i \nabla_z \kappa_i(y, z) \Big\rangle + \sum_{i=1}^{p} u_i \kappa_i(y, z)]\Big)$$
$$\geq \mathcal{G}\Big(x, y; \beta(x, y) \sum_{i=1}^{p} u_i \{\nabla_z h_i(y, z) - \lambda \nabla_z \kappa_i(y, z)\}\Big) + \sum_{i=1}^{p} u_i [\bar{\rho}_i(x, y) + \lambda \tilde{\rho}_i(x, y)] \|\theta(x, y)\|^2. \quad (3.6)$$

From the primal feasibility of x, dual feasibility of w, and (3.3), we find that

$$G_{j_m}(x, t^m) \leq 0 \leq G_{j_m}(y, t^m) + \mu_{j_m}(y, t^m, z) - \langle z, \nabla_z \mu_{j_m}(y, t^m, z) \rangle, \quad m \in \underline{\nu_0},$$

and hence using the properties of the functions $\hat{\phi}_m$, we have

$$\hat{\phi}_m(G_{j_m}(x, t^m) - [G_{j_m}(y, t^m) + \mu_{j_m}(y, t^m, z) - \langle z, \nabla_z \mu_{j_m}(y, t^m, z) \rangle]) \leq 0,$$

which from (ii) implies that $\mathcal{G}(x, y; \beta(x, y)[\nabla_z \mu_{j_m}(y, t^m, z)]) \leq -\hat{\rho}_m(x, y)\|\theta(x, y)\|^2$. As $v_m > 0$ for each $m \in \underline{\nu_0}$, the above inequality yield

$$\mathcal{G}\Big(x, y; \beta(x, y) \sum_{m=1}^{\nu_0} v_m [\langle z, \nabla_z \mu_{j_m}(y, t^m, z) \rangle]\Big) \leq -\sum_{m=1}^{\nu_0} v_m \hat{\rho}_m(x, y) \|\theta(x, y)\|^2. \quad (3.7)$$

Similarly, from the primal feasibility of x, dual feasibility of w, (3.4), and (iii) we deduce (since $v_m > 0$ for each $m \in \underline{\nu} \backslash \underline{\nu_0}$) that

$$\mathcal{G}\Big(x, y; \beta(x, y) \sum_{m=\nu_0+1}^{\nu} v_m [\nabla_z \psi_{j_m}(y, t^m, z)]\Big) \leq -\sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x, y) \|\theta(x, y)\|^2. \quad (3.8)$$

Now, based on the positivity of $\beta(x,y)$, sublinearity of $\mathcal{G}(x,y;\cdot)$, and (3.1), we conclude that

$$\mathcal{G}\Big(x,y;\beta(x,y)\sum_{i=1}^{p}u_i\{\nabla_z h_i(y,z)-\lambda\nabla_z \kappa_i(y,z)\}\Big)+\mathcal{G}(x,y;\beta(x,y)\sum_{m=1}^{\nu_0}v_m[\nabla_z \mu_{j_m}(y,t^m,z)])$$
$$+\quad \mathcal{G}\Big(x,y;\beta(x,y)\sum_{m=\nu_0+1}^{\nu}v_m[\nabla_z \psi_{j_m}(y,t^m,z)]\Big)\geq 0. \tag{3.9}$$

Next, applying (3.9) to (3.6), and then combining with (3.7) and (3.8) and using (iv), we have

$$\begin{aligned}
&\bar{\phi}\Big(\sum_{i=1}^{p}u_i[f_i(x)-f_i(y)]+\Big\langle z,\sum_{i=1}^{p}u_i\nabla_z h_i(y,z)\Big\rangle-\sum_{i=1}^{p}u_ih_i(y,z)\\
+\ &\lambda[\sum_{i=1}^{p}u_i[-g_i(x)+g_i(y)]-\Big\langle z,\sum_{i=1}^{p}u_i\nabla_z \kappa_i(y,z)\Big\rangle+\sum_{i=1}^{p}u_i\kappa_i(y,z)]\Big)\\
\geq\ &\mathcal{G}\Big(x,y;\beta(x,y)\sum_{i=1}^{p}u_i\{\nabla_z h_i(y,z)-\lambda\nabla_z \kappa_i(y,z)\}\Big)\\
+\ &\sum_{i=1}^{p}u_i[\bar{\rho}_i(x,y)+\lambda\tilde{\rho}_i(x,y)]\|\theta(x,y)\|^2\geq-\Big[\mathcal{G}(x,y;\beta(x,y)\sum_{m=1}^{\nu_0}v_m[\nabla_z \mu_{j_m}(y,t^m,z)])\\
+\ &\mathcal{G}\Big(x,y;\beta(x,y)\sum_{m=\nu_0+1}^{\nu}v_m[\nabla_z \psi_{j_m}(y,t^m,z)]\Big)\Big]\geq\sum_{m=1}^{\nu_0}v_m\hat{\rho}_m(x,y)\|\theta(x,y)\|^2+\sum_{m=\nu_0+1}^{\nu}\breve{\rho}_m(x,y)\|\theta(x,y)\|^2\\
+\ &\sum_{i=1}^{p}u_i[\bar{\rho}_i(x,y)+\lambda\tilde{\rho}_i(x,y)]\|\theta(x,y)\|^2=\sum_{m=1}^{\nu_0}v_m\hat{\rho}_m(x,y)\|\theta(x,y)\|^2\\
+\ &\sum_{m=\nu_0+1}^{\nu}\breve{\rho}_m(x,y)\|\theta(x,y)\|^2+\rho^*(x,y)\|\theta(x,y)\|^2\geq 0.
\end{aligned}$$

But $\bar{\phi}(a)\geq 0 \ \Rightarrow\ a\geq 0$ and hence because of (3.2) the above inequality reduces to

$$\sum_{i=1}^{p}u_i[f_i(x)-\lambda g_i(x)]\geq 0.$$

Finally, this inequality using Lemma 3.1 leads to the weak duality inequality as follows:

$$\varphi(x)=\max_{1\leq i\leq p}\frac{f_i(x)}{g_i(x)}=\max_{u\in U}\frac{\sum_{i=1}^{p}u_if_i(x)}{\sum_{i=1}^{p}u_ig_i(x)}\geq\lambda.$$

(b) - (e) : The proofs are similar to that of part (a). □

The following theorem is based on the $(\mathcal{G},\beta,h_i(\cdot,\cdot),\tilde{\rho}_i,\theta)$-univexities and quasiunivexities.

Theorem 3.2. *(Weak Duality) Let x and $w=(y,z,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})$ be arbitrary feasible solutions of (P) and (DI), respectively, and let us assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{G},\beta,h_i(\cdot,\cdot),\bar{\rho}_i,\theta)$-univex and $-g_i$ is $(\mathcal{G},\beta,\kappa_i(\cdot,\cdot),\tilde{\rho}_i,\theta)$-univex at y,

(ii) the function $\xi \to G_{j_m}(\xi,t^m)$ is $(\mathcal{G},\beta,\mu_m(\cdot,\cdot),\hat{\rho}_m,\theta)$-quasiunivex at y, for each $m \in \underline{\nu_0}$;

(iii) the function $\xi \to v_m H_{k_m}(\xi,s^m)$ is $(\mathcal{G},\beta,\psi_m(\cdot,\cdot),\breve{\rho}_m,\theta)$-quasi univex at y, for each $m \in \underline{\nu}\backslash\underline{\nu_0}$;

(iv) $\rho^*(x,y) + \sum_{m=1}^{\nu_0} v_m\hat{\rho}_m(x,y) + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x,y) \geq 0$ where $\rho^*(x,y) = \sum_{i=1}^{p} u_i[\bar{\rho}_i(x,y) + \lambda\tilde{\rho}_i(x,y)]$;

(b) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{G},\beta,h_i(\cdot,\cdot),\bar{\rho}_i,\theta)$-univex and $-g_i$ is $(\mathcal{G},\beta,\kappa_i(\cdot,\cdot),\tilde{\rho}_i,\theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$.

(ii) the function $\xi \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi,t^m)$ is $(\mathcal{G},\beta,\mu_m(\cdot,\cdot),\hat{\rho},\theta)$-quasiunivex at y.

(iii) the function $\xi \to v_m H_{k_m}(\xi,s^m)$ is $(\mathcal{G},\beta,\breve{\phi}_m,\psi_m(\cdot,\cdot),\breve{\rho}_m,\theta)$-quasiunivex at y.

(iv) $\rho^*(x,y) + \hat{\rho}(x,y) + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x,y) \geq 0$;

(c) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{G},\beta,h_i(\cdot,\cdot),\bar{\rho}_i,\theta)$-univex and $-g_i$ is $(\mathcal{G},\beta,\tilde{\rho}_i,\kappa_i(\cdot,\cdot),\theta)$-univex at y.

(ii) the function $\xi \to G_{j_m}(\xi,t^m)$ is $(\mathcal{G},\beta,\mu_m(\cdot,\cdot),\hat{\rho}_m,\theta)$-quasiunivex at y.

(iii) the function $\xi \to \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi,s^m)$ is $(\mathcal{G},\beta,\psi_m(\cdot,\cdot),\breve{\rho},\theta)$-quasiunivex at y.

(iv) $\rho^*(x,y) + \sum_{m=1}^{\nu_0} v_m\hat{\rho}_m(x,y) + \breve{\rho}(x,y) \geq 0$;

(d) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{G},\beta,h_i(\cdot,\cdot),\bar{\rho}_i,\theta)$-univex and $-g_i$ is $(\mathcal{G},\beta,\kappa_i(\cdot,\cdot),\tilde{\rho}_i,\theta)$-univex at y.

(ii) the function $\xi \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi,t^m)$ is $(\mathcal{G},\beta,\hat{\phi},\mu_m(\cdot,\cdot),\hat{\rho},\theta)$-quasiunivex at y.

(iii) the function $\xi \to \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi,s^m)$ is $(\mathcal{G},\beta,\breve{\phi},\psi_m(\cdot,\cdot),\breve{\rho},\theta)$-quasiunivex at y.

(iv) $\rho^*(x,y) + \hat{\rho}(x,y) + \breve{\rho}(x,y) \geq 0$;

(e) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{G},\beta,h_i(\cdot,\cdot),\bar{\rho}_i,\theta)$-univex and $-g_i$ is $(\mathcal{G},\beta,\kappa_i(\cdot,\cdot),\tilde{\rho}_i,\theta)$-univex at y.

(ii) the function $\xi \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi,t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi,s^m)$ is $(\mathcal{G},\beta,\tau_m,\hat{\rho},\theta)$-quasiunivex at y.

(iii) $\rho^*(x,y) + \hat{\rho}(x,y) \geq 0$.

Then $\varphi(x) \geq \lambda$.

Proof. (a): Applying (i), we have the following inequality:

$$\begin{aligned}&\sum_{i=1}^{p} u_i[f_i(x)-f_i(y)] + \Big\langle z, \sum_{i=1}^{p} u_i\nabla_z h_i(y,z)\Big\rangle - \sum_{i=1}^{p} u_i h_i(y,z)\\ &+\lambda[\sum_{i=1}^{p} u_i[-g_i(x)+g_i(y)] - \Big\langle z, \sum_{i=1}^{p} u_i\nabla_z \kappa_i(y,z)\Big\rangle + \sum_{i=1}^{p} u_i\kappa_i(y,z)]\\ &\geq \mathcal{G}\Big(x,y;\beta(x,y)\sum_{i=1}^{p} u_i\{\nabla_z h_i(y,z)-\lambda\nabla_z\kappa_i(y,z)\}\Big) + \sum_{i=1}^{p} u_i[\bar{\rho}_i(x,y)+\lambda\tilde{\rho}_i(x,y)]\|\theta(x,y)\|^2. \quad (3.10)\end{aligned}$$

From the primal feasibility of x, dual feasibility of w, and (3.3), we find that

$$G_{j_m}(x,t^m) \leq 0 \leq G_{j_m}(y,t^m) + \mu_{j_m}(y,t^m,z) - \langle z, \nabla_z \mu_{j_m}(y,t^m,z)\rangle, \quad m \in \underline{\nu_0}.$$

Then we have $G_{j_m}(x,t^m) - [G_{j_m}(y,t^m) + \mu_{j_m}(y,t^m,z) - \langle z,\nabla_z\mu_{j_m}(y,t^m,z)\rangle] \leq 0$, which from (ii) implies that $\mathcal{G}(x,y;\beta(x,y)[\nabla_z\mu_{j_m}(y,t^m,z)]) \leq -\hat{\rho}_m(x,y)\|\theta(x,y)\|^2$. As $v_m > 0$ for each $m \in \underline{\nu_0}$, the above inequalities yield

$$\mathcal{G}\Big(x,y;\beta(x,y)\sum_{m=1}^{\nu_0} v_m[\langle z,\nabla_z\mu_{j_m}(y,t^m,z)\rangle]\Big) \leq -\sum_{m=1}^{\nu_0} v_m\hat{\rho}_m(x,y)\|\theta(x,y)\|^2. \quad (3.11)$$

Similarly, from the primal feasibility of x, dual feasibility of w, (3.4), and (iii) we deduce that

$$\mathcal{G}\Big(x,y;\beta(x,y)\sum_{m=\nu_0+1}^{\nu} v_m[\nabla_z\psi_{j_m}(y,t^m,z)]\Big) \leq -\sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x,y)\|\theta(x,y)\|^2. \quad (3.12)$$

Now, based on the positivity of $\beta(x,y)$, sublinearity of $\mathcal{G}(x,y;\cdot)$, and applying (3.1), we conclude that

$$\begin{aligned}&\mathcal{G}\Big(x,y;\beta(x,y)\sum_{i=1}^{p} u_i\{\nabla_z h_i(y,z)-\lambda\nabla_z\kappa_i(y,z)\}\Big) + \mathcal{G}(x,y;\beta(x,y)\sum_{m=1}^{\nu_0} v_m[\nabla_z\mu_{j_m}(y,t^m,z)])\\ &+\ \mathcal{G}\Big(x,y;\beta(x,y)\sum_{m=\nu_0+1}^{\nu} v_m[\nabla_z\psi_{j_m}(y,t^m,z)]\Big) \geq 0. \quad (3.13)\end{aligned}$$

Next, applying (3.13) to (3.10), and then combining with (3.11) and (3.12) and using (iv), we have

$$\begin{aligned}&\Big(\sum_{i=1}^{p} u_i[f_i(x)-f_i(y)] + \Big\langle z,\sum_{i=1}^{p} u_i\nabla_z h_i(y,z)\Big\rangle - \sum_{i=1}^{p} u_i h_i(y,z)\\ +\ &\lambda[\sum_{i=1}^{p} u_i[-g_i(x)+g_i(y)] - \Big\langle z,\sum_{i=1}^{p} u_i\nabla_z\kappa_i(y,z)\Big\rangle + \sum_{i=1}^{p} u_i\kappa_i(y,z)]\Big)\\ \geq\ &\Big(\rho^*(x,y) + \sum_{m=1}^{\nu_0} v_m\hat{\rho}_m(x,y) + \sum_{m=\nu_0+1}^{\nu}\breve{\rho}_m(x,y)\Big)\|\theta(x,y)\|^2 \geq 0.\end{aligned}$$

Hence because of (3.2) the above inequality reduces to

$$\sum_{i=1}^{p} u_i[f_i(x) - \lambda g_i(x)] \geq 0.$$

Finally, this inequality using Lemma 3.1 leads to the weak duality inequality as follows:

$$\varphi(x) \equiv \max_{1\leq i\leq p} \frac{f_i(x)}{g_i(x)} = \max_{u\in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} \geq \lambda.$$

(b) - (e) : The proofs are similar to that of part (a). □

Theorem 3.3. *(Strict Converse Duality) Let x^* be a normal optimal solution of (P), let $\tilde{w} = (\tilde{x}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{\lambda}, \tilde{v}, \tilde{v}_0, J_{\tilde{v}_0}, K_{\tilde{v}\backslash\tilde{v}_0}, \bar{\tilde{t}}, \bar{\tilde{s}})$ be an optimal solution of (DI), and assume that any one of the following five sets of conditions is satisfied:*

(a) The assumptions specified in part (a) of Theorem 3.2 are satisfied for the feasible solution $\tilde{w}$ of (DI). Moreover, $\bar{\phi}(a) > 0 \Rightarrow a > 0$, f_i is strictly $(\mathcal{G}, \beta, \bar{\phi}, h(\cdot,\cdot), \bar{\rho}_i, \theta)$-univex at $\tilde{x}$ for at least one $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive, or $-g_i$ is strictly $(\mathcal{G}, \beta, \bar{\phi}, \kappa(\cdot,\cdot), \tilde{\rho}_i, \theta)$-univex at $\tilde{x}$ for at least one $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive (and $\tilde{\lambda} > 0$), or $\xi \to G_{j_m}(\xi, \tilde{t}^m)$ is strictly $(\mathcal{G}, \beta, \hat{\phi}_m, \mu(\cdot,\cdot), \hat{\rho}_m, \theta)$-pseudounivex at $\tilde{x}$ for at least one $m \in \underline{\tilde{v}_0}$, or $\xi \to \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ is strictly $(\mathcal{G}, \beta, \breve{\phi}_m, \psi(\cdot,\cdot), \breve{\rho}_m, \theta)$-pseudounivex at $\tilde{x}$ for at least one $m \in \underline{\tilde{v}}\backslash\underline{\tilde{v}_0}$, or

$$\rho^*(x^*, \tilde{x}) + \sum_{m=1}^{\tilde{v}_0} \tilde{v}_m \hat{\rho}_m(x^*, \tilde{x}) + \sum_{m=\tilde{v}_0+1}^{\tilde{v}} \tilde{v}_m \breve{\rho}_m(x^*, \tilde{x}) > 0,$$

where $\rho^*(x^*, \tilde{x}) = \sum_{i=1}^{p} \tilde{u}_i[\bar{\rho}_i(x^*, \tilde{x}) + \tilde{\lambda}\tilde{\rho}_i(x^*, \tilde{x})]$.

(b) The assumptions specified in part (b) of Theorem 3.2 are satisfied for the feasible solution $\tilde{w}$ of (DI). Moreover, $\bar{\phi}(a) > 0 \Rightarrow a > 0$, f_i is strictly $(\mathcal{G}, \beta, \bar{\phi}, h(\cdot,\cdot), \bar{\rho}_i, \theta)$-univex at $\tilde{x}$ for at least one $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive, or $-g_i$ is strictly $(\mathcal{G}, \beta, \bar{\phi}, \kappa(\cdot,\cdot), \tilde{\rho}_i, \theta)$-univex at $\tilde{x}$ for at least one $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive (and $\tilde{\lambda} > 0$), or $\xi \to \sum_{m=1}^{\tilde{v}_0} \tilde{v}_m G_{j_m}(\xi, \tilde{t}^m)$ is strictly $(\mathcal{G}, \beta, \hat{\phi}, \mu(\cdot,\cdot), \hat{\rho}, \theta)$-pseudounivex at $\tilde{x}$, or $\xi \to \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ is strictly $(\mathcal{G}, \beta, \breve{\phi}_m, \psi(\cdot,\cdot), \breve{\rho}_m, \theta)$-pseudounivex at $\tilde{x}$ for at least one $m \in \underline{\tilde{v}}\backslash\underline{\tilde{v}_0}$, or $\rho^*(x^*, \tilde{x}) + \hat{\rho}(x^*, \tilde{x}) + \sum_{m=\tilde{v}_0+1}^{\tilde{v}} \tilde{v}_m \breve{\rho}_m(x^*, \tilde{x}) > 0$.

(c) The assumptions specified in part (c) of Theorem 3.2 are satisfied for the feasible solution $\tilde{w}$ of (DI). Moreover, $\bar{\phi}(a) > 0 \Rightarrow a > 0$, f_i is strictly $(\mathcal{G}, \beta, \bar{\phi}, h(\cdot,\cdot), \bar{\rho}_i, \theta)$-univex at $\tilde{x}$ for at least one $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive, or $-g_i$ is strictly $(\mathcal{G}, \beta, \bar{\phi}, \kappa(\cdot,\cdot), \tilde{\rho}_i, \theta)$-univex at $\tilde{x}$ for at least one $i \in \underline{p}$ with the corresponding component $\tilde{u}_i$ of $\tilde{u}$ positive (and $\tilde{\lambda} > 0$), or $\xi \to G_{j_m}(\xi, \tilde{t}^m)$ is strictly $(\mathcal{G}, \beta, \hat{\phi}_m, \mu(\cdot,\cdot), \hat{\rho}_m, \theta)$-pseudounivex at $\tilde{x}$ for at least one $m \in \underline{\tilde{v}_0}$, or $\xi \to \sum_{m=\tilde{v}_0+1}^{\tilde{v}} \tilde{v}_m H_{j_m}(\xi, \tilde{s}^m)$ is strictly $(\mathcal{G}, \beta, \breve{\phi}, \psi(\cdot,\cdot), \breve{\rho}, \theta)$-pseudounivex at $\tilde{x}$, or $\rho^*(x^*, \tilde{x}) + \sum_{m=1}^{\tilde{v}_0} \tilde{v}_m \hat{\rho}_m(x^*, \tilde{x}) + \tilde{v}_m \breve{\rho}(x^*, \tilde{x}) > 0$.

(d) *The assumptions specified in part (d) of Theorem 3.2 are satisfied for the feasible solution* $\tilde{w}$ *of (DI). Moreover,* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, f_i *is strictly* $(\mathcal{G}, \beta, \bar{\phi}, h(\cdot,\cdot), \bar{\rho}_i, \theta)$*-univex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $-g_i$ *is strictly* $(\mathcal{G}, \beta, \bar{\phi}, \kappa(\cdot,\cdot), \tilde{\rho}_i, \theta)$*-univex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive (and* $\tilde{\lambda} > 0$*), or* $\xi \to \sum_{m=1}^{\tilde{\nu}_0} \tilde{v}_m G_{j_m}(\xi, \tilde{t}^m)$ *is strictly* $(\mathcal{G}, \beta, \hat{\phi}, \mu(\cdot,\cdot), \hat{\rho}, \theta)$*-pseudounivex at* $\tilde{x}$*, or* $\xi \to \sum_{m=\tilde{\nu}_0+1}^{\tilde{\nu}} \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ *is strictly* $(\mathcal{G}, \beta, \breve{\phi}, \psi(\cdot,\cdot), \breve{\rho}_m, \theta)$*-pseudounivex at* $\tilde{x}$*, or* $\rho^*(x^*, \tilde{x}) + \hat{\rho}(x^*, \tilde{x}) + \breve{\rho}(x^*, \tilde{x}) > 0$.

(e) *The assumptions specified in part (e) of Theorem 3.2 are satisfied for the feasible solution* $\tilde{w}$ *of (DI). Moreover,* $\bar{\phi}(a) > 0 \Rightarrow a > 0$, f_i *is strictly* $(\mathcal{G}, \beta, \bar{\phi}, h(\cdot,\cdot), \bar{\rho}_i, \theta)$*-univex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive, or* $-g_i$ *is strictly* $(\mathcal{G}, \beta, \bar{\phi}, \kappa(\cdot,\cdot), \tilde{\rho}_i, \theta)$*-univex at* $\tilde{x}$ *for at least one* $i \in \underline{p}$ *with the corresponding component* $\tilde{u}_i$ *of* $\tilde{u}$ *positive (and* $\tilde{\lambda} > 0$*), or* $\xi \to \sum_{m=1}^{\tilde{\nu}_0} \tilde{v}_m G_{j_m}(\xi, \tilde{t}^m) + \sum_{m=\tilde{\nu}_0+1}^{\tilde{\nu}} \tilde{v}_m H_{k_m}(\xi, \tilde{s}^m)$ *is strictly* $(\mathcal{G}, \beta, \hat{\phi}, \tau(\cdot,\cdot), \hat{\rho}, \theta)$*-pseudounivex at* $\tilde{x}$*, or* $\rho^*(x^*, \tilde{x}) + \hat{\rho}(x^*, \tilde{x}) > 0$.

Then $\tilde{x} = x^*$ *and* $\varphi(x^*) = \tilde{\lambda}$.

Proof. The proof is similar to that of Theorem 3.2. □

4. Specialization I

In this section, we consider two duality models with special constraint structures that allow the generalized $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-univexity reduce to second order generalized $(\mathcal{F}, \beta, \phi, \rho, \theta)$-univexity introduced and studied by Zalmai (see (Zalmai, 2012)) under which duality can be established.

Consider the following two problems:

(DII) $$\sup_{(y,z,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\setminus\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$$

subject to

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m)$$
$$+ \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla^2 H_{k_m}(y, s^m) \Big\} z = 0, \quad (4.1)$$

$$f_i(y) - \lambda g_i(y) - \frac{1}{2}\langle z, [\nabla^2 f_i(y) - \lambda \nabla^2 g_i(y)] z\rangle \geq 0, \quad i \in \underline{p}, \quad (4.2)$$

$$G_{j_m}(y, t^m) - \frac{1}{2}\langle z, \nabla^2 G_{j_m}(y, t^m) z\rangle \geq 0, \quad m \in \underline{\nu_0}, \quad (4.3)$$

$$v_m H_{k_m}(y, s^m) - \frac{1}{2}\langle z, v_m \nabla^2 H_{k_m}(y, s^m) z\rangle \geq 0, \quad m \in \underline{\nu} \backslash \underline{\nu_0}; \quad (4.4)$$

$(\tilde{D}II)$ $\sup_{(y,z,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$ subject to (3.3) and (4.2) - (4.4).

The next theorem shows that (DII) is a dual problem for (P).

Theorem 4.1. *(Weak Duality) Let x and $w = (y, z, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each $i \in \underline{p}$, f_i is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-sounivex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-sounivex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *the function $\xi \to G_{j_m}(\xi, t^m)$ is $(\mathcal{F}, \beta, \hat{\phi}_m, \hat{\rho}_m, \theta)$-quasisounivex at y, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;*

(iii) *the function $\xi \to v_m H_{k_m}(\xi, s^m)$ is $(\mathcal{F}, \beta, \breve{\phi}_m, \breve{\rho}_m, \theta)$-quasisounivex at y, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu}\backslash\underline{\nu_0}$;*

(iv) *$\rho^*(x, y) + \sum_{m=1}^{\nu_0} v_m\hat{\rho}_m(x, y) + \sum_{m=\nu_0+1}^{\nu} v_m\breve{\rho}_m(x, y) \geq 0$, where $\rho^*(x, y) = \sum_{i=1}^{p} u_i[\bar{\rho}_i(x, y) + \lambda\tilde{\rho}_i(x, y)]$;*

(b) (i) *for each $i \in \underline{p}$, f_i is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-sounivex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-sounivex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *the function $\xi \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi, t^m)$ is $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$-quasisounivex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;*

(iii) *the function $\xi \to v_m H_{k_m}(\xi, s^m)$ is $(\mathcal{F}, \beta, \breve{\phi}_m, \breve{\rho}_m, \theta)$-quasisounivex at y, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu}\backslash\underline{\nu_0}$;*

(iv) *$\rho^*(x, y) + \hat{\rho}(x, y) + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x, y) \geq 0$;*

(c) (i) *for each $i \in \underline{p}$, f_i is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-sounivex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-sounivex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *the function $\xi \to G_{j_m}(\xi, t^m)$ is $(\mathcal{F}, \beta, \hat{\phi}_m, \hat{\rho}_m, \theta)$-quasisounivex at y, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;*

(iii) *the function $\xi \to \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi, s^m)$ is $(\mathcal{F}, \beta, \breve{\phi}, \breve{\rho}, \theta)$-quasisounivex at y, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$;*

(iv) *$\rho^*(x, y) + \sum_{m=1}^{\nu_0} v_m\hat{\rho}_m(x, y) + \breve{\rho}(x, y) \geq 0$;*

(d) (i) *for each $i \in \underline{p}$, f_i is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-sounivex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-sounivex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;*

(ii) *the function $\xi \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi, t^m)$ is $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$-quasisounivex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;*

(iii) *the function* $\xi \to \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi, s^m)$ *is* $(\mathcal{F}, \beta, \breve{\phi}, \breve{\rho}, \theta)$*-quasisounivex at* y*,* $\breve{\phi}$ *is increasing, and* $\breve{\phi}(0) = 0$*;*

(iv) $\rho^*(x, y) + \hat{\rho}(x, y) + \breve{\rho}(x, y) \geq 0$*;*

(e) (i) *for each* $i \in \underline{p}$*,* f_i *is* $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-sounivex and* $-g_i$ *is* $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$*-sounivex at* y*,* $\bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) *the function* $\xi \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(\xi, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(\xi, s^m)$ *is* $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$*-quasisounivex at* y*,* $\hat{\phi}$ *is increasing, and* $\hat{\phi}(0) = 0$*;*

(iii) $\rho^*(x, y) + \hat{\rho}(x, y) \geq 0$.

Then $\varphi(x) \geq \lambda$.

Proof. The poof is similar to that of Theorem 3.2. □

5. Specializations II

In this section, we consider certain specializations of the $(\mathcal{G}, \beta, \phi, h(\cdot,\cdot), \rho, \theta)$-univexity to first order univexity under which first order duality (see (Zalmai & Zhang, 2007)) can be established. These duality models have the following forms:

$$(DIII) \quad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$$

subject to

$$\sum_{i=1}^{p} u_i[\nabla f_i(y) - \lambda \nabla g_i(y)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(y, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(y, s^m) = 0, \tag{5.1}$$

$$u_i[f_i(y) - \lambda g_i(y)] \geq 0, \quad i \in \underline{p}, \tag{5.2}$$

$$G_{j_m}(y, t^m) \geq 0, \quad m \in \underline{\nu_0}, \tag{5.3}$$

$$v_m H_{k_m}(y, s^m) \geq 0, \quad m \in \underline{\nu}\backslash\underline{\nu_0}; \tag{5.4}$$

$$(\tilde{D}III) \quad \sup_{(y,u,v,\lambda,\nu,\nu_0,J_{\nu_0},K_{\nu\backslash\nu_0},\bar{t},\bar{s})\in\mathbb{H}} \lambda$$

subject to (3.3) and (5.2) - (5.4).

Theorem 5.1. *(see (Zalmai & Zhang, 2007)) (Weak Duality) Let* x *and* $(y, u, v, \lambda, \nu, \nu_0, J_{\nu_0}, K_{\nu\backslash\nu_0}, \bar{t}, \bar{s})$ *be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following five sets of hypotheses is satisfied:*

(a) (i) *for each* $i \in \underline{p}$*,* f_i *is* $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$*-univex and* $-g_i$ *is* $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$*-univex at* y*,* $\bar{\phi}$ *is superlinear, and* $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$*;*

(ii) the function $z \to G_{j_m}(z, t^m)$ is $(\mathcal{F}, \beta, \hat{\phi}_m, \hat{\rho}_m, \theta)$-quasiunivex at y, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;

(iii) the function $z \to v_m H_{k_m}(z, s^m)$ is $(\mathcal{F}, \beta, \breve{\phi}_m, \breve{\rho}_m, \theta)$-quasiunivex at y, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \backslash \underline{\nu_0}$;

(iv) $\rho^* + \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m + \sum_{m=\nu_0+1}^{\nu} v_m \breve{\rho}_m \geq 0$, where $\rho^* = \sum_{i=1}^{p} u_i(\bar{\rho}_i + \lambda \tilde{\rho}_i)$;

(b) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-univex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) the function $z \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(z, t^m)$ is $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$-quasiunivex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iii) the function $z \to v_m H_{k_m}(z, s^m)$ is $(\mathcal{F}, \beta, \breve{\phi}_m, \breve{\rho}_m, \theta)$-quasiunivex at y, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \backslash \underline{\nu_0}$;

(iv) $\rho^* + \hat{\rho} + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m \geq 0$;

(c) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-univex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) the function $z \to G_{j_m}(z, t^m)$ is $(\mathcal{F}, \beta, \hat{\phi}_m, \hat{\rho}_m, \theta)$-quasiunivex at y, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;

(iii) the function $z \to \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(z, s^m)$ is $(\mathcal{F}, \beta, \breve{\phi}, \breve{\rho}, \theta)$-quasiunivex at y, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$;

(iv) $\rho^* + \sum_{m=1}^{\nu_0} v_m \hat{\rho}_m + \breve{\rho} \geq 0$;

(d) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-univex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) the function $z \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(z, t^m)$ is $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$-quasiunivex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iii) the function $z \to \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(z, s^m)$ is $(\mathcal{F}, \beta, \breve{\phi}, \breve{\rho}, \theta)$-quasiunivex at y, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$;

(iv) $\rho^* + \hat{\rho} + \breve{\rho} \geq 0$;

(e) (i) for each $i \in \underline{p}$, f_i is $(\mathcal{F}, \beta, \bar{\phi}, \bar{\rho}_i, \theta)$-univex and $-g_i$ is $(\mathcal{F}, \beta, \bar{\phi}, \tilde{\rho}_i, \theta)$-univex at y, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) the function $z \to \sum_{m=1}^{\nu_0} v_m G_{j_m}(z, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m H_{k_m}(z, s^m)$ is $(\mathcal{F}, \beta, \hat{\phi}, \hat{\rho}, \theta)$-quasiunivex at y, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iii) $\rho^* + \hat{\rho} \geq 0$.

Then $\varphi(x) \geq \lambda$.

6. Concluding Remarks

The duality results established in this communication encompass a fairly large number of second-order dual problems and duality theorems that were investigated previously for several classes of nonlinear programming problems. Furthermore, the methods utilized in this paper could lead to extend and generalize results to other classes of mathematical programming problems based on general univexity assumptions.

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