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ICONIC REPRESENTATION AS STUDENT'S SUCCESS FACTOR IN ALGEBRAIC GENERALISATIONS

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Abstract: *Problems of finding general rule in the case of linear correspondences are often encountered in the research papers, where they are supplied with iconic means which support induction. And as the researchers report it, students have a tendency to focus solely on the numeric data even when visual patterns are given. The objective of our research is the examination of the influence of various iconic representations designed to help students establish and express general relations between quantities. This research is of empirical nature. The research was based on the testing technique. The obtained results show that iconic representation of the structure of algebraic contents influences greatly students' ability of making generalisations and establishing algebraic relations and simple rules of correspondence. The very activity of drawing and iconic representing of the pattern members can also be seen as potentially significant for the discovering of the rule of correspondence and its generalisation.*

Keywords: *algebra; generalization; pattern; iconic representation.*

Introduction

Many researchers see the use of patterns to be a means of promoting and encouraging generalisations as a significant algebraic activity (Hargreaves, Shorrocks-Taylor, and Threlfal 1998; Lee 1996; Warren and Pierce 2004; Warren 2005; Warren and Cooper 2008; Specht 2005; Mason, Johnston-Wilder, and Graham 2005; Moss and Beatty 2006; etc). Some researchers particularly emphasize the fact that generalisation of numeric patterns and symbolic formulation of relations between variables cause specific problems for beginner-level students (MacGregor and Stacey 1993; Mason 1996; Lee 1996; etc). Houssart investigated nature of teachers' understanding of numerical patterns. He concluded that teachers do not express enough understanding of this contents (2000). However, as Waren states it (2005), few reference books focus on generalisation of patterns and younger students' expression and justification of such generalisations. On the other hand, certain studies show that children are capable of functional thinking even at an early age (Blanton and Kaput 2004).

The impact of the use of visual means on the development of students' mathematical abilities is an interesting field of research, although it seems that no consensus has been reached in this area. Many studies stress the importance of visualization in the problem-solving processes, while certain results suggest that visualization must also be accompanied by analytical reasoning. Numerous authors consider the use of different representations for illustrating problem situations as the important component of algebraic thinking (Kieran 1996; Duval 1999; Rivera 2010). In that manner, Kieran defines algebraic thinking as "an approach to quantitative situations that emphasizes the general relational aspects with tools that are not necessarily letter-symbolic, but which can ultimately be used as cognitive support for introducing and for sustaining the more traditional discourse of school algebra" (1996, 275). Patterson and al. (2004) point out that algebraic representation is not always optimal for learning, so alternative representations (graphical and tabular) that might be functional need to be investigated.

MacGregor and Stacey (1993) based their research on the assessment of generalisation of linear patterns made by students aged 9-13. They reported on the strategies which the students used in "close generalisation"(the activities pertaining to drawing closer pattern members) and "distant generalisation"(the one that includes discovering the rules) and concluded that drawing was a decisive factor in their strategies for exploring patterns. Garcia Cruz and Martinon (as quoted in Barbosa, Palhares, and Vale 2007) conducted research with students aged 15-16, with the aim of establishing and analysing the manner in which students discovered generalisations, i.e. whether they preferred numeric or geometric strategies. This research showed that drawing had a double role in the abstraction and generalisation process. Drawing represents a certain environment for students using visual strategies to make generalisations, and on the other hand, it represents a mode of assessing the correct reasoning of the students who prefer numeric strategies. Mason (1996) noticed that, when the patterns are introduced, although they are given "geometrically", the emphasis is still laid on constructing value tables from which the formula is derived, which is tested on one or two examples. Mason, Johnston-Wilder and Graham (2005) promoted the use of the "look what you are doing" strategy. Each drawing activity, when expressed as an instruction on how to "draw" a pattern, represents potential data generalisation.

Barbosa, Palhares, and Vale (2007) conducted a research with sixth-grade students (aged 11-12) with the aim of examining the strategies that the students used when they worked with patterns. The authors believe that the results of this research verified the claim that students preferred analytic approaches to mathematical activities and that they "turn into numbers" even the problems which are visual in nature.

Stalo and associates (2006) examined the role of verbal, visual and symbolic representations of patterns in students' success when working with patterns of different levels of complexity. The results of this research indicate that when working with complex patterns, visual representation enables students to predict expressions in the distance and form generalisation in relation to the verbal representation form. Visual representation enables these activities since it helps students recognise the relations that are not visible in verbal representation. It has been revealed that representation role had a smaller significance in simple patterns, probably because the students managed to recognise the same pattern which they have experienced before in the case different representations.

Warren and Cooper give three major reasons for exploring geometric growing patterns in the elementary school classroom: “they are visual representations of number patterns, they can be used as an informal introduction to the concept of a variable, and they can be used to generate equivalent expressions“ (2008, 113).

Concerning current techology, Pierce and al. (2007) cite research results according to which majority of teachers believe that the use of CAS as a tool to support learning algebra, is of the most benefit for high ability students, while presenting an obstacle for low ability students.

In the assessment of students’ abilities to work with patterns, Radford focused on semiotic and cultural perspective, which is based on the idea that learning is achieved with the use of different semiotic systems. Radford (2000) carried out an experiment in which he tested eighth-grade students, who were working on examples of patterns in small groups. He analysed students’ activities which lead them to express meanings of algebraic generalisations and he noticed that such activities develop in three steps:

(1) Arithmetic testing;
(2) The expression of generalisations in a natural language (in the message form);
(3) The use of standard algebraic symbols for expressing generalisations.

Radford concludes that “to learn to generalise geometric-numeric patterns amounts to learning to see and to interpret a finite number (usually very few) of concrete objects or signs in a different way“ (Radford, Bardini, and Sabena 2005, 685).

Specht (2005) conducted research with the aim of clarifying the reasoning process of younger students when working with patterns. In her research with fourth-grade students, she concluded that understanding the tasks which demanded generalisation of the change of quantities required four levels of understanding variables above all:

(1) Recognition and understanding that symbol x could be used to express general solutions and relations (which means that it is not solely reduced to the use and expression of an unknown number);
(2) Acceptance of a symbolic (algebraic) expression as the solution to the task;
(3) Understanding of the meaning “x could be any number“;
(4) The development of the equality sign concept is not limited to the understanding with the meaning “to find a solution”.

Having conducted research with fourth-grade students, this author states that students of that age are perhaps not capable of expressing generalisations using algebraic symbols.

This claim was refuted in the research conducted by Warren (2005). She carried out a teaching experiment in which 45 children (from two classes) took part all of whom were on average nine years and six months old. Based on this research, the author (Warren 2005) concluded that students had a tendency to seek an additional strategy when they were looking for a rule in value table. Generalisations expressed by students could be put in four categories:

- The use of big numbers to express generalisations;
- Simple comparison of numbers and counting (for instance, two n's put together give one m);
- Verbal expression of generalisation;
- Formal record (m = 2n).

The results showed that younger children were not only capable of considering the relation between two sets of data, but also of expressing such relation in a very abstract form. These results oppose the results of research conducted by Pytlak (2011). In the research carried out on a group of fourth-grade students (aged 9-10), the author examines whether the students of that age are capable of discovering and applying certain rules. The results of research reveal that students can notice rules and express them verbally, but that symbolic expression of rules is difficult for students at that age.

Gray and Tall defined proceptual thinking which include: "flexible facility to view symbolism either as a trigger for carrying out a procedure or as the representation of a mental object that may be decomposed, recomposed, and manipulated at a higher level" (1994, 125). They also emphasized the importance of understanding algebraic expressions not only as concept, but as process.

Pitta and Gray conducted a research with "low achiever" and "high achiever" students aged between eight and 12. They concluded that: "The mathematics of the low achievers remains abstract; its symbols need concretizing and its pictures focusing. By not understanding the nature of the abstract nouns or the symbolic nature of icons and numerical symbols we suggest they may not form the generalisations and relationships that are the hallmarks of proceptual thinking" (1996, 42).

The research of visualization process and the role of mental images in mathematical reasoning reveal the importance of selected representation in the development of concepts. Patterns can be given to students in different manners: symbolic, verbal, visual. Based on results, it cannot be claimed for sure that visual representation of data helped in discovering the rules of a given pattern and encouraged generalisation. However, we tend to believe that such results (often obtained with examinees in final high school grades) are mostly conditioned by students' previous experience, which is based on working with numbers. We believe that adequately selected and structured images, which represent numeric models, can assist in discovering rules and suggest meaning. Therefore, in the case of more complex patterns, we believe that it is useful to link numeric data to appropriate images.

Research Method

The objective of our research is the examination of the influence of various iconic representations designed to help students establish and express general relations between quantities. There is a traditional approach to patterns in Serbian mathematics coursebooks. This approach means stating several members of the patterns whereby students are expected to continue the sequence, notice and express the rule of correspondence. The aim of this research is to determine the following:

(1) How much the process of visualization and iconic representation of the structure of sequence members influence the success of students when revealing and generalising the rules of correspondence.

(2) Which strategies students use in the process of discovering rules and which means they prefer in expressing generalisations.

According to the treated topic, this research was conducted in the field of mathematics teaching methodology and, in terms of the applied research methodology, it is of empirical nature. The research was based on the testing technique. For the purpose of research, we constructed knowledge tests (which are standardized). We obtained an adequate sample: realization that requires cooperation with teachers and their commitment, and therefore the research was conducted in cooperation with teachers who showed willingness to participate. The sample comprises two classes (58 students) of fourth-graders (aged nine to ten) from a primary school in Belgrade.

Students did two tests in three days. The first test consists of three specially programmed tasks. Patterns were given numerically, i.e. by stating the first four pattern members of a sequence (3, 8, 15, 24…).

In the second test (parallel version) the task were modified in the way that the structure of the pattern members of a sequence is represented in the iconic form (example in Figure 1). Iconic symbols are the bearers of meaning of concepts. Our aim was to examine in what manner and to what extent they contribute to the understanding of wider content.

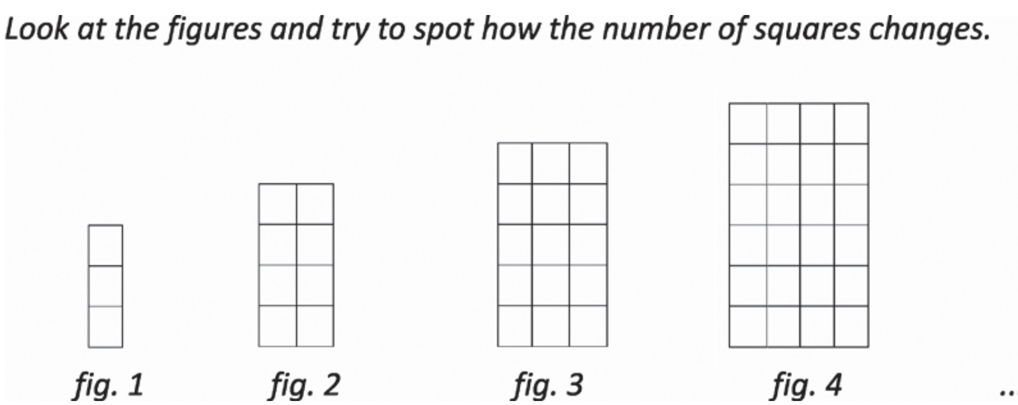

Figure 2 Iconic representations of patterns - example of a task

The tasks were created in such a way that the students had to go through the following phases in order to express generalisations:

- The expression of generalisation by revelation of the members' structure on "distant positions" using numerical expressions;

- The expression of generalisations in the terms of the natural language;
- The use of standard and algebraic symbols for expressing generalisations.

Research results

Comparing students' test results, we have concluded that there is statistically significant difference at the level of 0.01 (Table 1) in students' achievement in this study on generalisation by revealing members in distant positions using numerical expressions.

Table 1. Numerical and iconic representations of patterns – difference in achievement when revealing members of a sequence in distant position (McNemar test)

	I1a & II1a	I2a & II2a	I3a & II3a
N – sample volume	58	58	58
p - value	.000	.000	.000

Among the studied differences in students' achievement regarding pairs of task components, statistically significant differences were found in the following:

(1) I1a and II1a (p = 0,000<0,01): a significantly greater number of students (N = 34) solved the I1a, than II1a (N = 17);

(2) I2a and II2a (p = 0,000<0,01): a significantly greater number of students (N = 36) solved the I2a, than II2a (N = 15);

(3) I3a and II3a (p = 0,000<0,01): a significantly greater number of students (N = 36) solved the I3a, than II3a (N = 15).

Comparing students' test results, we have concluded that there is statistically significant difference at the level of 0.01 (Table 2) in students' achievement in this study on generalisation by natural language.

Table 2. Numerical and iconic representations of patterns – difference in achievement of expressing generalisations in the terms of the natural language (McNemar test)

	I1b & II1b	I2b & II2b	I3b & II3b
N – sample volume	58	58	58
p – value	.000	.000	.000

Among the studied differences in students' achievement regarding pairs of task components, statistically significant differences were found in the following:

(1) I1b and II1b (p = 0,000<0,01): a significantly greater number of students (N = 32) solved the I1b, than II1b (N = 8);

(2) I2b and II2b (p = 0,000<0,01): a significantly greater number of students (N = 33) solved the I2b, than II2b (N = 9);

(3) I3b and II3b (p = 0,000<0,01): a significantly greater number of students (N = 32) solved the I3b, than II3b (N = 7).

Comparing students' test results, we have concluded that there is statistically significant difference at the level of 0.01 (Table 3) in students' achievement in this study on generalisation by algebraic symbols.

Table 3. Numerical and iconic representations of patterns – difference in achievement of expressing generalisations of algebraic symbols (McNemar test)

	I1c & II1c	I2c & II2c	I3c & II3c
N – sample volume	58	58	58
p – value	.000	.000	.000

Among the studied differences in students' achievement regarding pairs of task components, statistically significant differences were found in the following:

(1) I1c and II1c (p = 0,000<0,01): a significantly greater number of students (N = 35) solved the I1c , than II1c (n = 0);

(2) I1c and II2c (p = 0,000<0,01): a significantly greater number of students (N = 33) solved the I1c, than II1c (N = 1);

(3) I3c and II3c (p = 0,000<0,01): a significantly greater number of students (N = 36) solved the I3c, than II3c (N = 3).

The obtained results show that iconic representation of the structure of algebraic contents influences greatly students' ability of making generalisations and establishing algebraic relations and simple rules of correspondence.

The discussion of the results

Let's analyse the first task which pertains to the expression of generalisations in patterns if pattern members of a sequence are represented in the iconic form (example in Figure 2).

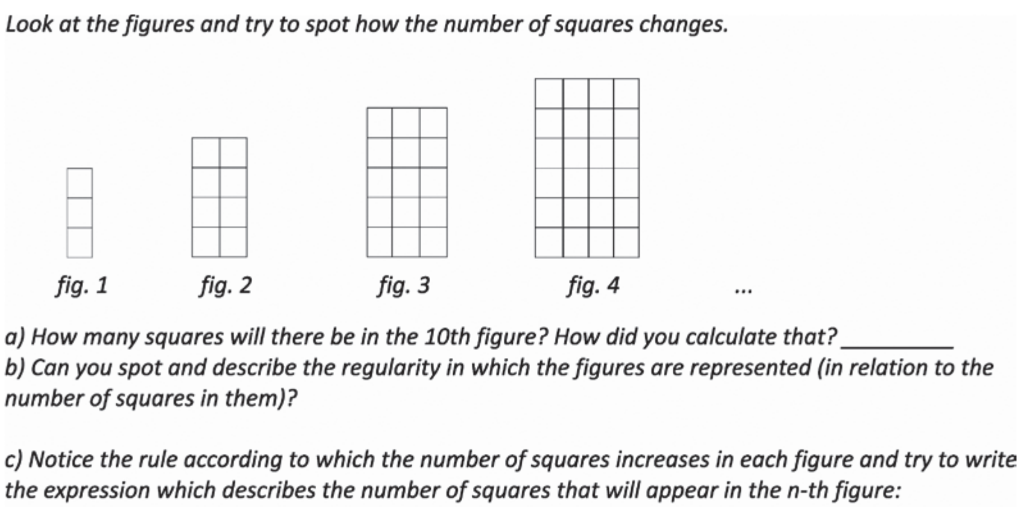

Figure 3 Expression of generalisations in patterns if pattern members of a sequence are represented in the iconic form

High percentage of students (83.6%) successfully expressed the structure of members shown in the picture with the use of numeric expressions. The best way to encourage the generalisation is the writing of the first numbers of squares in the form: 3, 2·4, 3·5... The students also successfully (81,8%) expressed the generalisation by revealing the structure of members at "distant positions" with the use of numeric expressions, i.e. they correctly concluded that at the 10th place there will be 10 · 12 squares. The fact that iconic representation of structure of pattern ·members is highly important for the understanding of structure and the formation of necessary mental images and representations is enhanced by the fact that the students, although they were not required to do so in the task, they independently draw the next pattern (Example - Figure 3). The activity of drawing and iconic representation of the structure of pattern members is a potential generalisation.

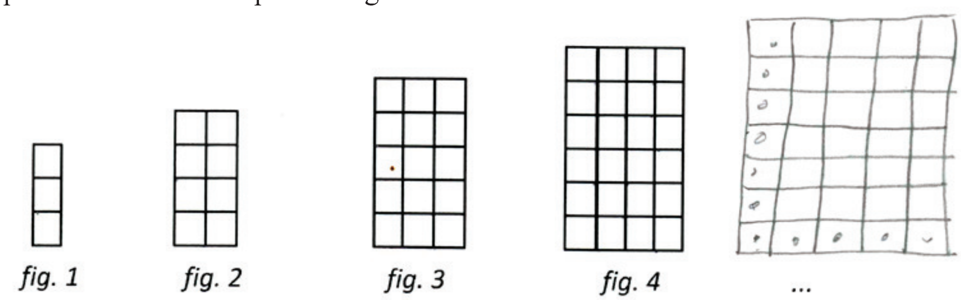

Figure 4 Expression of generalization by drawing

In the next step, the students (58.2%) generalised the rule of sequencing pattern members (expressing generalisation in a natural language), i.e. they verbalized their observations as a result of intuitive understanding in the field of images and arithmetic research. Based on this, the students used standard algebraic symbols to successfully (60%) express the rule of sequencing pattern members. The results showed that students can express their generalisations in various manners: expression of rules by drawing members at "closer positions", discovering members at "distant positions", verbal description of the rules, the expression of rules in abstract symbolic language ($n\cdot(n+2)$). The recognition and formulation of stated generalisations provide an opportunity for genuine and significant mathematical activity.

When pattern is represented in the numerical form (3, 8, 15, 24…), students (29.3%) made generalisation revealing members of a sequence in distant position. But, in this case the students have written number (120), and not an expression that suggests a generalisation. Based on this, we believe that link between structured images, numbers and symbolic expressions may help developing proceptual thinking. Images refer, that correspondence is not only a concept, but a process.

Students described that generalisation in the terms of the natural language (13.8%), as adding of successive odd numbers, beginning with five. None of the students could generalise the rule and to expression of generalisations of algebraic symbols.

The obtained results show that iconic representation of the structure of algebraic contents influences greatly students' ability of making generalisations and establishing algebraic relations and simple rules of correspondence.

Conclusions

Working with patterns is regarded as a significant algebraic activity. The discovery of an unknown rule and its generalisation gives an opportunity for students to develop abstraction and generalisation abilities and it is also a manner of active work with a variable. Students' success in this research indicates that they can deal with certain algebraic generalisations which significantly surpass the content envisaged by the syllabus. Our research supports the results which show that younger children are capable of learning this content in a certain framework and in certain ways. Students can express their generalisations in various manners: by drawing, arithmetically, verbally and symbolically. In the process of developing the meaning of symbolically expressed generalisations, iconic representation has a significant role. The activity of visual representation of quantitative relations is of great importance for development of meaning of algebraic symbols and procedures. So, when introducing new mathematical terms, teachers have to turn attention to the creation of meaning, and by carriers of meaning we consider iconic symbols. Therefore, operating with symbols has meaning when it is accompanied by evocation of mental images or drawing iconic symbols which represents full meaning. Conventional notation helps abstraction and generalisation. Symbolic mathematical language is precise and concise. However, if symbols are introduced without adequate basis which give meaning for symbol manipulations, students can develop early formalization and for them symbolic language can become semantically empty. A premature use of symbols independently of their meaning always leads to formalism. Results of our research showed that students are mostly successful in working with tasks where mathematical structure has schematic representation. Question is, whether the students would use visual models for representing relations in process of solving other algebraic problems. Also it is significant to answer the question what is the character of students spontaneously incurred visual models.

Patterns are such contents which offer the possibility of developing abstraction and generalisation skills, as the development of efficient schemes for problem-solving, which students can apply on generalised relations between quantities in algebraic problems. Working with numeric patterns which have iconic representation students are able to develop better understanding of number expressions, expressions with variable and functions. The concept of numeric patterns (we refer to the patterns in which the rule is expressed by a linear expression or a very simple quadratic one) can have its place in the first grades of elementary school with the expectation that the real effects of such lessons will be visible in higher grades. Elaborating this type of problems, we hope to develop students' abilities preparing them for the following stages of learning mathematics. Therefore, this type of exercises should not be considered to be either formal or insignificant.

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