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An Implementation of the ELECTRE II Method Using Fuzzy Numbers

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Abstract

In this paper there are presented fuzzy numbers, ELECTRE II method, and graphs theory notions which are used to develop an F# application to rank alternatives in multi-criteria decision-making problems. A usage example is shown in Section 8. The source code of the application can be found here.

Keywords: ELECTRE II, multi-criteria decision-making, fuzzy sets, membership functions, linguistic variables, transitive closure matrix, Hamiltonian path, Yu Chen’s theorem.
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1. Introduction

This application is an implementation of a method which derives the collective opinion of a group of members as expressed in a grading process in which individual group members evaluate alternatives by assigning linguistic variables. A linguistic variable is a variable whose values are words or sentences in artificial or natural language. Linguistic terms are converted into fuzzy numbers (Babak & Erol, 2012).

The multi-criteria decision-making (MCDM) problem is concerned with the task of ranking a finite number of decision alternatives, each of which is explicitly described in terms of different characteristics called attributes, decision criteria, or objectives, which have to be taken into account simultaneously. ELECTRE method (Elimination and Decision Translating the Truth) was presented by B. Roy in 1960. It is a comprehensive evaluation approach that tries to rank a number of alternatives each one of which is described in terms of a number of criteria. The main idea is the proper utilization of what are called ”outranking relations” (Tihomir, 1997). The result of ELECTRE method is the comprehensive dominance matrix which can be considered the adjacency matrix of a directed graph. To be able to rank the alternatives, the graph must contain a Hamiltonian path. This application use the Yu Chen’s theorem to rank the alternatives.

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2. Definition of Fuzzy Sets.

Fuzzy sets (Mitchels *et al.*, 2006) are generalizations of conventional set theory introduced by Zadeh in 1965 as a mathematical way to represent vagueness in everyday life. A fuzzy set assigns to each possible individual in the universe of discourse a value representing its grade of membership in the set. Fuzzy sets play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, decision making and abstraction.

Definition 2.1. Let X be a conventional set called the universe of discourse. A *fuzzy set* $\tilde{X}$ associated to X is the set of ordered pairs: $\tilde{X} = \{(x, \mu_{\tilde{X}(x)}) | x \in X\}$ where $\mu : X \to M$, is a mapping which assigns to each element $x \in X$ a degree of membership $\mu(x)$ to the fuzzy set $\tilde{X}$. The set of all fuzzy sets of X is denoted by $\mathcal{F}(X)$

3. Types of Membership Functions

For applications it is sufficient to consider only a few basic forms of convex fuzzy sets, so that a fuzzy set can be specified uniquely by a few parameters. Typical example of such a fuzzy set is:

3.1. Triangular membership function :

$\Lambda : X \to [0, 1]$

$$\Lambda(x; l, m, u) := \begin{cases} 0, & if \quad x \le l \\ \dfrac{x-l}{m-l}, & if \quad l \le x \le m \\ \dfrac{m-x}{-m+u} + 1, & if \quad m \le x \le u \\ 0, & if \quad x > u \end{cases} \tag{3.1}$$

where $l < m < u$.
If $l = 2, m = 12.5$ and $u = 20$, we obtain:

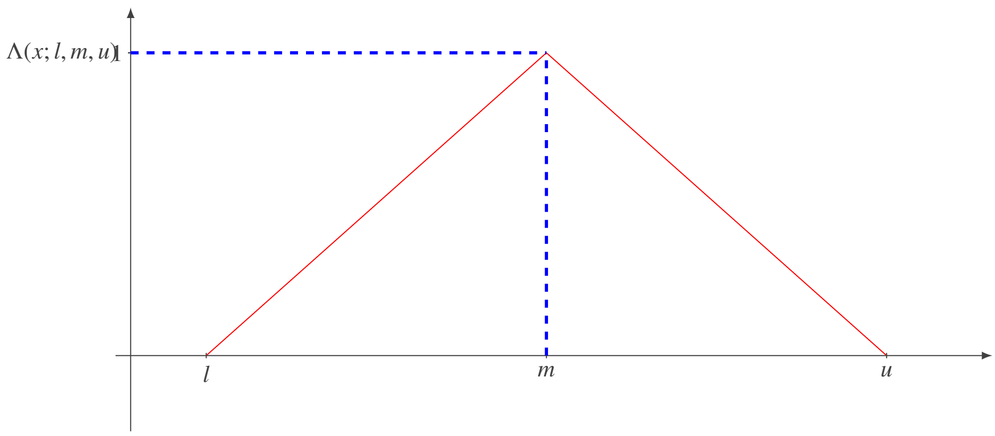

Figure 3.1. Triangular membership function.

4. Fuzzy Numbers

Fuzzy numbers are fuzzy sets defined on the real line $\mathbb{R}$ (Cheng, 2004).

Definition 4.1. A *fuzzy number* F is a fuzzy set constrained by a membership function :

$$\mu_F : \mathbb{R} \rightarrow [0, 1] \tag{4.1}$$

that satisfies:
(1) F is normal, if there exists a real number m, such that $\mu_F(m) = 1$.
(2) F is fuzzy convex, if for any pair x, y belonging to support of F,

$$\mu_F(\lambda x + (1 - \lambda)y) \geq min\{\mu_F(X), \mu_F(Y)\}, \quad \text{for} \quad \text{all} \quad \lambda \in [0, 1]. \tag{4.2}$$

(3) F is upper semicontinuous, if for each $\alpha \in (0, 1)$, the α-level set $[F]_\alpha = \{x \in \mathbb{R} | \mu(x) \geq \alpha\}$ is closed.

For example if the mode, left endpoint, and right endpoint are denoted by m, l, and u, respectively, the triangular fuzzy number can be defined as :
$f(x, l, m, u) = 5x - 4$, if $l \leq x \leq m$, left membership function
$f(x, l, m, u) = -5x + 6$, if $m < x \leq u$, right membership function
$f(x, l, m, u) = 0$, if $x < l$ and $x > u$,
This triangular fuzzy number can also be denoted as (l, m, u).

5. Linguistic Variables

Definition 5.1. (Adeel *et al.*, 2019). *Linguistic variables* are variables whose values are words or sentences in a natural or artificial language. If these words are described by fuzzy sets that are defined over a universal set, then the variables are called *fuzzy linguistic variables*.

Definition 5.2. (Adeel *et al.*, 2019). A *linguistic term* set is defined by means of an ordered structure providing the term set that is distributed on a scale at which a total order has been defined.

For example, a set S of six terms could be written as follows:
S={Extremly good=9; Very good=7; Good=5; Medium bad=3; Bad=2; Very bad=1}.

6. Electre Method

Definition 6.1. (Fei *et al.*, 2019) *Preference* in ELECTRE method is modeled based on binary outranking relations, S, whose meaning is "at least as good as". Considering two alternatives x and y, four cases could arise:
(i) xSy and not ySx, i.e., xPy (x is strictly preferred to y),
(ii) ySx and not xSy, i.e., yPx (y is strictly preferred to x),
(iii) xSy and ySx, i.e., xIy (x is indifferent to y),
(iv) not xSy and not ySx (x is incomparable to y).

Incomparability preference is a significant relation to account for cases in which two alternatives cannot be compared.

Definition 6.2. (Fei *et al.*, 2019) According to ELECTRE method, for given two alternatives x and y, their outranking relation depends on two major aspects, namely, the *concordance* and *discordance*. The following statements provide insights into these concepts.
1) In the concept of concordance, a sufficient majority of the criteria should be in favor of the assertion for an outranking xSy to be validated.
2) In the concept of discordance, when the concordance condition holds, none of the criteria in the minority should oppose too strongly to the assertion xSy.

These two situations must be implemented for validating the assertion xSy.

7. Graph Theory

The result of ELECTRE method is the comprehensive dominance matrix that can be used to select the best compromise alternative. We can construct a directed graph $G = (V, U)$, where V is the set of vertices and U is the set of arcs. Each alternative is treated as a vertex, and an arc exists between alternatives A_i and A_j if either A_i is preferred to A_j or A_i is indifferent to A_j. Alternative A_i outranks A_j if an arc exists between A_i and A_j and the arrow goes from A_i to A_j. The relationship is incomparable if no arc exists between alternative A_i and A_j. They are indifferent if an arc exists between A_i and A_j and an arrow exists in both directions.

To be able to rank all the alternatives, the graph G must contain a Hamiltonian path. The problem of finding the Hamiltonian paths in a directed graph and without cycles has been solved by Yu Chen (Vasiliu, 2017).

Theorem 7.1 (Chen)**.** *A graph $G = (X, U)$ with n vertices, without cycles, contains a Hamiltonian path if and only if:*

$$\sum_{i=1}^{n} p_i = \frac{n(n-1)}{2} \tag{7.1}$$

where p_i is the power of reaching for vertex i, and n is the number of vertices.

Theorem 7.2. *If a directed graph $G = (X, U)$, without cycles, contains a Hamiltonian path then this path is unique.*

The Chen's algorithm for finding the Hamiltonian paths in a directed graph, without cycles, consist in following steps:
Step 1. Generate the adjacency matrix (comprehensive dominance matrix),
Step 2. Build the transitive closure matrix D using Roy-Warshall algorithm,
Step 3. If there exist i such that $d_{ii} = 1$ then the graph has cycles. Chen's algorithm can't be applied. Stop program.
else go to step 4.
Step 4. Find the powers of reaching for each vertex p_i.
Step 5. If

$$\sum_{i=1}^{n} p_i \neq \frac{n(n-1)}{2} \tag{7.2}$$

then the graph does not contain a Hamiltonian path. Stop program.
else go to step 6.
Step 6. Sort in descending order the powers of reaching the vertices and obtain the Hamiltonian path.
In steps 5 and 6 the best alternative is the alternative with the highest power of reaching.

8. Example

8.1. The rating of the projects

Suppose we have to rank three alternatives for a project. There are four experts that evaluates the alternatives using a set of four criteria. The results are in Table 1 where $\{A1, A2, A3\}$ is the set of alternatives, and $\{C1, C2, C3, C4\}$ is the set of criteria.

Alternative	Criteria	Decision maker 1	Decision maker 2	Decision maker 3	Decision maker 4
A1	C1	VG	VG	G	G
	C2	G	VG	MB	MB
	C3	VG	G	B	VG
	C4	VG	VG	G	G
A2	C1	G	VG	MB	B
	C2	VG	VG	G	MB
	C3	VG	VG	B	B
	C4	VG	VG	MB	G
A3	C1	VG	VG	G	VG
	C2	VG	G	G	VG
	C3	VG	G	VG	G
	C4	VG	VG	VG	VG

Table 8.1. The rating of the projects

8.2. Converting linguistic terms

We can convert linguistic terms into numbers using Table 2:

Linguistic term	Scale
Extremely good -EG	9
Very good -VG	7
Good -G	5
Medium bad -MB	3
Bad -B	2
Very bad -VB	1

Table 8.2. The decision makers opinion and the linguistic terms.

8.3. Constructing the decision matrix

We can construct the decision matrix X using Table 1 and Table 2 :
where $x_{1,1} = \dfrac{(VG)7 + (VG)7 + (G)5 + (G)5]}{4} = 6.00$ is for alternative $A1$ and criteria $C1$, etc.

8.4. Constructing fuzzy decision matrix

We can construct fuzzy decision matrix FX using decision matrix X (Table 1) and left (22) and right (23) membership functions:

	C1	C2	C3	C4
A1	6.00	4.50	5.25	6.00
A2	4.25	5.50	4.50	5.50
A3	6.50	6.00	6.00	7.00

Table 8.3. Decision matrix.

	C1	C2	C3	C4
A1	(4.80 ; 6.00 ; 7.20)	(3.60 ; 4.50 ; 5.40)	(4.25 ; 5.25 ; 6.30)	(4.80 ; 6.00 ; 7.20)
A2	(3.40 ; 4.25 ; 5.40)	(4.40 ; 5.50 ; 6.60)	(3.60 ; 4.50 ; 5.40)	(4.40 ; 5.50 ; 6.60)
A3	(5.20 ; 6.50 ; 7.80)	(4.80 ; 6.00 ; 7.20)	(4.80 ; 6.00 ; 7.20)	(5.60 ; 7.00 ; 8.40)

Table 8.4. Fuzzy decision matrix.

8.5. Aggregated fuzzy importance weight

Calculation of the aggregated fuzzy importance weight for each criterion : $w_j = (l_j, m_j, u_j)$. l_j , m_j and u_j are geometric means. For example:
$l_1 = (4.80 * 3.40 * 5.20)^{1/3}$; $m_1 = (6.00 * 4.25 * 6.50)^{1/3}$; $u_1 = (7.20 * 5.40 * 7.80)^{1/3}$ so $w_1 = (4.394483445; 5.493104306; 6.591725167)$.
The aggregated fuzzy importance weight array is :

	C1	C2	C3	C4
w	(4.394 ; 5.493 ; 6.5912)	(4.236 ; 5.296 ; 6.355)	(4.171 ; 5.214 ; 6.257)	(4.909 ; 6.136 ; 7.363)

Table 8.5. Aggregated fuzzy importance weight array.

8.6. The normalized decision matrix

The normalized decision matrix is calculated in the following way:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, \quad i = 1, 2, ..., m \quad j = 1, 2, ..., n \tag{8.1}$$

$$r_{ij} = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \dots & \dots & \dots & \dots \\ r_{m1} & r_{m2} & \dots & r_{mn} \end{pmatrix}$$

The normalized decision matrix R is :

	C1	C2	C3	C4
A1	0.6113781625	0.4838429975	0.5734623444	0.5588957194
A2	0.4330595318	0.5913636636	0.4915391523	0.5123210761
A3	0.6623263427	0.6451239967	0.6553855364	0.652045006

Table 8.6. The normalized decision matrix.

8.7. The normalized aggregated fuzzy importance weight

The fuzzy geometric mean of the fuzzy priority value is calculated with normalization priorities for factors using the following:

$$\hat{w}_i = \frac{(l_j, m_j, u_j)}{\left(\sum_{j=1}^{k} l_j, \sum_{j=1}^{k} m_j, \sum_{j=1}^{k} u_j\right)} = \left(\frac{l_j}{\sum_{j=1}^{k} u_j}, \frac{m_j}{\sum_{j=1}^{k} m_j}, \frac{u_j}{\sum_{j=1}^{k} l_j}\right) \tag{8.2}$$

The normalized aggregated fuzzy importance weight array is :

	C1	C2	C3	C4
$\hat{w}$	(0.165 ; 0.248 ; 0.372)	(0.159 ; 0.239 ; 0.359)	(0.157 ; 0.236 ; 0.353)	(0.185 ; 0.277 ; 0.416)

Table 8.7. The normalized aggregated fuzzy importance weight array .

8.8. The weighted normalized decision matrix

The weighted normalized decision matrix based on the normalized matrix is constructed as follows:

$$\tilde{V} = \left(\tilde{v}_{ij}\right)_{m\times n}, \quad i = 1, 2, ..., m, \quad j = 1, 2, ..., n \tag{8.3}$$

where $\tilde{v}_{ij}$: normalized positive triangular FN's.
$\tilde{v}_{ij} = r_{ij} \times \hat{w}_j$

The weighted normalized decision matrix $V1$ is :

	C1	C2	C3	C4
A1	0.1011321214	0.07715687457	0.09004106245	0.1032672929
A2	0.07163525265	0.09430284669	0.07717805353	0.0946616852
A3	0.1095597982	0.1028758328	0.1029040714	0.1204785084

Table 8.8. The weighted normalized decision matrix $V1$.

The weighted normalized decision matrix $V2$ is :

	C1	C2	C3	C4
A1	0.1516981821	0.1157353119	0.1350615937	0.1549009394
A2	0.107452879	0.14145427	0.1157670803	0.1419925278
A3	0.1643396973	0.1543137491	0.1543561071	0.1807177626

Table 8.9. The weighted normalized decision matrix $V2$.

The weighted normalized decision matrix $V3$ is :

	C1	C2	C3	C4
A1	0.2275472731	0.1736029678	0.2025923905	0.2323514091
A2	0.1611793185	0.2121814051	0.1736506204	0.2129887917
A3	0.2465095459	0.2314706237	0.2315341606	0.271076644

Table 8.10. The weighted normalized decision matrix $V3$.

8.9. Calculation of concordance indexes

The concordance index $C_{a_1a_2}$ represents the degree of confidence in pairwise judgement $(A_{a_1} \rightarrow A_{a_2})$ accordingly. The concordance index to satisfy the measured problem can be written with the following formula:

$$C^1_{a_1a_2} = \sum_{J^*} w_{j1}, \quad C^2_{a_1a_2} = \sum_{J^*} w_{j2}, \quad C^3_{a_1a_2} = \sum_{J^*} w_{j3}, \tag{8.4}$$

where J^* are the attributes contained in the concordance set $C_{a_1a_2}$.
The concordance set is composed as follows:

$$C(a_1, a_2) = \{j | v_{a_{1j}} \geq v_{a_{2j}}\}, \ (a_1, a_2 = 1, 2, ..., m \text{ and } a_1 \neq a_2) \tag{8.5}$$

$C(a_1, a_2)$ is the collection of attributes where A_{a_1} is better than, or equal, to A_{a_2}.
The defuzzified concordance index is calculated using the following formula :

$$C_{a_1a_2} = (C^1_{a_1a_2} * C^2_{a_1a_2} * C^3_{a_1a_2})^{1/3} \tag{8.6}$$

The defuzzified concordance matrix C is:

	A1	A2	A3
A1	None	Some 16.61530156	Some 0.0
A2	Some 5.223952615	None	Some 0.0
A3	Some 21.83925417	Some 21.83925417	None

Table 8.11. The defuzzified concordance matrix.

where "None" means there isn't a value.

8.10. Calculation of discordance indexes

The preference of the dissatisfaction can be measured by discordance indexes. The discordance index $D_{a_1a_2}$ represents the degree of disagreement in $(A_{a_1} \rightarrow A_{a_2})$, as follows:

$$D^1_{a_1a_2} = \frac{\sum_{j^+} |v^1_{a_{1J^+}} - v^1_{a_{2J^+}}|}{\sum_{j} |v^1_{a_{1J}} - v^1_{a_{2J}}|}, \quad D^2_{a_1a_2} = \frac{\sum_{j^+} |v^2_{a_{1J^+}} - v^2_{a_{2J^+}}|}{\sum_{j} |v^2_{a_{1J}} - v^2_{a_{2J}}|}, \quad D^3_{a_1a_2} = \frac{\sum_{j^+} |v^3_{a_{1J^+}} - v^3_{a_{2J^+}}|}{\sum_{j} |v^3_{a_{1J}} - v^3_{a_{2J}}|} \tag{8.7}$$

J^+ are the attributes contained in the discordance set $D(a_1, a_2)$ and v_{ij} is the weighted normalized evaluation of the alternative i on the criterion j.
The discordance set is composed as follows:

$$D(a_1, a_2) = \{j | v_{a_{1j}} < v_{a_{2j}}\}, \ (a_1, a_2 = 1, 2, ..., m, \text{ and } a_1 \neq a_2) \tag{8.8}$$

The defuzzified discordance index is calculated using the following formula :

$$D_{a_1a_2} = (D^1_{a_1a_2} * D^2_{a_1a_2} * D^3_{a_1a_2})^{1/3} \tag{8.9}$$

The defuzzified discordance matrix D is:

	A1	A2	A3
A1	None	Some 0.5812810939	Some 1.0
A2	Some 1.0	None	Some 1.0
A3	Some 0.0	Some 0.0	None

Table 8.12. The defuzzified discordance matrix.

8.11. Concordance dominance matrix.

In order to develop the concordance dominance matrix $F = [f_{kl}]$ (Sabzi & King, 2015), a threshold value is calculated based on the concordance index values of concordance matrix which is as follows:

$$\bar{c} = \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{C_{kl}}{m(m-1)} \tag{8.10}$$

where $k \neq l$; $f_{kl} = 1$ if $C_{kl} \geq \bar{c}$ and $f_{kl} = 0$ if $C_{kl} < \bar{c}$.
The concordance dominance matrix F is:

	A1	A2	A3
A1	None	Some 1	Some 0
A2	Some 0	None	Some 0
A3	Some 1	Some 1	None

Table 8.13. The concordance dominance matrix.

8.12. Discordance dominance matrix.

In order to develop the discordance dominance matrix $G = [g_{kl}]$ (Sabzi & King, 2015), a threshold value is calculated based on the discordance index values of discordance matrix which is as follows:

$$\bar{d} = \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{D_{kl}}{m(m-1)} \tag{8.11}$$

where $k \neq l$; $g_{kl} = 1$ if $D_{kl} \geq \bar{d}$ and $g_{kl} = 0$ if $D_{kl} < \bar{d}$.
The discordance dominance matrix G is:

8.13. The comprehensive dominance matrix.

The elements of the comprehensive dominance matrix $E = [e_{kl}]$ (Sabzi & King, 2015) are equal to mutual multiplying of F and G elements. Here, the E matrix is $m \times m$ dimensional depending upon the F and G matrices and again comprise 1 or 0 values.

$$e_{ij} = f_{ij} * g_{ij} \tag{8.12}$$

The comprehensive dominance matrix E is:

	A1	A2	A3
A1	None	Some 1	Some 0
A2	Some 0	None	Some 0
A3	Some 1	Some 1	None

Table 8.14. The discordance dominance matrix.

	A1	A2	A3
A1	None	Some 1	Some 0
A2	Some 0	None	Some 0
A3	Some 1	Some 1	None

Table 8.15. The comprehensive dominance matrix.

8.14. The transitive closure matrix.

The comprehensive dominance matrix E can be considered an adjacency matrix of a directed graph, so the transitive closure matrix can be calculated with Roy-Warshall algorithm. The transitive closure matrix is a boolean matrix :

	A1	A2	A3
A1	false	true	false
A2	false	false	false
A3	true	true	false

Table 8.16. The transitive closure matrix.

The power of reaching for vertex 1 is $p_1 = 1$.
The power of reaching for vertex 2 is $p_2 = 0$.
The power of reaching for vertex 3 is $p_3 = 2$.
Because the sum of the powers of reaching $p_1 + p_2 + p_3 = \frac{n(n-1)}{2}$ the graph contains a Hamiltonian path.
The Hamiltonian path is: $A3 \to A1 \to A2$. It follows that the best alternative is $A3$ and the worst is $A2$.

9. Conclusion.

Analysts and engineers can use this application to understand and evaluate 'what if' case scenarios. Input data can be read from a text file with the following structure:
Number of projects : 3
Number Of Criteria : 4
Number of Decision Makers : 4
Coeficients of left function :5.0 -4.0
Coeficients of right function :-5.0 6.0
VG,VG,G,G
G,VG,MB,MB
VG,G,B,VG
VG,VG,G,G
G,VG,MB,B

VG,VG,G,MB
VG,VG,B,B
VG,VG,MB,G
VG,VG,G,VG
VG,G,G,VG
VG,G,VG,G
VG,VG,VG,VG

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