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Lacunary Statistical Convergence for Sets of Triple Sequences via Orlicz Function

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Abstract

The concept of statistical convergence was presented by Fast in 1951. This concept was extended to the sequences of sets by Nuray and Rhoades in 2012. In this manuscript, we study the concepts of Wijsman statistical $\widetilde{\phi}$-convergence, Wijsman lacunary statistical $\widetilde{\phi}$-convergence and Wijsman strongly lacunary statistical $\widetilde{\phi}$-convergence for the sets of triple sequences. We investigate some of its basic properties. Also, we give some the relations between these new concepts.

Keywords: Statistical convergence, lacunary sequence, sets of triple sequence, Wijsman convergence, Orlicz function.
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1. Introduction

The notion of a statistical convergence was presented in (Fast, 1951). Later on it was further investigated from a sequence space point of view and linked with the summability theory in (Fridy, 1985), (Tripathy, 2003), (Gürdal, 2004) and many others (see (Gürdal & Huban, 2012, 2017; Nabiev *et al.*, 2019; Savaş & Gürdal, 2014)). This concept was extended to the double sequences in (Mursaleen & Edely, 2003). A lot of useful developments of double sequences in summability methods can be found in (Altay & Başar, 2005; Dündar & Akin, 2020; Gürdal & Şahiner, 2008; Nuray *et al.*, 2016*a*; Patterson & Savaş, 2005; Tortop & Dündar, 2008; Ulusu & Dündar, 2016; Yegül & Dündar, 2018). The study on the statistical convergence of triple sequences due to (Sahiner *et al.*, 2007) and the other researches continued in (Dutta *et al.*, 2013), (Esi & Savaş, 2015), (Esi & Subramanian, 2017), (Esi *et al.*, 2018), (Huban & Gürdal, 2021), (Huban *et al.*,

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2020), (Subramanian & Esi, 2017*a*), (Subramanian & Esi, 2017*b*) and (Subramanian & Esi, 2018). The idea of convergence of sequences of points has been extended to convergence of sequences of sets by several researchers (see (Nuray & Rhoades, 2012; Ulusu & Dündar, 2014; Ulusu & Nuray, 2012; Wijsman, 1966)). Nuray and Rhoades extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems (Nuray & Rhoades, 2012). Studies on this subject continue to be actively studied recently.

Since sequence convergence plays a very important role in the fundamental theory of mathematics, there are many convergence concepts in summability theory, in classical measure theory, in approximation theory and in probability theory, and the relationships between them are discussed. The interested reader may consult (Gürdal & Huban, 2014), (Hazarika *et al.*, 2020) and (Mohiuddine & Alamri, 2019), the monographs (Başar, 2012) and (Mursaleen & Başar, 2020) for the background on the sequence spaces and related topics. Inspired by this, in this paper, a further investigation into the mathematical properties of triple sequences will be made. Section 2 recalls some definitions and theorems in summability theory. In Section 3, we study the concepts of Wijsman statistical convergent, Wijsman lacunary statistical convergent, Wijsman lacunary convergent and Wijsman strongly lacunary convergent triple sequences of sets using the Orlicz function $\widetilde{\phi}$, and investigate the relationship among them.

2. Definitions and notations

The notion of statistical convergence depends on the density of the subsets of the set $\mathbb{N}$ of positive integers.

If K is a subset of $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ also denotes the cardinality of the set K_n. The natural density of K given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |K_n| .$$

It is said that a sequence $x = (x_k)_{k\in\mathbb{N}}$ is statistically convergent to a point L, which provided that

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. If $(x_k)_{k\in\mathbb{N}}$ is statistically convergent to L and is written as $st\text{-}\lim x_k = L$.

The statistical convergence of any real sequences is defined in some articles relative to absolute value. The absolute value of real numbers is special of an Orlicz function (Rao & Ren, 2002). That is, the function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ which is a continuous on $\mathbb{R}$ and is even, non-decreasing on $\mathbb{R}^+$, and satisfying

$$\widetilde{\phi}(x) = 0 \text{ if and only if } x = 0 \text{ and } \widetilde{\phi}(x) \to \infty, \text{ as } x \to \infty.$$

An Orlicz function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$ condition if there is a positive real number M while $\widetilde{\phi}(2x) \leq M.\widetilde{\phi}(x)$ for each $x \in \mathbb{R}^+$. Few examples of Orlicz functions are given below:

Example 1. (i) For a fixed $r \in \mathbb{N}$, the function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = |x|^r$ is an Orlicz function.

(ii) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = x^2$ is an Orlicz function satisfying the $\triangle_2$ condition.

(iii) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = e^{|x|} - |x| - 1$ is an Orlicz function not satisfying the $\triangle_2$ condition.

(iv) The function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ defined as $\widetilde{\phi}(x) = x^3$ is not an Orlicz function.

Definition 2.1. (Savaş & Debnath, 2019) Let be the Orlicz function $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$. The sequence $x = (x_n)$ is said to be statistically $\widetilde{\phi}$-convergent to L if for all $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \widetilde{\phi}\left(x_k - L\right) \geq \varepsilon \right\} \right| = 0.$$

Now, we recall the basic definitions and concepts (see (Esi & Savaş, 2015), (Nuray *et al.*, 2016*b*), (Sahiner *et al.*, 2007), (Wijsman, 1966)).

For non-empty subset A of X, and for any point $x \in X$, the distance $d(x, A)$ from x to A is defined by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Let (X, ρ) be a metric space and A, A_k as closed non-empty subsets of X. If the sequence $\{A_k\}$ is Wijsman convergent to A then

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for all $x \in X$ and is written $W - \lim A_k = A$.

The sequence $\{A_k\}$ is Wijsman statistical convergent to A, if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| = 0$$

and is written $st - \lim_W A_k = A$.

We now recall that the concept of statistical convergence for triple sequences was as follows:

The function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$) is called a real (complex) triple sequence. For every $\varepsilon > 0$, $n_0(\varepsilon) \in \mathbb{N}$ such that $\left| x_{jkl} - L \right| < \varepsilon$ whenever $j, k, l \geq n_0$. Thus a triple sequence $\left(x_{jkl} \right)$ converges to L in Pringsheim's sense.

A subset K of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have natural density $\delta_3(K)$ if

$$\delta_3(K) = P - \lim_{n,k,l \to \infty} \frac{|K_{nkl}|}{nkl}$$

exists, where the vertical bars denote the number of (n, k, l) in K such that $p \leq n$, $q \leq k$, $r \leq l$. Then, a real triple sequence $x = (x_{nkl})$ is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\delta_3 \left(\{ (n, k, l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |x_{nkl} - L| \geq \varepsilon \} \right) = 0.$$

Throughout the paper, we let (X, ρ) be a metric space and A, A_{kjl} be any closed non-empty subsets of X.

We now introduce our definitions.

Definition 2.2. The triple sequence $\left\{A_{kjl}\right\}$ is Wijsman convergent to A, if for each $x \in X$

$$P - \lim_{k,j,l\to\infty} d(x, A_{kjl}) = d(x, A) \text{ or } \lim_{k,j,l\to\infty} d(x, A_{kjl}) = d(x, A),$$

and is written $W_3 - \lim A_{kjl} = A$.

Definition 2.3. The triple sequence $\left\{A_{kjl}\right\}$ is Wijsman statistically convergent to A, if for every $\varepsilon > 0$ and for all $x \in X$,

$$\lim_{m,n,b\to\infty} \frac{1}{mnb} \left|\left\{k \leq m, j \leq n, l \leq b : \left|d(x, A_{kjl}) - d(x, A)\right| \geq \varepsilon\right\}\right| = 0,$$

that is

$$\left|d(x, A_{kjl}) - d(x, A)\right| < \varepsilon, \text{ almost all } (k, j, l).$$

In this case, we write $st_3 - \lim_{W_3} A_{kjl} = A$.

Wijsman's set of statistically convergent triple sequences is as follows:

$$(W_3S) := \left\{\left\{A_{kjl}\right\} : st_3 - \lim_{W_3} A_{kjl} = A\right\}.$$

A new type of sequence has been previously introduced, called the triple lacunary sequence. A triple sequence $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ is called a triple lacunary sequence if there are three increasing integer sequences as follows:

$$k_0 = 0, \ h_r = k_r - k_{r-1} \to \infty, \text{ as } r \to \infty,$$

$$j_0 = 0, \ h_u = j_u - j_{u-1} \to \infty, \text{ as } u \to \infty$$

and

$$l_0 = 0, \ h_p = l_p - l_{p-1} \to \infty, \text{ as } p \to \infty$$

Let $k_{r,u,p} = k_r j_u l_p, h_{r,u,p} = h_r h_u h_p$, and $\theta_{p,q,r}$ is determined by

$$I_{r,u,p} = \left\{(k, j, l) : k_{r-1} < k \leq k_r, j_{u-1} < j \leq j_u \text{ and } l_{p-1} < l \leq l_p\right\},$$

$$q_r = \frac{k_r}{k_{r-1}}, q_u = \frac{j_u}{j_{u-1}} \text{ and } q_p = \frac{l_p}{l_{p-1}} \text{ and } q_{r,u,p} = q_r q_u q_p.$$

3. Main results

Here, we will look at the concepts of Wijsman lacunary convergence, Wijsman strongly lacunary convergence and Wijsman lacunary statistical convergence for sets of triple sequences using the Orlicz function $\widetilde{\phi}$. In addition, the relationship between Wijsman statistical convergence and Wijsman lacunary statistical convergence for sets of triple sequences will be analyzed using the Orlicz function $\widetilde{\phi}$.

Definition 3.1. Let $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ be a triple lacunary sequence. The triple sequence $\left\{A_{kjl}\right\}$ is Wijsman lacunary convergent to A, if for every $x \in X$

$$\lim_{r,u,p\to\infty} \frac{1}{h_{r,u,p}} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} \sum_{l=l_{p-1}+1}^{l_p} d(x, A_{kjl}) = d(x, A).$$

In this case $A_{kjl} \overset{(W_3 N_\theta)}{\rightarrow} A$.

Definition 3.2. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ be a triple lacunary sequence. The triple sequence $\left\{A_{kjl}\right\}$ is Wijsman strongly lacunary convergent to A, if for each $x \in X$,

$$\lim_{r,u,p\to\infty} \frac{1}{h_{r,u,p}} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} \sum_{l=l_{p-1}+1}^{l_p} \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) = 0.$$

In this case $A_{kjl} \overset{[W_3 N_\theta]}{\rightarrow} A$.

Definition 3.3. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. We say that the triple sequence $\left\{A_{kjl}\right\}$ is Wijsman statistically $\widetilde{\phi}$-convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n,b\to\infty} \frac{1}{mnb} \left|\left\{k \leq m, j \leq n, l \leq b : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right| = 0,$$

that is

$$\widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) < \varepsilon, \text{ almost all } (k, j, l),$$

and is written $st_3 - \widetilde{\phi} - \lim_{W_3} A_{kjl} = A$.

Wijsman's set of statistically convergent triple sequences is as follows :

$$(W_3 S)_{\widetilde{\phi}} := \left\{\left\{A_{kjl}\right\} : st_3 - \widetilde{\phi} - \lim_{W_3} A_{kjl} = A\right\}.$$

Remark. If we take $\widetilde{\phi}(x) = |x|$, then $st_3 - \widetilde{\phi}$ convergence concepts coincide with Wijsman statistically convergence in Definition 2.3.

Definition 3.4. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ be a triple lacunary sequence. The triple sequence $\left\{A_{kjl}\right\}_{k,j,l\in\mathbb{N}}$ is Wijsman lacunary statistically $\widetilde{\phi}$-convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r,u,p\to\infty} \frac{1}{h_{r,u,p}} \left|\left\{(k, j, l) \in I_{r,u,p} : \widetilde{\phi}\left(\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right| = 0$$

and is written $st_3 - \widetilde{\phi} - \lim_{W_{\theta_3}} A_{kjl} = A$.

The set of Wijsman lacunary statistically $\widetilde{\phi}$-convergent triple sequences will be denoted by

$$(W_3S_\theta)_{\widetilde{\phi}} := \left\{\left\{A_{kjl}\right\} : st_3 - \widetilde{\phi} - \lim_{W_{\theta_3}} A_{kjl} = A\right\}.$$

Example 2. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function, $X = \mathbb{R}$ and we denote a triple sequence $\left\{A_{kjl}\right\}$ as:

$$A_{kjl} := \begin{cases} \left\{(x,y,z) \in \mathbb{R}^3 : 2 \le x \le k_r - k_{r-1}, 2 \le y \le j_u - j_{u-1}, 2 \le z \le l_p - l_{p-1}\right\} \\ \qquad \text{if } k,j,l \ge 2 \text{ and } k,j,l \text{ is cube integer,} \\ \{(1,1,1)\} \quad \text{otherwise.} \end{cases}$$

This sequence is not Wijsman lacunary summable. However, this sequence is Wijsman lacunary statistically $\widetilde{\phi}$-convergent to the set $A = \{(1,1,1)\}$.

Definition 3.5. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ be a triple lacunary sequence. The triple sequence $\left\{A_{kjl}\right\}$ is Wijsman strongly lacunary statistically $\widetilde{\phi}$-convergent to A, if for each $x \in X$,

$$\lim_{r,u,p\to\infty} \frac{1}{h_{r,u,p}} \left|\left\{(k,j,l) \in I_{r,u,p} : \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} \sum_{l=l_{p-1}+1}^{l_p} \widetilde{\phi}\left(d(x,A_{kjl}) - d(x,A)\right) \ge \varepsilon\right\}\right| = 0.$$

In this case we write $A_{kjl} \overset{[W_3N_\theta]_{\widetilde{\phi}}}{\to} A$.

Theorem 3.1. (i) *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ be a triple lacunary sequence. When $\left\{A_{kjl}\right\}_{k,j,l\in\mathbb{N}}$ is Wijsman strongly lacunary $\widetilde{\phi}$-convergent to A, $\left\{A_{kjl}\right\}_{k,j,l\in\mathbb{N}}$ is Wijsman lacunary statistical $\widetilde{\phi}$-convergent to A*

(ii) *A suitable subset of $(W_3S_\theta)_{\widetilde{\phi}}$ is $[W_3N_\theta]_{\widetilde{\phi}}$.*

Proof. (i) For every $\varepsilon > 0$ and if $A_{kjl} \to A\left([W_3N_\theta]_{\widetilde{\phi}}\right)$ we write

$$\begin{aligned} &\sum_{(k,j,l)\in I_{r,u,p}} \widetilde{\phi}\left(d(x,A_{kjl}) - d(x,A)\right) \\ &\ge \sum_{\substack{(k,j,l)\in I_{r,u,p} \\ \widetilde{\phi}\left(d(x,A_{kjl})-d(x,A)\right)\ge\varepsilon}} \widetilde{\phi}\left(d(x,A_{kjl}) - d(x,A)\right) \end{aligned}$$

$$\ge \varepsilon. \left|\left\{(k,j,l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x,A_{kjl}) - d(x,A)\right) \ge \varepsilon\right\}\right|$$

which gives the conclusion.

(ii) Define a sequence $\left\{A_{kjl}\right\}_{k,j,l\in\mathbb{N}}$ and give θ_3 to show that it is appropriate the inclusion $[W_3N_\theta]_{\widetilde{\phi}} \subset (W_3S_\theta)_{\widetilde{\phi}}$ in (i):

$$A_{kjl} := \begin{cases} \{(k,j,l)\}, & \text{if } k_{r-1} < k \le k_{r-1} + \left[\sqrt{h_r}\right], j_{u-1} < j \le j_{u-1} + \left[\sqrt{h_u}\right], \\ & \quad l_{p-1} < l \le l_{p-1} + \left[\sqrt{h_p}\right], (r,u,p = 1,2,...) \\ \{(0,0,0)\}, & \text{otherwise.} \end{cases}$$

The boundedness of $\left\{A_{kjl}\right\}$ will be considered. For every $\varepsilon > 0$ and for each $x \in X$,

$$\frac{1}{h_r h_u h_p}\left|\left\{(k,j,l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, \{(0,0,0)\})\right) \geq \varepsilon\right\}\right|$$
$$= \frac{\left[\sqrt{h_r}\right]\left[\sqrt{h_u}\right]\left[\sqrt{h_p}\right]}{h_{r,u,p}} \to 0, \text{ as } r,u,p \to \infty,$$

obtained i.e., $A_{kjl} \to \{(0,0,0)\}\,(W_3 S_\theta)_{\widetilde{\phi}}$. However,

$$\frac{1}{h_{r,u,p}}\left\{(k,j,l) \in I_{r,u,p} : \sum_{(k,j,l)\in I_{r,u,p}} \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, \{(0,0,0)\})\right)\right\}$$
$$= \frac{1}{h_{r,u,p}} \frac{\left(\left[\sqrt{h_r}\right].\left(\left[\sqrt{h_r}\right]+1\right)\right)\left(\left[\sqrt{h_u}\right].\left(\left[\sqrt{h_u}+1\right]\right)\right)\left(\left[\sqrt{h_p}\right].\left(\left[\sqrt{h_p}\right]+1\right)\right)}{8}$$
$$\to \frac{1}{8} \neq 0.$$

Therefore, $A_{kjl} \nrightarrow \{(0,0,0)\}\,[W_3 N_\theta]_v$. □

Definition 3.6. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A triple sequence $\left\{A_{kjl}\right\}$ is said to be bounded if there exists $M > 0$ such that $\widetilde{\phi}\left(A_{jkl}\right) \leq M$ for all $j,k,l \in \mathbb{N}$.

We denote the space of all bounded triple sequences by ℓ^3_∞.

Theorem 3.2. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and for any triple lacunary sequence $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_s, l_p\right)\right\}$, when $\left\{A_{kjl}\right\} \in \ell^3_\infty$ and $\left\{A_{kjl}\right\}$ is Wijsman lacunary statistical $\widetilde{\phi}$-convergent to A, $\left\{A_{kjl}\right\}$ is Wijsman strongly lacunary $\widetilde{\phi}$-convergent to A.*

Proof. For each $x \in X$ and for every (k,j,l), what if that $\left\{A_{kjl}\right\} \in \ell^3_\infty$ and $A_{kjl} \to A\,(W_3 S_\theta)_{\widetilde{\phi}}$, say $\widetilde{\phi}\left(d(x, A_{jkl}) - d(x, A)\right) \leq M$. Taken $\varepsilon > 0$, we have

$$\frac{1}{h_{r,u,p}} \sum_{(k,j,l)\in I_{r,u,p}} \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right)$$
$$= \frac{1}{h_{r,u,p}} \sum_{\substack{(k,j,l)\in I_{r,u,p} \\ \widetilde{\phi}\left(d(x,A_{kjl})-d(x,A)\right)\geq\varepsilon}} \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right)$$
$$+ \frac{1}{h_{r,u,p}} \sum_{\substack{(k,j,l)\in I_{r,u,p} \\ \widetilde{\phi}\left(d(x,A_{kjl})-d(x,A)\right)<\varepsilon}} \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right)$$
$$\leq \frac{M}{h_{r,u,p}}\left|\left\{(k,j,l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2},$$

for every $x \in X$. Therefore, we give the conclusion. □

Theorem 3.3. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function and* $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ *be a triple lacunary sequence. Then,* $\left\{(W_3 S_\theta)_{\widetilde{\phi}}\right\} \cap \ell_\infty^3 = \left\{[W_3 N_\theta]_{\widetilde{\phi}}\right\} \cap \ell_\infty^3$.

Proof. Omitted. □

Theorem 3.4. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function and for any triple lacunary sequence* $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_s, l_p\right)\right\}$, *if* $\liminf_{r,u,p} q_{r,u,p} > 1$, *then* $st_3 - \widetilde{\phi} - \lim_{W_3} A_{kjl} = A$ *implies* $st_3 - \widetilde{\phi} - \lim_{W_{\theta_3}} A_{kjl} = A$.

Proof. Suppose that $\liminf_{r,u,p} q_{r,u,p} > 1$ then there exist $\gamma > 0$ such that $q_{r,u,p} \geq 1 + \gamma$ for sufficiently large r, u, p which implies

$$\frac{h_{r,u,p}}{k_{r,u,p}} \geq \frac{\gamma}{(1+\gamma)}.$$

If $st_3 - \widetilde{\phi} - \lim_{W_3} A_{kjl} = A$, then for all $\varepsilon > 0$, for sufficiently large r, u, p we have

$$\begin{aligned}
&\frac{1}{k_r j_u l_p} \left|\left\{k \leq k_r, j \leq j_u, l \leq l_p : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right| \\
&\geq \frac{1}{k_r j_u l_p} \left|\left\{(k, j, l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right| \\
&\geq \frac{\gamma}{(1+\gamma)} . \frac{1}{h_{r,u,p}} \left|\left\{(k, j, l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right|.
\end{aligned}$$

Then, for each $x \in X$. Therefore, $st_3 - \widetilde{\phi} - \lim_{W_{\theta_3}} A_{kj} = A$. This completes the proof. □

Theorem 3.5. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function and for any triple lacunary sequence* $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_s, l_p\right)\right\}$, *when* $\limsup_{r,u,p} q_{r,u,p} < \infty$, $st_3 - \widetilde{\phi} - \lim_{W_{\theta_3}} A_{kjl} = A$ *implies* $st_3 - \widetilde{\phi} - \lim_{W_3} A_{kjl} = A$.

Proof. When $\limsup_{r,u,p} q_{r,u,p} < \infty$ then there is an $M > 0$ such that $q_{r,u,p} < M$ for all r, u, p. Assume that $st_3 - \widetilde{\phi} - \lim_{W_{\theta_3}} A_{kjl} = A$ and let

$$F_{rup} = F(r, u, p, x) := \left|\left\{(k, j, l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right|,$$

for $\varepsilon > 0$ there is an $r_0, u_0, p_0 \in \mathbb{N}$ such that

$$\frac{F_{rup}}{h_{r,u,p}} < \varepsilon, \text{ for all } r > r_0, u > u_0, p > p_0.$$

Now let

$$U := \max\left\{F_{rup} : 1 \leq r \leq r_0, 1 \leq u \leq u_0, 1 \leq p \leq p_0\right\}$$

and let t, v and f be three integers with satisfying $k_{r-1} < t \leq k_r$, $j_{u-1} < v \leq j_u$ and $l_{p-1} < f \leq l_p$. Then we have

$$\begin{aligned}
&\frac{1}{tvf}\left|\left\{k \leq t, j \leq v, l \leq f : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right| \\
&\leq \frac{1}{k_{r-1} j_{u-1} l_{p-1}}\left|\left\{k \leq k_r, j \leq j_u, l \leq l_p : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right| \\
&= \frac{1}{k_{r-1} j_{u-1} l_{p-1}}\left\{\left|\left\{(k, j, l) \in I_{1,1,1} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right|\right. \\
&\left. + ... + \frac{1}{k_{r-1} j_{u-1} l_{p-1}}\left|\left\{(k, j, l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, A)\right) \geq \varepsilon\right\}\right|\right\} \\
&= \frac{1}{k_{r-1} j_{u-1} l_{p-1}}\left\{F_{111} + F_{222} + ... + F_{r_0 u_0 p_0} + F_{r_0+1 u_0+1 p_0+1} + ... + F_{rup}\right\} \\
&\leq \frac{U}{k_{r-1} j_{u-1} l_{p-1}} r_0 u_0 p_0 \\
&+ \frac{1}{k_{r-1} j_{u-1} l_{p-1}}\left\{h_{r_0+1,u_0+1,p_0+1}\frac{F_{r_0+1 u_0+1 p_0+1}}{h_{r_0+1,u_0+1,p_0+1}} + ... + h_{r,u,p}\frac{F_{rup}}{h_{r,u,p}}\right\} \\
&\leq \frac{r_0 u_0 p_0 . U}{k_{r-1} j_{u-1} l_{p-1}} \\
&+ \frac{1}{k_{r-1} j_{u-1} l_{p-1}}\left(\sup_{r>r_0, u>u_0, p>p_0}\frac{F_{rup}}{h_{r,u,p}}\right)\left\{h_{r_0+1,u_0+1,p_0+1} + ... + h_r h_u h_p\right\} \\
&\leq \frac{r_0 u_0 p_0 . U}{k_{r-1} j_{u-1} l_{p-1}} + \varepsilon . \frac{(k_r - k_{r_0})(j_u - j_{u_0})\left(l_p - l_{p_0}\right)}{k_{r-1} j_{u-1} l_{p-1}} \\
&\leq \frac{r_0 u_0 p_0 . U}{k_{r-1,u-1,p-1}} + \varepsilon . q_{r,u,p} \\
&\leq \frac{r_0 u_0 p_0 . U}{k_{r-1,u-1,p-1}} + \varepsilon . M
\end{aligned}$$

and this completes the proof of of theorem. □

Theorem 3.6. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function. For any triple lacunary sequence* $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_s, l_p\right)\right\}$, *if* $1 < \liminf_{r,u,p} q_{r,u,p} \leq \limsup_{r,u,p} q_{r,u,p} < \infty$ *then* $(W_3 S)_{\widetilde{\phi}} = (W_3 S_\theta)_{\widetilde{\phi}}$.

Proof. This follows from Theorem 3.4 and Theorem 3.5. □

Theorem 3.7. *Let* $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ *be an Orlicz function and* $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ *be a triple lacunary sequence. When* $\left\{A_{kjl}\right\} \in (W_3 S)_{\widetilde{\phi}} \cap (W_3 S_\theta)_{\widetilde{\phi}}$, $st_3 - \widetilde{\phi} - \lim_{W_{\theta_3}} A_{kjl} = st_3 - \widetilde{\phi} - \lim_{W_3} A_{kjl}$.

Proof. Assume first that $st_3 - \widetilde{\phi} - \lim_{W_3} A_{kjl} = A$ and $st_3 - \widetilde{\phi} - \lim_{W_{\theta_3}} A_{kjl} = B$ and $A \neq B$. For

$$\frac{1}{2}\widetilde{\phi}\left(d(x, A) - d(x, B)\right) > \varepsilon$$

and every $x \in X$ we get

$$\lim_{m,n,b\to\infty} \frac{1}{mnb} \left|\left\{k \le m, j \le n, l \le b : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right| = 1.$$

Considering the $k_t j_v l_w$th term of the statistical limit expression

$$\frac{1}{mnb} \left|\left\{k \le m, j \le n, l \le b : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right| :$$

$$\begin{aligned}
&\frac{1}{k_t j_v l_w} \left|\left\{k \le k_t, j \le j_v, l \le l_w : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right| \\
&= \frac{1}{k_t j_v l_w} \left|\left\{(k, j, l) \in \bigcup_{r,u,p=1,1,1}^{t,v,w} I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right| \\
&= \frac{1}{k_t j_v l_w}. \sum_{r,u,p=1,1,1}^{t,v,w} \left|\left\{(k, j, l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right| \\
&= \frac{1}{\sum_{r,u,p=1,1,1}^{t,v,w} h_r h_u h_p}. \sum_{r,u,p=1,1,1}^{t,v,w} \left|\left\{(k, j, l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right| \\
&= \frac{1}{\sum_{r,u,p=1,1,1}^{t,v,w} h_{r,u,p}}. \sum_{r,u,p=1,1,1}^{t,v,w} h_{r,u,p} s_{rup},
\end{aligned}$$

where

$$s_{rup} = \frac{1}{h_{r,u,p}} \left|\left\{(k, j, l) \in I_{r,u,p} : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right| \to 0$$

that $A_{kjl} \to B\,(W_3 S_\theta)_{\widetilde{\phi}}$. Since θ_3 is a triple lacunary sequence,

$$\frac{1}{\sum_{r,u,p=1,1,1}^{t,v,w} h_{r,u,p}}. \sum_{r,u,p=1,1,1}^{t,v,w} h_{r,u,p} s_{rup}$$

is the regular weighted mean transform of s_{rup}, (see (Fridy & Orhan, 1993)), and hence it also approaches zero as $t, v, w \to \infty$. Furthermore, since this is a subsequence of

$$\left\{\frac{1}{mnb} \left|\left\{k \le m, j \le n, l \le b : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right|\right\}_{m,n,b=1,1,1}^{\infty,\infty,\infty}$$

herefrom

$$\lim_{m,n,b\to\infty} \frac{1}{mnb} \left|\left\{k \le m, j \le n, l \le b : \widetilde{\phi}\left(d(x, A_{kjl}) - d(x, B)\right) \ge \varepsilon\right\}\right| \ne 1,$$

and this is a contradiction. This shows that we cannot have $A \ne B$. □

From (Nuray & Rhoades, 2012), we introduced the notion of strongly almost $\widetilde{\phi}$-convergence for sets of triple sequences as follows :

Definition 3.7. Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ be a triple lacunary sequence. The triple sequence $\left\{A_{kjl}\right\}$ is Wijsman strongly almost $\widetilde{\phi}$-convergent to A, if for each $x \in X$,

$$\lim_{p,q,r\to\infty} \frac{1}{pqr} \sum_{k=1}^{p} \sum_{j=1}^{q} \sum_{l=1}^{r} \widetilde{\phi}\left(d(x, A_{k+m,j+n,l+o}) - d(x, A)\right) = 0$$

uniformly in m, n, o, and is written $A_{kjl} \overset{[W_3AC]_{\widetilde{\phi}}}{\longrightarrow} A$.

The set of Wijsman strongly almost convergent sequences will be denoted

$$[W_3AC]_{\widetilde{\phi}} := \left\{\left\{A_{kjl}\right\} : \lim_{p,q,r\to\infty} \frac{1}{pqr} \sum_{k=1}^{p} \sum_{j=1}^{q} \sum_{l=1}^{r} \widetilde{\phi}\left(d(x, A_{k+m,j+n,l+o}) - d(x, A)\right) = 0\right\}.$$

Corollary 3.8. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ be a triple lacunary sequence. Then $\left\{[W_3AC]_{\widetilde{\phi}}\right\} = \cap \{[W_3N_\theta]\}$.*

From Corollary 3.8, the relations $[W_3AC]_{\widetilde{\phi}} \subset \ell^3_\infty$ and Theorem 3.3, we have following theorem:

Theorem 3.9. *Let $\widetilde{\phi} : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and $\theta_3 = \theta_{r,u,p} = \left\{\left(k_r, j_u, l_p\right)\right\}$ be a triple lacunary sequence. If Φ_3 denotes the set of all triple lacunary sequences, then*

$$\left\{[W_3AC]_{\widetilde{\phi}}\right\} = \ell^3_\infty \cap \left(\cap_{\theta_3\in\Phi_3} \left\{(WS_\theta)_{\widetilde{\phi}}\right\}\right).$$

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