On the Growth of Solutions of Higher Order Complex Differential Equations with finite \([p, q]\)-Order

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Abstract

In this paper, we study the growth of entire solutions of higher order linear complex differential equations with entire coefficients of finite \([p, q]\)-order. We give another conditions that generalize some results due to (Belaïdi, 2015), (Liu et al., 2010) and (Li & Cao, 2012).

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1. Introduction

In this article, we use the standard notation and fundamental results of the Nevanlinna value distribution theory of meromorphic functions, see (Hayman, 1964; Laine, 1993; Yang & Yi, 2003). We define, for \(r \in [0, +\infty)\), \(\exp_0 r := r\), \(\exp_1 r := e^r\) and \(\exp_{n+1} r := \exp(\exp_n r), n \in \mathbb{N}\). For all \(r\) sufficiently large, we define \(\log_0 r := r\), \(\log_1 r := \log r\) and \(\log_{n+1} r := \log(\log_n r), n \in \mathbb{N}\). Moreover, we denote by \(\exp_{-1} r := \log r\) and \(\log_{-1} r := \exp_1 r\).

For a meromorphic function \(f\) in complex plane \(\mathbb{C}\), the order of growth is defined by

\[
\sigma(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r},
\]

where \(T(r, f)\) is the Nevanlinna characteristic function of \(f\). The exponents of convergence of sequence of the zeros and distinct zeros of \(f\) are respectively defined by

\[
\lambda(f) = \limsup_{r \to +\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}, \quad \overline{\lambda}(f) = \limsup_{r \to +\infty} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r},
\]

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where \( N(r, \frac{1}{f}) \) (resp. \( N(r, \frac{1}{f}) \)) is the integrated counting function of zeros (resp. distinct zeros) of \( f(z) \) in the disc \( \{ z : |z| \leq r \} \).

(Juneja et al., 1976, 1977) have investigated some properties of entire functions of \([p, q]\)-order and obtained some results about their growth. In order to maintain accordance with general definitions of the entire function \( f \) of iterated \( p \)-order\(^1\), (Liu et al., 2010) gave a minor modification of the original definition of the \([p, q]\)-order given by (Juneja et al., 1976, 1977).

We recall the following definition,

**Definition 1.1.** (Kinnunen, 1998) Let \( p \geq 1 \) be an integer. The iterated \( p \)-order \( \sigma_p(f) \) of a meromorphic function \( f \) is defined by

\[
\sigma_p(f) = \limsup_{r \to +\infty} \frac{\log_p T(r, f)}{\log r}.
\]

Now, we shall introduce the definition of meromorphic functions of \([p, q]\)-order, where \( p, q \) are positive integers satisfying \( p \geq q \geq 1 \) or \( 2 \leq q = p + 1 \). In order to keep accordance with Definition 1.1, (Li & Cao, 2012; Belaïdi, 2015) have gave a minor modification to the original definition of \([p, q]\)-order (e.g. see, (Juneja et al., 1976, 1977)). We recall the following definitions

**Definition 1.2.** (Belaïdi, 2015; Li & Cao, 2012; Liu et al., 2010) Let \( p \geq q \geq 1 \) or \( 2 \leq q = p + 1 \) be integers. If \( f(z) \) is a transcendental meromorphic function, then the \([p, q]\)-order is defined by

\[
\sigma_{[p, q]}(f) = \limsup_{r \to +\infty} \frac{\log_p T(r, f)}{\log_q r}.
\]

It is easy to see that \( 0 \leq \sigma_{[p, q]}(f) \leq +\infty \). If \( f(z) \) is rational, then \( \sigma_{[p, q]}(f) = 0 \) for any \( p \geq q \geq 1 \).

By Definition 1.2, we note that \( \sigma_{[1, 1]}(f) = \sigma(f) \) (order of growth), \( \sigma_{[2, 1]}(f) = \sigma_2(f) \) (hyper-order), \( \sigma_{[1, 2]}(f) = \sigma_{\log}(f) \) (logarithmic order) and \( \sigma_{[p, 1]}(f) = \sigma_p(f) \) (iterated \( p \)-order).

**Definition 1.3.** (Belaïdi, 2015; Li & Cao, 2012) Let \( p \geq q \geq 1 \) or \( 2 \leq q = p + 1 \) be integers. The \([p, q]\) convergence exponent of the sequence of zeros of a meromorphic function \( f(z) \) is defined by

\[
\lambda_{[p, q]}(f) = \limsup_{r \to +\infty} \frac{\log_p N\left(r, \frac{1}{f}\right)}{\log_q r}.
\]

Similarly, the \([p, q]\) convergence exponent of the sequence of distinct zeros of \( f(z) \) is defined by

\[
\lambda_{[p, q]}(f) = \limsup_{r \to +\infty} \frac{\log_p \overline{N}\left(r, \frac{1}{f}\right)}{\log_q r}.
\]

\(^1\)see (Kinnunen, 1998), for the definition of the iterated \( p \)-order.
We recall also the following definitions. The linear measure of a set \( E \subset (0, +\infty) \) is defined as
\[
m(E) = \int_0^{+\infty} \chi_E(t) dt
\]
and the logarithmic measure of a set \( F \subset (1, +\infty) \) is defined as
\[
\ell m(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt,
\]
where \( \chi_H(t) \) is the characteristic function of the set \( H \). The upper density of a set \( E \subset (0, +\infty) \) is defined by
\[
\overline{\text{dens}}(E) = \limsup_{r \to +\infty} \frac{m(E \cap [0, r])}{r}.
\]
The upper logarithmic density of a set \( F \subset (1, +\infty) \) is defined by
\[
\overline{\text{log dens}}(F) = \limsup_{r \to +\infty} \frac{\ell m(F \cap [1, r])}{\log r}.
\]

**Proposition 1.1.** (Belaïdi, 2015) For all \( H \subset [1, +\infty) \) the following statements hold:
(i) If \( \ell m(H) = \infty \), then \( m(H) = \infty \),
(ii) if \( \overline{\text{dens}}(H) > 0 \), then \( m(H) = \infty \),
(iii) if \( \overline{\text{log dens}}(H) > 0 \), then \( \ell m(H) = \infty \).

For \( a \in \mathbb{C} \), the deficiency of \( a \) with respect to a meromorphic function \( f \) is defined by
\[
\delta(a, f) = \liminf_{r \to +\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \limsup_{r \to +\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.
\]

Consider the differential equation
\[
f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0. \tag{1.1}
\]

(Liu et al., 2010) studied the growth of solutions of the homogeneous differential equation (1.1) with coefficients that are entire functions of finite \([p, q]\)-order and obtained following result

**Theorem 1.1.** (Liu et al., 2010) Let \( A_j(z) \) \( (j = 0, 1, \ldots, k - 1) \) be entire functions satisfying \( \max \{\sigma_{[p, q]}(A_j) : j \neq s\} < \sigma_{[p, q]}(A_s) < \infty \). Then every solution \( f(z) \) of (1.1) satisfies \( \sigma_{[p+1, q]}(f) \leq \sigma_{[p, q]}(A_s) \). Furthermore, at least one solution of (1.1) satisfies \( \sigma_{[p+1, q]}(f) = \sigma_{[p, q]}(A_s) \).

**Theorem 1.2.** (Liu et al., 2010) Let \( A_0, A_1, \ldots, A_{k-1} \) be entire functions, and let \( s \in \{0, \ldots, k - 1\} \) be the largest index for which \( \sigma_{[p, q]}(A_s) = \max_{0 \leq j \leq k-1} \sigma_{[p, q]}(A_j) \). Then there are at least \( k - s \) linearly independent solutions \( f(z) \) of (1.1) such that \( \sigma_{[p+1, q]}(f) = \sigma_{[p, q]}(A_s) \). Moreover, all solutions of (1.1) satisfy \( \sigma_{[p+1, q]}(f) \leq \rho \) if and only if \( \sigma_{[p, q]}(A_j) \leq \rho \) for all \( j = 0, 1, \ldots, k - 1 \).
Theorem 1.3. (Liu et al., 2010) Let $H$ be a set of complex numbers satisfying $\log \text{dens} \{|z| : z \in H\} > 0$ and let $A_j(z)$ ($j = 0, 1, \ldots, k - 1$) be entire functions satisfying

$$\max \{\sigma_{[p,q]}(A_j) : j = 0, 1, \ldots, k - 1\} \leq \alpha.$$  

Suppose that there exists a positive constant $\beta$ satisfying $\beta < \alpha$ such that any given $\varepsilon$ ($0 < \varepsilon < \alpha - \beta$), we have

$$|A_0(z)| \geq \exp_{p+1} \left((\alpha - \varepsilon) \log_q r\right)$$  and

$$|A_j(z)| \leq \exp_{p+1} \left(\beta \log_q r\right) \quad (j = 1, \ldots, k - 1)$$

for $z \in H$. Then, every solution $f \neq 0$ of the equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \alpha$.

Recently, (Belaidi, 2015) has obtained the following results which generalize and improve Theorem 1.3 and also improve some results due to (Li & Cao, 2012).

Theorem 1.4. (Belaidi, 2015) Let $H$ be a set of complex numbers satisfying $\log \text{dens} \{|z| : z \in H\} > 0$ and let $A_j(z)$ ($j = 0, 1, \ldots, k - 1$) be meromorphic functions satisfying

$$\max \{\sigma_{[p,q]}(A_j) : j = 0, 1, \ldots, k - 1\} \leq \rho, \quad 0 < \rho < +\infty.$$  

Suppose that there exist two real numbers $\alpha$ and $\beta$ satisfying $0 \leq \beta < \alpha$ such that

$$|A_0(z)| \geq \exp_p \left(\alpha \left[\log_{q-1} r\right]^\rho\right)$$  (1.2)  

and

$$|A_j(z)| \leq \exp_p \left(\beta \left[\log_{q-1} r\right]^\rho\right), \quad (j = 1, \ldots, k - 1)$$  (1.3)  

as $|z| = r \to +\infty$ for $z \in H$. Then the following statements hold:

(i) If $p \geq q \geq 2$ or $3 \leq q = p + 1$, then every meromorphic solution $f \neq 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.

(ii) If $p = 1, q = 2$, then every meromorphic solution $f \neq 0$ of equation (1.1) satisfies $\sigma_{[2,2]}(f) \geq \rho$.

Theorem 1.5. (Belaidi, 2015) Let $H$ be a set of complex numbers satisfying $\log \text{dens} \{|z| : z \in H\} > 0$ and let $A_j(z)$ ($j = 0, 1, \ldots, k - 1$) be meromorphic functions satisfying

$$\max \{\sigma_{[p,q]}(A_j) : j = 0, 1, \ldots, k - 1\} \leq \rho, \quad 0 < \rho < +\infty.$$  

Suppose that there exist two positive constants $\alpha$ and $\beta$ such that, we have

$$m(r, A_0) \geq \exp_{p-1} \left(\alpha \left[\log_{q-1} r\right]^\rho\right)$$  (1.4)  

and

$$m(r, A_j) \leq \exp_{p-1} \left(\beta \left[\log_{q-1} r\right]^\rho\right), \quad (j = 1, \ldots, k - 1)$$  (1.5)  

as $|z| = r \to +\infty$ for $z \in H$. Then the following statements hold:

(i) If $p \geq q \geq 2$ and $0 \leq \beta < \alpha$, then every meromorphic solution $f \neq 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,q]}(f) = \rho$.

(ii) If $3 \leq q = p + 1$, $0 \leq \beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \neq 0$ whose poles are uniformly bounded multiplicities or $\delta(\infty, f) > 0$ of equation (1.1) satisfies $\sigma_{[p+1,p+1]}(f) = \rho$.

(iii) If $p = 1, q = 2$, $0 \leq (k - 1)\beta < \alpha$ and $\rho > 1$, then every meromorphic solution $f \neq 0$ of equation (1.1) satisfies $\sigma_{[2,2]}(f) \geq \rho$.  

2. Main results

Now, a natural question is whether somewhat similar results to Theorem 1.4 and Theorem 1.5 could be obtained for the differential equation (1.1), where \( A_j(z) \) (\( j = 0, 1, \ldots, k \)) are entire functions and the dominant coefficient is some \( A_j(z) \) (\( 0 \leq s \leq k - 1 \)) instead of \( A_0(z) \)? The main purpose of this article is to answer the above question and improving and generalizing the previous results.

**Theorem 2.1.** Let \( H \) be a set of complex numbers satisfying \( \log \, \text{dens} \{ |z| : z \in H \} > 0 \). Let \( A_j(z) \) (\( j = 0, 1, \ldots, k - 1 \)) be entire functions satisfying

\[
\max \{ \sigma_{[p,q]}(A_j) : j = 0, 1, \ldots, k - 1 \} \leq \rho, \quad 0 < \rho < +\infty.
\]

Suppose that there exist two real numbers \( \alpha \) and \( \beta \) satisfying \( 0 < \beta < \alpha \) and let \( s \in \{0, \ldots, k - 1 \} \) be an integer for which

\[
|A_s(z)| \geq \exp_p \left( \alpha \left[ \log_{q-1} r \right]^p \right), \quad 0 \leq s \leq k - 1 \quad (2.1)
\]

and

\[
|A_j(z)| \leq \exp_p \left( \beta \left[ \log_{q-1} r \right]^p \right), \quad j \neq s, \quad (2.2)
\]

as \( |z| = r \to +\infty, z \in H \). Then,

(i) If \( p \geq q \geq 1 \), then every polynomial solution \( f \neq 0 \) of equation (1.1) is of \( \deg f \leq s - 1 \) (\( s \geq 1 \)) and every transcendental solution \( f \) of equation (1.1) satisfies \( \sigma_{[p,q]}(f) = \rho \).

(ii) If \( 2 \leq q = p + 1, p > 1 \), then every polynomial solution \( f \neq 0 \) of equation (1.1) is of \( \deg f \leq s - 1 \) (\( s \geq 1 \)) and every transcendental solution \( f \) of equation (1.1) satisfies \( \rho \leq \sigma_{[p+1,p+1]}(f) \leq \rho + 1 \).

**Corollary 2.1.** Let \( H \) be a set of complex numbers satisfying \( \log \, \text{dens} \{ |z| : z \in H \} > 0 \). Let \( F(z) \neq 0, A_j(z) \) (\( j = 0, 1, \ldots, k - 1 \)) be entire functions. Suppose that \( H, A_j(z) \) (\( j = 0, 1, \ldots, k - 1 \)) satisfy the hypotheses in Theorem 2.1. Consider the equation

\[
f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = F. \quad (2.3)
\]

(i) Let \( p \geq q \geq 1 \), if \( \sigma_{[p+1,q]}(F) \leq \rho \), then every transcendental solution \( f \) of equation (2.3) satisfies \( \sigma_{[p+1,q]}(f) = \rho \) with at most one exceptional solution \( f_0 \) satisfying \( \sigma_{[p+1,q]}(f_0) < \rho \); if \( \rho_{[p+1,q]}(F) > \rho \), then every transcendental solution \( f \) of equation (2.3) satisfies \( \rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F) \).

(ii) Let \( 2 \leq q = p + 1 \) and \( p > 1 \), if \( \sigma_{[p+1,p+1]}(F) \leq \rho \), then every transcendental solution \( f \) of equation (2.3) satisfies \( \sigma_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(f) = \sigma_{[p+1,p+1]}(f) = \rho \) with at most one exceptional solution \( f_0 \) satisfying \( \sigma_{[p+1,q]}(f_0) < \rho \); if \( \rho_{[p+1,p+1]}(F) > \rho \), then every transcendental solution \( f \) of equation (2.3) satisfies \( \rho_{[p+1,p+1]}(f) = \rho_{[p+1,p+1]}(F) \).

**Theorem 2.2.** Let \( H \) be a set of complex numbers satisfying \( \log \, \text{dens} \{ |z| : z \in H \} > 0 \). Let \( A_j(z) \) (\( j = 0, 1, \ldots, k - 1 \)) be entire functions satisfying

\[
\max \{ \sigma_{[p,q]}(A_j) : j = 0, 1, \ldots, k - 1 \} \leq \rho, \quad 0 < \rho < +\infty.
\]
Suppose that there exist two real numbers \( \alpha \) and \( \beta \) satisfying \( 0 \leq \beta < \alpha \) and let \( s \in \{0, \ldots, k-1\} \) be an integer for which

\[
m(r, A_j) \geq \exp_{p-1} \left( \alpha \left[ \log_{q-1} r \right]^p \right), \quad 0 \leq s \leq k-1
\]

and

\[
m(r, A_j) \leq \exp_{p-1} \left( \beta \left[ \log_{q-1} r \right]^p \right), \quad j \neq s,
\]
as \( |z| = r \to +\infty, z \in H \). Then the following statements hold:

(i) If \( p \geq q \geq 1 \) and \( 0 \leq \beta < \alpha \), then every polynomial solution \( f \neq 0 \) of (1.1) is of \( \deg f \leq s - 1 \) \( (s \geq 1) \), and every transcendental solution \( f \) satisfies \( \sigma_{[p, q]}(f) \geq \rho \geq \sigma_{[p+1, q]}(f) \).

(ii) If \( 2 \leq q = p + 1 \) and \( 0 \leq (k - 1) \beta < \alpha \), then every polynomial solution \( f \neq 0 \) of (1.1) is of \( \deg f \leq s - 1 \) \( (s \geq 1) \), and every transcendental solution \( f \) satisfies \( \rho \leq \sigma_{\{p, p+1\}}(f) \) and \( \rho \leq \sigma_{[p+1, p+1]}(f) \).

3. Some preliminary lemmas

**Lemma 3.1.** (Gundersen, 1988) Let \( f \) be a transcendental meromorphic function, and let \( \alpha > 1 \) be a given constant. Then there exists a set \( E_1 \subset (1, \infty) \) with finite logarithmic measure and a constant \( B > 0 \) that depends only on \( \alpha \) and \( s, j(0 \leq s < j) \), such that for all \( z \) satisfying \( |z| = r \notin E_1 \cup [0, 1] \)

\[
\left| \frac{f^{(j)}(z)}{f^{(s)}(z)} \right| \leq B \left( \frac{T(\alpha r, f)}{r} \left( \log^{\alpha} r \right) \log T(\alpha r, f) \right)^{j-s}.
\]

**Lemma 3.2.** (Gundersen, 1988) Let \( f \) be a meromorphic function, and let \( j \) be a given positive integer, and let \( \alpha > 1 \) be a real constant. Then there exists a constant \( R > 0 \) such that for all \( r \geq R \) we have

\[
T \left( r, f^{(j)} \right) \leq (j + 2) T \left( \alpha r, f \right).
\]

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function, \( \mu_j(r) \) be the maximum term, i.e., \( \mu_j(r) = \max \{|a_n| r^n; n = 0, 1, \ldots\} \), and let \( v_j(r) \) be the central index of \( f \), i.e., \( v_j(r) = \max \{m; \mu_j(r) = |a_m| r^m\} \).

**Lemma 3.3.** (Hayman, 1974) Let \( f(z) \) be a transcendental entire function, and let \( z \) be a point with \( |z| = r \) at which \( |f(z)| = M(r, f) \). Then for all \( |z| = r \) outside a set \( E_2 \) of \( r \) of finite logarithmic measure, we have

\[
\frac{f^{(j)}(z)}{f(z)} = \left( \frac{v_j(r)}{z} \right)^j (1 + o(1)), \quad j \in \mathbb{N},
\]
where \( v_j(r) \) is the central index of \( f(z) \).

**Lemma 3.4.** (Juneja et al., 1976) Let \( f(z) \) be an entire function of \( [p, q] \)-order, and let \( v_j(r) \) be the central index of \( f(z) \). Then

\[
\sigma_{[p, q]}(f) = \limsup_{r \to +\infty} \frac{\log_p v_j(r)}{\log_q r}.
\]
Lemma 3.5. Let $A_0(z), \ldots, A_{k-1}(z)$ be entire functions of finite $[p, q]$-order. Then, 

(i) If $p \geq q \geq 1$, then every solution $f \not= 0$ of equation (1.1) satisfies 

$$
\sigma_{[p+1,q]}(f) \leq \max \left\{ \sigma_{[p,q]}(A_j) : j = 0, 1, \ldots, k-1 \right\}.
$$

(ii) If $2 \leq q = p + 1$, then every solution $f \not= 0$ of equation (1.1) satisfies 

$$
\sigma_{[p+1,p+1]}(f) \leq \max \left\{ \sigma_{[p,p+1]}(A_j) : j = 0, 1, \ldots, k-1 \right\} + 1.
$$

Proof. We prove only (ii). For the proof of (i) see (Liu et al., 2010). Let $f \not= 0$ be a solution of equation (1.1). By (1.1), we have 

$$
\left| \frac{f^{(k)}(z)}{f} \right| \leq |A_{k-1}| \frac{f^{(k-1)}}{f} + |A_{k-2}| \frac{f^{(k-2)}}{f} + \cdots + |A_1| \frac{f'}{f} + |A_0|.
$$

(3.1)

Set $\max \left\{ \sigma_{[p,p+1]}(A_j) : j = 0, 1, \ldots, k-1 \right\} = \rho$. For any given $\varepsilon > 0$, when $r$ is sufficiently large, we have 

$$
|A_j(z)| \leq \exp_{p+1} \left( (\rho + \varepsilon) \left[ \log_{p+1} r \right] \right), \quad j = 0, 1, \ldots, k-1.
$$

(3.2)

By Lemma 3.3, there exists a set $E_2 \subset [1, +\infty)$ with logarithmic measure $\ell m E_2 < \infty$, we can choose $z$ satisfying $|z| = r \not\in [0, 1] \cup E_2$ and $|f(z)| = M(r, f)$, such that 

$$
\frac{f^{(j)}(z)}{f(z)} = \left( \frac{\nu_f(r)}{|z|} \right)^j (1 + o(1)), \quad j = 1, \ldots, k
$$

(3.3)

holds. Substituting (3.2) and (3.3) into (3.1), we obtain 

$$
\left( \frac{\nu_f(r)}{|z|} \right)^k |1 + o(1)| \leq k \exp_{p+1} \left( (\rho + \varepsilon) \left[ \log_{p+1} r \right] \right) \left( \frac{\nu_f(r)}{|z|} \right)^{k-1} |1 + o(1)|,
$$

(3.4)

where $z$ satisfies $|z| = r \not\in [0, 1] \cup E_2$ and $|f(z)| = M(r, f)$. By (3.4), we get 

$$
\nu_f(r) |1 + o(1)| \leq kr |1 + o(1)| \exp_{p+1} \left( (\rho + \varepsilon) \left[ \log_{p+1} r \right] \right).
$$

(3.5)

So, from (3.5), we obtain 

$$
\limsup_{r \to +\infty} \frac{\log_{p+1} \nu_f(r)}{\log_{p+1} r} \leq \rho + 1 + \varepsilon.
$$

(3.6)

Since $\varepsilon > 0$ is arbitrary, by (3.6) and Lemma 3.4 we have $\sigma_{[p+1,p+1]}(f) \leq \rho + 1$. \hfill \Box

Remark. Lemma 3.5 (ii) has been proved for $p = 1$ and $q = 2$ by (Cao et al., 2013).

Lemma 3.6. (Chen & Shon, 2004) Let $f(z)$ be a transcendental entire function. Then there is a set $E_3 \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point $z$ satisfying $|z| = r \not\in [0, 1] \cup E_3$ and $|f(z)| = M(r, f)$, we have 

$$
\left| \frac{f(z)}{f^{(s)}(z)} \right| \leq 2r^s, \quad s \in \mathbb{N}.
$$
Lemma 3.7. Let $f$ be a transcendental meromorphic function of finite $[p,q]$-order. Then the following statements hold:

(i) If $p \geq q \geq 1$, then $\rho_{[p,q]}(f') = \rho_{[p,q]}(f)$.

(ii) If $2 \leq q = p + 1$, then $\rho_{[p,p+1]}(f') = \rho_{[p,p+1]}(f)$.

Proof. We prove only (ii). For the proof of (i) see (Belaidi, 2015). Let $f$ be a transcendental meromorphic function of finite $[p,q]$-order. By lemma of logarithmic derivative $^2$, we have

$$T(r, f') = m(r, f') + N(r, f') \leq m(r, f) + m\left(r, \frac{f'}{f}\right) + 2N(r, f)$$

$$\leq 2T(r, f) + m\left(r, \frac{f'}{f}\right) \leq 2T(r, f) + O(\log T(r, f) + \log r)$$

holds outside of an exceptional set $E_4 \subset (0, +\infty)$ with finite linear measure. By (3.7), it is easy to see that $\rho_{[p,p+1]}(f') \leq \rho_{[p,p+1]}(f)$ if $2 \leq q = p + 1$. On the other hand, by (Chuang, 1951), ((Yang & Yi, 2003), p. 35), we have for $r \to +\infty$

$$T(r, f) < O(2T(r, f') + \log r).$$

Hence, by using (3.8) we obtain $\rho_{[p,p+1]}(f') \leq \rho_{[p,p+1]}(f)$ if $2 \leq q = p + 1$. Thus, $\rho_{[p,p+1]}(f') = \rho_{[p,p+1]}(f)$ if $2 \leq q = p + 1$. □

Remark.Lemma 3.7 (ii) has been proved for $p = 1$ and $q = 2$ by (Chern, 2006).

Lemma 3.8. (Belaidi, 2015) Let $A_j (j = 0, 1, \ldots, k - 1)$, $F \neq 0$ be meromorphic functions. Then the following statements hold:

(i) If $p \geq q \geq 1$, then every every meromorphic solution $f$ of equation (2.3) such that

$$\max\{\sigma_{[p,q]}(A_j) : \sigma_{[p,q]}(F) : j = 0, 1, \ldots, k - 1\} < \sigma_{[p,q]}(f)$$

satisfies $\lambda_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

(ii) If $2 \leq q = p + 1$, then every meromorphic solution $f$ of equation (2.3) such that

$$\max\{1; \sigma_{[p,q]}(A_j) : \sigma_{[p,q]}(F) : j = 0, 1, \ldots, k - 1\} < \sigma_{[p,q]}(f)$$

satisfies $\lambda_{[p,p+1]}(f) = \lambda_{[p,p+1]}(f) = \rho_{[p,p+1]}(f)$.

4. Proofs of main results

Proof of Theorem 2.1 It’s should be noticed that the case $s = 0$ returns to Theorem 1.4. So, we will prove Theorem 2.1 in case $s > 0$. 

\(^2\) see, (Hayman, 1964; Yang & Yi, 2003).
(i) Case : \( p \geq q \geq 1 \). Suppose that \( f \not\equiv 0 \) is a polynomial solution of the equation (1.1), let \( f(z) = a_nz^n + \cdots + a_0, a_n \neq 0 \) and suppose that \( n \geq s \), i.e., \( f^{(s)}(z) \neq 0 \). Then from (1.1), we have
\[
|A_s| A_n^s |a_n| r^{n-s} (1 + o(1)) \leq |A_s| |f^{(s)}(z)| \leq \sum_{j=0}^{k} |A_j| |f^{(j)}(z)| \leq \sum_{j=0}^{k} |A_j| A_n^j |a_n| r^{n-j} (1 + o(1)),
\]
where \( A_k \equiv 1 \) and \( A_n^j = n(n-1) \cdots (n-j+1) \). It follows from (4.1), (2.1) and (2.2) that
\[
\exp_p \left( \alpha \left[ \log_{\psi-1} r \right]^\beta \right) r^{-s} \leq O \left( \exp_p \left( \beta \left[ \log_{\psi-1} r \right]^\alpha \right) \right).
\]
(4.2)
Since \( \alpha > \beta \), we see that (4.2) is a contradiction as \( r \to +\infty \). Then \( \deg f \leq s-1 \).
Now, suppose that \( f \) is a transcendental solution of the equation (1.1). From the conditions of Theorem 2.1, there is a set \( H \) of complex numbers satisfying \( \log \text{den}(|z| : z \in H) > 0 \), and there exists \( A_s (0 \leq s \leq k-1, k \geq 2) \) such that for all \( z \in H \) we have (2.1) and (2.2) as \( |z| \to +\infty \). Set \( H_1 = \{ |z| : z \in H \} \), since \( \log \text{den}(|z| : z \in H) > 0 \), then \( H_1 \) is a set with \( \ell \text{m}(H_1) = \infty \).
From (1.1), we have
\[
-A_s = \frac{f^{(s)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \cdots + A_{s+1} \frac{f^{(s+1)}}{f}
\]
\[
+ A_{s-1} \frac{f^{(s-1)}}{f} + \cdots + A_1 \frac{f'}{f} + A_0.
\]
(4.3)
By Lemma 3.1, there exists a set \( E_1 \subset (1, \infty) \) with finite logarithmic measure and a constant \( B > 0 \), such that for all \( z \) satisfying \( |z| = r \not\in E_1 \cup [0, 1] \)
\[
\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B \left( T(2r, f) \right)^{j+1}, \quad j = 1, 2, \ldots, k-1.
\]
(4.4)
By Lemma 3.6, there is a set \( E_3 \subset (1, +\infty) \) having finite logarithmic measure such that when we take a point \( z \) satisfying \( |z| = r \not\in [0, 1] \cup E_3 \) and \( |f(z)| = M(r, f) \), we have
\[
\left| \frac{f(z)}{f^{(j)}(z)} \right| \leq 2r^s.
\]
(4.5)
It follows from (4.3) – (4.5), (2.1) and (2.2) that
\[
\exp_p \left( \alpha \left[ \log_{\psi-1} r \right]^\beta \right) \leq 2kB \left( T(2r, f) \right)^{k+1} r^s \exp_p \left( \beta \left[ \log_{\psi-1} r \right]^\alpha \right).
\]
(4.6)
for all \( |z| = r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_3) \) and \( |f(z)| = M(r, f) \). Then by (4.6), we obtain \( \rho \leq \sigma_{\{p+1,q\}}(f) \). On the other hand, by Lemma 3.5 (i), we have \( \sigma_{\{p+1,q\}}(f) \leq \rho \). Hence, every transcendental solution \( f \) of the equation (1.1) satisfies \( \sigma_{\{p+1,q\}}(f) = \rho \).
(ii) Case : \( 2 \leq q = p + 1, \rho > 1 \). Suppose that \( f \not\equiv 0 \) is a polynomial solution of the equation (1.1), let \( f(z) = a_nz^n + \cdots + a_0, a_n \neq 0 \) and suppose that \( n \geq s \), i.e. \( f^{(s)}(z) \not\equiv 0 \). From (4.2), we have
\[
\exp_p \left( \alpha \left[ \log_{\psi-1} r \right]^\beta \right) r^{-s} \leq O \left( \exp_p \left( \beta \left[ \log_{\psi-1} r \right]^\alpha \right) \right).
\]
(4.7)
Since $\alpha > \beta$, we see that (4.7) is a contradiction as $r \to +\infty$. Then $\deg f \leq s - 1$.

Now, suppose that $f$ is a transcendental. Then from (4.6) we have

$$\exp_p (r \left[ \log_p |r| \right]^\rho) \leq 2kB^r [T (2r, f)]^{k+1} \exp_p (\beta \left[ \log_p |r| \right]^\rho)$$

holds for all $z$ satisfying $|z| = r \in H_1 \setminus ([0, 1] \cup E_1 \cup E_3)$, as $r \to +\infty$. By (4.8), every transcendental solution $f$ of equation (1.1) satisfies $\sigma_{[p+1,p+1]} (f) \geq \rho$, and by Lemma 3.5 (ii), we have $\sigma_{[p+1,p+1]} (f) \leq \rho + 1$, thus $\rho \leq \sigma_{[p+1,p+1]} (f) \leq \rho + 1$.

**Proof of Corollary 2.1** (i) (a) Let $p \geq q \geq 1$. Let $f$ be a transcendental solution of the equation (2.3) and $\{f_1, f_2, \ldots, f_k\}$ is a solution base of the corresponding homogeneous equation (1.1) of (2.3). By Theorem 2.1, we know that for $j = 1, 2, \ldots, k$

$$\sigma_{[p+1,q]} (f_j) = \rho.$$

Then $f$ can be expressed in the form

$$f(z) = B_1 (z) f_1 (z) + B_2 (z) f_2 (z) + \cdots + B_k (z) f_k (z),$$

where $B_1, B_2, \ldots, B_k$ are suitable meromorphic functions satisfying

$$B_j' = F \cdot G_j (f_1, f_2, \ldots, f_k) \cdot (W (f_1, f_2, \ldots, f_k))^{-1}, \quad j = 1, 2, \ldots, k,$$

where $G_j (f_1, f_2, \ldots, f_k)$ are differential polynomials in $f_1, f_2, \ldots, f_k$ and their derivatives with constant coefficients, thus

$$\sigma_{[p+1,q]} (G_j) \leq \max_{j=1,2,\ldots,k} \sigma_{[p+1,q]} (f_j) = \rho, \quad j = 1, 2, \ldots, k.$$  

(4.11)

Since the Wronskian $W (f_1, f_2, \ldots, f_k)$ is a differential polynomial in $f_1, f_2, \ldots, f_k$, it is easy to deduce also that

$$\sigma_{[p+1,q]} (W) \leq \max_{j=1,2,\ldots,k} \sigma_{[p+1,q]} (f_j) = \rho.$$  

(4.12)

Since $\sigma_{[p+1,q]} (F) \leq \rho$, then by using Lemma 3.7 (i) and (4.10) – (4.12) we get for $j = 1, 2, \ldots, k$

$$\sigma_{[p+1,q]} (B_j) = \sigma_{[p+1,q]} (B_j') \leq \max \left\{ \sigma_{[p+1,q]} (F) : \rho \right\} = \rho.$$  

(4.13)

Then by (4.9) and (4.13), we obtain

$$\sigma_{[p+1,q]} (f) \leq \max_{j=1,2,\ldots,k} \left\{ \sigma_{[p+1,q]} (f_j) ; \sigma_{[p+1,q]} (B_j) \right\} = \rho.$$  

(4.14)

Now, we assert that every transcendental solution $f$ of (2.3) satisfies $\sigma_{[p+1,q]} (f) = \rho$ with at most one exceptional solution $f_0$ satisfying $\sigma_{[p+1,q]} (f_0) < \rho$. In fact, if $f^*$ is another transcendental solution with $\sigma_{[p+1,q]} (f^*) < \rho$ of (2.3), then $\sigma_{[p+1,q]} (f_0 - f^*) < \rho$, but $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.1), and this is a contradiction with the results of Theorem 2.1. Then, $\sigma_{[p+1,q]} (f) = \rho$ holds for every transcendental solution $f$ of (2.3) with at
most one exceptional solution $f_0$ satisfying $\sigma_{[p+1,q]}(f_0) < \rho$. By Lemma 3.8, every transcendental solution $f$ of (2.3) with $\sigma_{[p+1,q]}(f) = \rho$ satisfies $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \rho$.

(b) If $\rho < \rho_{[p+1,q]}(F)$, then by using Lemma 3.7 (i), (4.11) and (4.12), we have from (4.10) for $j = 1, 2, \cdots, k$

$$\rho_{[p+1,q]}(B_j) = \rho_{[p+1,q]}(B'_j)$$

$$\leq \max \left\{ \rho_{[p+1,q]}(F), \rho_{[p+1,q]}(f_j) : j = 1, 2, \cdots, k \right\} = \rho_{[p+1,q]}(F).$$  \hspace{1cm} (4.15)

Then from (4.15) and (4.9), we get

$$\rho_{[p+1,q]}(f) \leq \max \left\{ \rho_{[p+1,q]}(f_j), \rho_{[p+1,q]}(B_j) : j = 1, 2, \cdots, k \right\} \leq \rho_{[p+1,q]}(F).$$  \hspace{1cm} (4.16)

On the other hand, if $\rho < \rho_{[p+1,q]}(F)$, it follows from equation (2.3) that a simple consideration of $[p, q]$–order implies $\rho_{[p+1,q]}(f) \geq \rho_{[p+1,q]}(F)$. By this inequality and (4.16) we obtain $\rho_{[p+1,q]}(f) = \rho_{[p+1,q]}(F)$.

(ii) For $2 \leq q = p+1, \rho > 1$, by the similar proof in case (i), we can also obtain that the conclusions of case (ii) hold.

**Proof of Theorem 2.2** Suppose that $f \neq 0$ is a solution of the equation (1.1). From the conditions the Theorem 2.2, there is a set $H$ of complex numbers satisfying $\log\text{dens } |z| : z \in H > 0$, and there exists $A_s (0 \leq s \leq k-1, k \geq 2)$ such that for all $z \in H$ we have (2.4) and (2.5) as $|z| \to +\infty$. Set $H_1 = \{ |z| : z \in H \}$, since $\log\text{dens } |z| : z \in H > 0$ then $H_1$ is a set with $\ell m(H_1) = \infty$.

(i) Let $p \geq q \geq 1$ and $0 \leq \beta < \alpha$. Suppose that $f \neq 0$ is a polynomial with $\deg f = n \geq s$, then $f^{(s)} \neq 0$, implies that $\frac{f^{(s)}}{f^{(m)}} (j = 0, 1, \ldots, k)$ is a rational, hence $T \left( r, \frac{f^{(s)}}{f^{(m)}} \right) = O(\log r)$ for $r$ sufficiently large. From (4.3) we have

$$T(r, A_s) \leq \sum_{j=0}^{k-1} T \left( r, A_j \right) + O(\log r).$$  \hspace{1cm} (4.17)

It follows by (4.17), (2.4) and (2.5) that

$$\exp_{p-1} \left( \alpha \left[ \log_{q-1} r \right]^n \right) \leq O \left( \exp_{p-1} \left( \beta \left[ \log_{q-1} r \right]^n \right) \right)$$  \hspace{1cm} (4.18)

which is a contradiction since $\alpha > \beta$ and $r \to +\infty$. Then, every polynomial solution $f \neq 0$ of (1.1) is of $\deg f \leq s - 1$.

Now, suppose that $f$ is a transcendental solution of (1.1). By using the first main theorem of Nevanlinna and properties of the characteristic function, we obtain from (4.3)

$$T(r, A_s) \leq T \left( r, f^{(k)} \right) + kT \left( r, f^{(s)} \right) + \sum_{j=0}^{k-1} T \left( r, f^{(j)} \right) + \sum_{j=0}^{k-1} T \left( r, A_j \right) + O(1).$$  \hspace{1cm} (4.19)
By Lemma 3.2, there exists a constant $R > 0$ such that for all $z$ satisfying $|z| = r > R$, we rewrite (4.19) as follows

$$m(r, A_s) = T(r, A_s) \leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) + \sum_{j=0, j\neq s}^{k-1} T(r, A_j) + O(1)$$

$$= \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f) + \sum_{j=0, j\neq s}^{k-1} m(r, A_j) + O(1). \quad (4.20)$$

It follows by (4.20), (2.4) and (2.5) that

$$\exp_{p-1}\left(\alpha \left[\log_{q-1} r\right]^\beta\right) \leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f)$$

$$+ (k - 1) \exp_{p-1}\left(\beta \left[\log_{q-1} r\right]^\beta\right) + O(1) \quad (4.21)$$

holds for all $z$ satisfying $|z| = r \in H_1$ as $r \to +\infty$. Then, by (4.21), every transcendental solution $f$ of equation (1.1) satisfies $\sigma_{[p, q]}(f) \geq \rho$, and by Lemma 3.5 (i), we have $\sigma_{[p+1, q]}(f) \leq \rho$. Thus, $\sigma_{[p, q]}(f) \geq \rho \geq \sigma_{[p+1, q]}(f)$.

(ii) Let $2 \leq q = p+1$ and $0 \leq (k - 1)\beta < \alpha$. Suppose that $f \not= 0$ is a polynomial with $\deg f = n \geq s$, then $f^{(s)} \not= 0$. By the same reasoning as in the proof in case (i), it is clear that $f(z)$ is a polynomial with $\deg f \leq s - 1$.

Now, suppose that $f$ is a transcendental solution of (1.1). Then by (4.21)

$$\exp_{p-1}\left(\alpha \left[\log_p r\right]^\beta\right) \leq \left(\frac{3}{2}k^2 + \frac{7}{2}k\right) T(2r, f)$$

$$+ (k - 1) \exp_{p-1}\left(\beta \left[\log_p r\right]^\beta\right) + O(1) \quad (4.22)$$

holds for all $z$ satisfying $|z| = r \in H_1$ as $r \to +\infty$. Then, by (4.22), every transcendental solution $f$ of equation (1.1) satisfies $\sigma_{[p, p+1]}(f) \geq \rho$, and by Lemma 3.5 (ii), we have $\sigma_{[p+1, p+1]}(f) \leq \rho + 1$. Hence, $\rho \leq \sigma_{[p, p+1]}(f)$ and $\sigma_{[p+1, p+1]}(f) \leq \rho + 1$.

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References


