Subordination Properties of Certain Subclasses of Multivalent Functions Defined By Srivastava-Wright Operator

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Abstract

Some subordination properties are investigated for functions belonging to each of the subclasses $V(\lambda, A, B)$ and $W(\lambda, A, B)$ of analytic $p$-valent functions involving the Srivastava-Wright operator in the open unit disk, \( U \) with suitable restrictions on the parameters \( \lambda, A \) and \( B \). The authors also derive certain subordination results involving the Hadamard product (or convolution) of the associated functions. Relevant connections of the main results to various known results are established.

Keywords: Multivalent function, Srivastava-Wright Operator, Convex function, Differential subordination, Argument estimates.

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1. Introduction

Let \( \mathcal{A}_k (p) \) be the class of functions of the form

\[
f(z) = z^p + \sum_{n=k}^{\infty} a_n z^n \quad (p < k; p, k \in \mathbb{N} := \{1, 2, 3, \ldots \}),
\]

which are analytic and \( p \)-valent in the unit disc, \( \mathbb{U} := \mathbb{U}(1) \), where \( \mathbb{U}(r) = \{ z \in \mathbb{C} : |z| < r \} \). Also, let \( \mathcal{A}(p) = \mathcal{A}_{p+1}(p) \) and \( \mathcal{A} = \mathcal{A}(1) \). For the functions \( f \in \mathcal{A}_k (p) \) of the form (1.1) and \( g \in \mathcal{A}_k (p) \) given by \( g(z) = z^p + \sum_{n=k}^{\infty} b_n z^n \), the Hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[
(f \ast g)(z) := z^p + \sum_{n=k}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}.
\]
If \( f \) and \( g \) are two analytic functions in \( \mathbb{U} \), we say that \( f \) is subordinate to \( g \), written symbolically as \( f(z) \prec g(z) \), if there exists a Schwarz function \( w \), which (by definition) is analytic in \( \mathbb{U} \), with \( w(0) = 0 \), and \( |w(z)| < 1 \) for all \( z \in \mathbb{U} \), such that \( f(z) = g(w(z)), z \in \mathbb{U} \).

If the function \( g \) is univalent in \( \mathbb{U} \), then we have the following equivalence, (c.f (Miller & Mocanu, 1981, 2000)):

\[
f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).
\]

Let \( \alpha_1, A_1, \ldots, \alpha_q, A_q \) and \( \beta_1, B_1, \ldots, \beta_s, B_s (q, s \in \mathbb{N}) \) be positive and real parameters such that

\[
1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{q} A_i > 0.
\]

The Wright generalized hypergeometric function

\[
_1\Psi_1[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = \sum_{n=0}^{\infty} \prod_{i=1}^{q} \Gamma(\alpha_i + nA_i) \frac{z^n}{n!} (z \in \mathbb{U}).
\]

If \( A_i = 1(i = 1, \ldots, q) \) and \( B_i = 1(i = 1, \ldots, s) \), we have the following relationship:

\[
_1\Psi_1[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] = z^{p} _qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),
\]

where \( _qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \) is the generalized hypergeometric function and

\[
\Omega = \frac{\prod_{i=1}^{q} \Gamma(\beta_i)}{\prod_{i=1}^{q} \Gamma(\alpha_i)} \quad (1.2)
\]

Now we define a function \( \mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}, z] \) by

\[
\mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}, z] = \Omega z^{p} _q\Psi_1[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z]
\]

and also consider the following linear operator

\[
\theta_p^{\alpha,\beta}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z] : \mathcal{A}_k(p) \rightarrow \mathcal{A}_k(p)
\]

defined using the convolution

\[
\theta_p^{\alpha,\beta}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}, f(z)] = \mathcal{WH}_p[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}, z] * f(z).
\]

We note that, for a function \( f \) of the form (1.1), we have

\[
\theta_p^{\alpha,\beta}[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}, z] f(z) = z^{p} + \sum_{n=k}^{\infty} \Omega \sigma_{n,p}(\alpha_1) a_n z^n, \quad (1.3)
\]
where $\Omega$ is given by (1.2) and $\sigma_{n,p}(\alpha_1)$ is defined by
\[ \sigma_{n,p}(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(n - p)) \ldots \Gamma(\alpha_q + A_q(n - p))}{\Gamma(\beta_1 + B_1(n - p)) \ldots \Gamma(\beta_s + B_s(n - p))(n - p)!}. \] (1.4)

If for convenience, we write
\[ \theta_p^{\alpha_1}(1)f(z) = \theta_p^{\alpha_1}[(\alpha_1, A_1) \ldots (\alpha_q, A_q); (\beta_1, B_1) \ldots (\beta_s, B_s)]f(z) \]
then we can easily verify from (1.3) that
\[ zA_1(\theta^{\alpha_1}_p f(z))' = \alpha_1 \theta^{\alpha_1}_p (\alpha_1 + 1)f(z) - (\alpha_1 - pA_1) \theta^{\alpha_1}_p f(z) (A_1 > 0). \] (1.5)

For $A_i = 1(i = 1, \ldots, q)$ and $B_i = 1(i = 1, \ldots, s)$, we obtain $\theta^{\alpha_1}_p [\alpha_1]f(z) = H_{p,q,s} f(z)$, which is known as the Dziok-Srivastava operator; it was introduced and studied by Dziok and Srivastava (Dziok & Srivastava, 1999, 2003). Also, for $f(z) \in \mathcal{A}$, the linear operator $\theta^{\alpha_1}_p [\alpha_1] f(z) = \theta[\alpha_1]$ is popularly known in the current literature as the Srivastava-Wright operator; it was systematically and firmly investigated by Srivastava (Srivastava, 2007). (see also (Kiryakova, 2011; Dziok & Raina, 2004) and (Aouf et al., 2010)).

**Remark.** For $f \in \mathcal{A}(p), A_i = 1(i = 1, 2, \ldots, q), B_i = 1(i = 1, 2, \ldots, s), q = 2$ and $s = 1$ by specializing the parameters $\alpha_1, \alpha_2$ and $\beta_1$ the operator $\theta^{\alpha_1}_p (\alpha_1)$ gets reduced to the following familiar operators:

(i) $\theta^{\alpha_1}_p [a, 1; c] f(z) = L_p(a, c) f(z)$ [see Saitoh (Saitoh, 1996)];
(ii) $\theta^{\alpha_1}_p [\mu + p, 1; 1] f(z) = D^{\mu+p-1} f(z)$ ($\mu > -p$), where $D^{\mu+p-1}$ is the $\mu + p - 1$- the order Ruscheweyh derivative of a function $f \in \mathcal{A}(p)$, [see Kumar and Shukla (Kumar & Shukla, 1984a,b)];
(iii) $\theta^{\alpha_1}_p [1 + p, 1; 1 + p - \mu] f(z)$, where the operator $\Omega^{\alpha_1}_p f$ is defined by [see Srivastava and Aouf (Srivastava & Aouf, 1992)];
\[ \Omega^{\alpha_1}_p f(z) = \frac{\Gamma(1 + p - \mu)}{\Gamma(1 + p)} a^{-\mu} D^{\mu}_z f(z) (0 \leq \mu < 1; p \in \mathbb{N}), \]
where $D^{\mu}_z$ is the fractional derivative operator.
(iv) $\theta^{\alpha_1}_p [v + p, 1; v + p + 1] f(z) = J_{v,p} f(z)$, where $J_{v,p}$ is the generalized Bernardi-Libera-Livingston-integral operator (see (Bernardi, 1996; Libera, 1969; Livingston, 1966));
(v) $\theta^{\alpha_1}_p [\lambda + p, a; c] f(z) = I^{a}_{p} f(z)$ ($a, c \in \mathbb{R}\setminus\mathbb{Z}_0; \lambda > -p$), where $I^{a}_{p} f(z)$ is the Cho-Kwon-Srivastava operator (Cho et al., 2004);

**Definition 1.1.** For the fixed parameters A and B, with $0 \leq B < 1, -1 \leq A < B$ and $0 \leq \lambda < p, p \in \mathbb{N}$ and for a analytic $p$-valent function of the form (1.1) we define the following subclasses:
\[ \mathcal{V}(\lambda, A, B) = \left\{ f \in \mathcal{A}(p) : \frac{1}{p - \lambda} \left| \frac{z}{\theta^{\alpha_1}_p (\alpha_1)} f(z) \right| < \frac{1 + Az}{1 + Bz} \right\}. \] (1.6)
and
\[ W(\lambda, A, B) = \left\{ f \in \mathcal{A}_k(p) : \frac{1}{p - \lambda} \left( 1 + \frac{z[\theta_p^{\alpha_1}(\alpha_1)f(z)]'}{[\theta_p^{\alpha_1}(\alpha_1)f(z)]'} - \lambda \right) < \frac{1 + Az}{1 + Bz} \right\}. \tag{1.7} \]

The subclass \( V(\lambda, A, B) \) was discussed by Aouf et al., (Aouf et al., 2010) for multivalent analytic functions with negative coefficients, also coefficients estimates, distortion theorem, the radii of \( p \)-valent starlikeness and \( p \)-valent convexity and modified Hadamard products were investigated. In (Murugusundaramoorthy & Aouf, 2013) Murugusundaramoorthy and Aouf obtained similar results for the meromorphic equivalent of the class \( W(\lambda, A, B) \). Sarkar et al., (Sarkar et al., 2013) presented certain inclusion and convolution results involving the operator \( \theta_p^{\alpha_1}(\alpha_1) \) for functions belonging to certain favoured classes of analytic \( p \)-valent functions.

Motivated by the aforementioned works, in the present study we obtain certain strict subordination relationship involving the subclasses \( V(\lambda, A, B) \) and \( W(\lambda, A, B) \). Some subordination properties involving the linear operator defined in (1.3) are also considered. An argument estimate result is also obtained.

2. Preliminaries

Let \( \mathcal{P}_m \) denote the class of function of the form
\[ f(z) = 1 + a_m z^m + a_{m+1} z^{m+1} + \ldots \tag{2.1} \]
that are analytic in the unit disc, \( \mathbb{U} \). In proving our main results, we need each of the following definitions and lemmas.

**Definition 2.1.** (Wilf, 1961)
A sequence \( \{b_n\}_{n=1}^{\infty} \) of complex numbers is said to be a subordination factor sequence if for each function \( f(z) = \sum_{k=0}^{\infty} a_k z^k, z \in \mathbb{U}, \) from the class of convex (univalent) functions in \( \mathbb{U} \), denoted by \( S^c \), we have
\[ \sum_{n=1}^{\infty} b_n a_n z^n < f(z) \quad (\text{where} \quad a_1 = 1). \]

**Lemma 2.1.** (Wilf, 1961) A sequence \( \{b_n\} \) is a subordinating factor sequence if and only if
\[ \text{Re} \left( 1 + 2 \sum_{n=1}^{\infty} b_n z^n \right) > 0, \ z \in \mathbb{U}. \tag{2.2} \]

**Lemma 2.2.** (Miller & Mocanu, 1981, 2000) Let the function \( h \) be analytic and convex (univalent) in \( \mathbb{U} \) with \( h(0) = 1 \). Suppose also that the function \( \phi \) given by (2.1). If
\[ \phi(z) + \frac{z\phi'(z)}{\gamma} < h(z) \quad (\text{Re} \gamma \geq 0, \ \gamma \in \mathbb{C}^*), \tag{2.3} \]
then
\[ \phi(z) < \psi(z) = \frac{\gamma}{m} z^{-\frac{n}{m}} \int_0^z t^{\frac{n}{m}-1} h(t) \, dt < h(z) \]

and \( \psi \) is the best dominant.

Lemma 2.3. (Nunokawa, 1993)
Let the function \( p \) be analytic in \( U \), such that \( p(0) = 1 \) and \( p(z) \neq 0 \) for all \( z \in U \). If there exists a point \( z_0 \in U \) such that
\[ |\arg p(z)| < \frac{\pi \delta}{2} \]
for \( |z| < |z_0| \)
and
\[ |\arg p(z_0)| = \frac{\pi \delta}{2} \quad (\delta > 0), \]
then we have
\[ \frac{z_0 p'(z_0)}{p(z_0)} = ik\delta, \]
where
\[ k \geq \frac{1}{2} \left( c + \frac{1}{c} \right), \quad \text{when} \quad \arg p(z_0) = \frac{\pi \delta}{2} \]
and
\[ k \leq -\frac{1}{2} \left( c + \frac{1}{c} \right), \quad \text{when} \quad \arg p(z_0) = -\frac{\pi \delta}{2}, \]
where
\[ p(z_0)^{1/\delta} = \pm ic, \quad \text{and} \quad c > 0. \]

Lemma 2.4. (Whittaker & Watson, 1927)
For the complex numbers \( a, b \) and \( c \), with \( c \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\} \), the following identities hold:
\[ \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} \, dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \, 2F_1(a,b;c;z), \quad z \in U, \quad (2.4) \]
\[ \text{for} \ Re c > Reb > 0, \quad (2.5) \]
\[ 2F_1(a,b;c;z) = (1-z)^{-a} \, 2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \quad z \in U, \quad (2.6) \]
and
\[ (b+1) \, 2F_1(1,b;b+1;z) = (b+1) + bz \, 2F_1(1,b+1;b+2;z), \quad z \in U. \quad (2.7) \]
3. Coefficient estimates and subordination results for the function classes \( \mathcal{W}(\lambda, A, B) \) and \( \mathcal{V}(\lambda, A, B) \)

Unless otherwise mentioned, we shall assume throughout the sequel that \( 0 \leq \lambda < p, p \in \mathbb{N} \) and \( 0 \leq B < 1 \). First, we will give sufficient conditions for a function to be in the classes \( \mathcal{W}(\lambda, A, B) \).

**Lemma 3.1.** A sufficient condition for an analytic \( p \)-valent function \( f \) of the form (1.1), to be in the class \( \mathcal{W}(\lambda, A, B) \) is

\[
\sum_{n=k}^{\infty} \gamma_{n,p}|a_n| \leq p(B - A)(p - \lambda) \tag{3.1}
\]

where

\[
\gamma_{n,p} = \Omega \tau_{n,p}(\alpha_1)n[(n - p)(1 + B) - (A - B)(p - \lambda)], \quad (n \geq k). \tag{3.2}
\]

**Proof.** An analytic \( p \)-valent function \( f \) of the form (1.1) belongs to the class \( \mathcal{W}(\lambda, A, B) \), if and only if there exists a *Schwarz function* \( w \), such that

\[
\frac{1}{p - \lambda} \left( 1 + \frac{z[\theta_p^{\alpha_1}(\alpha_1)f(z)]''}{[\theta_p^{\alpha_1}(\alpha_1)f(z)]'} - \lambda \right) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad z \in \mathbb{U}.
\]

Since \( |w(z)| \leq |z| \) for all \( z \in \mathbb{U} \), the above relation is equivalent to

\[
\left| \frac{[\theta_p^{\alpha_1}(\alpha_1)f(z)]' + z[\theta_p^{\alpha_1}(\alpha_1)f(z)]'' - p[\theta_p^{\alpha_1}(\alpha_1)f(z)]'}{([\theta_p^{\alpha_1}(\alpha_1)f(z)]' + z[\theta_p^{\alpha_1}(\alpha_1)f(z)]'' - p[\theta_p^{\alpha_1}(\alpha_1)f(z)]')B - (p - \lambda)(A - B)[\theta_p^{\alpha_1}(\alpha_1)f(z)]'} \right| < 1.
\]

Thus it is sufficient to show that

\[
\left| ([\theta_p^{\alpha_1}(\alpha_1)f(z)]' + z[\theta_p^{\alpha_1}(\alpha_1)f(z)]'' - p[\theta_p^{\alpha_1}(\alpha_1)f(z)]')B - (p - \lambda)(A - B)[\theta_p^{\alpha_1}(\alpha_1)f(z)]' \right| < 0, \quad z \in \mathbb{U}.
\]

Indeed, letting \( |z| = r \) \((0 < r < 1)\) and using (3.1), we have

\[
\left| ([\theta_p^{\alpha_1}(\alpha_1)f(z)]' + z[\theta_p^{\alpha_1}(\alpha_1)f(z)]'' - p[\theta_p^{\alpha_1}(\alpha_1)f(z)]') - ([\theta_p^{\alpha_1}(\alpha_1)f(z)]' + z[\theta_p^{\alpha_1}(\alpha_1)f(z)]'' - p[\theta_p^{\alpha_1}(\alpha_1)f(z)]')B - (p - \lambda)(A - B)[\theta_p^{\alpha_1}(\alpha_1)f(z)]' \right| \leq \sum_{n=k}^{\infty} n(n - p)\Omega \tau_{n,p}(\alpha_1)|a_n|r^n - (B - A)p(p - \lambda) \ r^{p-1}
\]

\[
+ \sum_{n=k}^{\infty} n(n - p)B - (A - B)(p - \lambda)|\Omega \tau_{n,p}(\alpha_1)|a_n|r^n = r^{p-1} \left( \sum_{n=k}^{\infty} \gamma_{n,p}|a_n|r^{p^n - p + 1} - (B - A)p(p - \lambda) \right) < 0.
\]

Hence \( f \in \mathcal{W}(\lambda, A, B) \). \( \square \)
Similarly, we have the following Lemma which gives sufficient condition for a function to be in the class $V(\lambda, A, B)$.

**Lemma 3.2.** A sufficient condition for an analytic $p$-valent function $f$ of the form (1.1), to be in the class $V(\lambda, A, B)$ is

$$
\sum_{n=k}^{\infty} \delta^*_n |a_n| \leq (B - A)(p - \lambda)
$$

(3.3)

where

$$
\delta^*_n = \Omega \sigma_{n,p}(\alpha_1) [(n - p)(1 + B) - (A - B)(p - \lambda)], \; (n \geq k).
$$

(3.4)

Our next result provides a sharp subordination result involving the functions of the class $W(\lambda, A, B)$.

**Theorem 3.1.** Let the sequence $\{\gamma_{n,p}\}_{n \in \mathbb{N}}$ defined in (3.2) be a nondecreasing sequence. If a function $f$ of the form (1.1) belong to the class $W(\lambda, A, B)$, and $g \in S^c$, then

$$
(\epsilon(z^{1-p}) * g)(z) \prec g(z),
$$

(3.5)

and

$$
\operatorname{Re}(z^{1-p} f(z)) > -\frac{1}{2\epsilon}, \; z \in \mathbb{U},
$$

(3.6)

whenever \( \epsilon = \frac{\gamma_{k,p}}{2[(B - A)p(p - \lambda)] + \gamma_{k,p}} \).

Moreover, if $(k - p)$ is even, then the number $\epsilon$ cannot be replaced by a larger number.

**Proof.** Supposing that the function $g \in S^c$ is of the form

$$
g(z) = \sum_{n=1}^{\infty} b_n z^n, \; z \in \mathbb{U} \quad \text{(where } b_1 = 1),
$$

then

$$
\sum_{n=1}^{\infty} d_n b_n z^n = (\epsilon(z^{1-p}) * g)(z) < g(z),
$$

where

$$
d_n = \begin{cases} 
\epsilon, & \text{if } n = 1, \\
0, & \text{if } 2 \leq n \leq k - p, \\
\epsilon a_{n+p-1}, & \text{if } n > k - p.
\end{cases}
$$
Now, using the Definition 2.1, the subordination result in (3.5) holds if \( \{d_n\} \) is a subordinating factor sequence. Since \( \{\gamma_n,p\}_{n \in \mathbb{N}} \) is a nondecreasing sequence we have,

\[
\Re \left( 1 + 2 \sum_{n=1}^{\infty} d_n z^n \right) = \Re \left( 1 + \frac{\gamma_k,p}{p(p - \lambda) (B - A) + \gamma_k,p} z^+ \right)
\]

(3.7)

\[
\sum_{n=k}^{\infty} \frac{\gamma_k,p}{p(p - \lambda) (B - A) + \gamma_k,p} a_n z^{-n-p} \geq 1 - \frac{\gamma_k,p}{p(p - \lambda) (B - A) + \gamma_k,p} r^{-r} \cdot \frac{r}{p(p - \lambda) (B - A) + \gamma_k,p} \sum_{n=k}^{\infty} \delta_n,p |a_n|, |z| = r < 1.
\]

Thus, by using Lemma 3.1 in (3.7) we obtain

\[
\Re \left( 1 + 2 \sum_{n=1}^{\infty} c_n z^n \right) \geq 1 - \frac{\gamma_k,p}{p(B - A)(p - \lambda) + \gamma_k,p} r^{-r} \cdot \frac{r}{p(B - A)(p - \lambda) + \gamma_k,p} (B - A) p(p - \lambda) > 0, z \in \mathbb{U},
\]

which proves the inequality (2.2), hence also the subordination result asserted by (3.5). The inequality (3.6) asserted by Theorem 3.1 would follow from (3.5) upon setting

\[
g(z) = \frac{z}{1 - z} = \sum_{n=1}^{\infty} z^n, z \in \mathbb{U}.
\]

We also observe that whenever the functions of the form

\[
f_{n,p}(z) = z^p + \frac{(B - A) p(p - \lambda)}{\gamma_n,p} z^n, z \in \mathbb{U} (n \geq k),
\]

belongs the class \( \mathcal{W}(\lambda, A, B) \) and if \((k - p)\) is a even number, then

\[
z^{1-p} f_{k,p}(z) \big|_{z=-1} = -\frac{1}{2 \epsilon}.
\]

and the constant \( \epsilon \) is the best estimate. \( \square \)

Using the same techniques as in the proof of Theorem 3.1, we have the following result.

**Theorem 3.2.** Let the sequence \( \{\delta_n,p\}_{n \in \mathbb{N}} \) defined by (3.4) be a nondecreasing sequence. If the function \( g \) of the form (1.1) belongs to the class \( \mathcal{V}(\lambda, A, B) \) and \( h \in \mathcal{S}^c \), then

\[
\left( \mu \left( z^{1-p} f \right) * h \right)(z) < h(z),
\]

(3.8)
\[ Re \left( z^{1-p} f(z) \right) > -\frac{1}{2\mu}; \quad z \in \mathbb{U}, \quad (3.9) \]

where
\[ \mu = \frac{\delta_{k,p}^*}{2 [(B-A)P - \lambda]} + \delta_{k,p}^*. \]

Moreover, if \((k - p)\) is even, then the number \(\mu\) cannot be replaced by a larger number.

4. Subordination Properties of the operator \(\theta^q_{\alpha^1}(P)\)

In this section we obtain certain subordination properties involving the operator \(\theta^q_{\alpha^1}(P)\).

**Theorem 4.1.** For \(f \in \mathcal{A}_k(P)\) let the operator \(Q\) be defined by
\[
Qf(z) := \left[ 1 - \frac{(\alpha_1 - pA_1)}{A_1} \theta^q_{\alpha^1}(P) f(z) \right] + \frac{\tau A_1}{A_1} \left[ \theta^q_{\alpha^1}(P + 1) f(z) \right], \quad (4.1)
\]
for \(A_1 \neq 0\) and \(\tau > 0\).

(i) If
\[
\frac{Q^{(j)} f(z)(p - j)!}{z^j p!} < (1 - \tau p) \frac{1 + Az}{1 + Bz} \quad (0 \leq j \leq p), \quad (4.2)
\]

, then
\[
\frac{\left[ \theta^q_{\alpha^1}(P) f(z)(p - j)! \right]^{(j)}}{z^j p!} < \tilde{g}(z) < \frac{1 + Az}{1 + Bz}, \quad (4.3)
\]

where for \(m\) positive, \(\tilde{g}\) is given by
\[
\tilde{g}(z) = \begin{cases} 
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} \sum_{m=0}^{\infty} \frac{\left(1 - \frac{A}{B}\right)(1 + Bz)^{-1}}{\left(1 - \tau + \tau(m + p)\right) \Gamma(1 + \frac{m}{\tau} + 1)} & \text{if } B \neq 0, \\
1 + \frac{Az(1 - \tau + \tau p)}{1 - \tau + \tau(m + p)} & \text{if } B = 0,
\end{cases}
\]
and \(\tilde{g}\) is the best dominant of \((4.3)\).

(ii)
\[
Re \left( \frac{Q^{(j)} f(z)}{z^p} \right) > \frac{p!}{(p - j)!} \sigma, \quad z \in \mathbb{U}, \quad (4.4)
\]

where
\[
\sigma = \begin{cases} 
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 - B)^{-1} \sum_{m=0}^{\infty} \frac{\left(1 - \frac{A}{B}\right)(1 - B)^{-1}}{\left(1 - \tau + \tau(p + m)\right) \Gamma(1 + \frac{m}{\tau} + 1)} & \text{if } B \neq 0, \\
1 - \frac{A(1 - \tau + \tau p)}{1 - \tau + \tau (p + m)} & \text{if } B = 0.
\end{cases}
\]

The inequality \((4.4)\) is the best possible.
Proof. From (1.5) and (4.1) we easily obtain
\[ Q^{(j)} f(z) = (1 - \tau + \tau j) \left[ \theta^{\alpha_1}(\alpha_1) f(z) \right]^{(j)} + \tau z \left[ \theta^{\alpha_1}(\alpha_1) f(z) \right]^{(j+1)}, \quad z \in \mathbb{U}. \] (4.5)

Letting
\[ g(z) := \frac{\left[ \theta^{\alpha_1}(\alpha_1) f(z) \right]^{(j)} (p - j)!}{z^{p-j} p!} \]
with \( f \in \mathcal{A}_k(p) \), then \( g \) is analytic in \( \mathbb{U} \) and has the form (2.1). Also, note that
\[ (1 - \tau + \tau p) \left[ g(z) + \frac{\tau}{1 - \tau + \tau p} zg'(z) \right] = \frac{Q^{(j)} f(z)(p - j)!}{z^{p-j} p!}. \] (4.6)

Then, by (4.2) we have
\[ g(z) + \frac{\tau}{1 - \tau + \tau p} zg'(z) < \frac{1 + Az}{1 + Bz}. \]

Now, by using Lemma 2.2 for \( \gamma = \frac{1 - \tau + \tau p}{\tau} \) and whenever \( \gamma > 0 \), by a changing of variables followed by the use of the identities (2.5), (2.6) and (2.7), we deduce that
\[
\left[ \theta^{\alpha_1}(\alpha_1) f(z) \right]^{(j)} (p - j)! \frac{1}{z^{p-j} p!} < \tilde{g}(z) = \int_0^\tau \frac{1}{z^{\tau m - 1}} \frac{1}{1 + Bt} \, dt
= \begin{cases} 
\frac{A}{B} + \left( 1 - \frac{A}{B} \right) (1 + Bz)^{-1} _2F_1 \left( 1, 1; \frac{1 - \tau + \tau p}{\tau m} + 1; \frac{Bz}{1 + Bz} \right), & \text{if } B \neq 0, \\
1 + \frac{A(1 - \tau + \tau p)}{1 - \tau + \tau(p + m)} z, & \text{if } B = 0,
\end{cases}
\]
which proves the assertion (4.3) of our Theorem.

Next, in order to prove the assertion (4.4), it suffices to show that
\[ \inf \{ \Re \tilde{g}(z) : z \in \mathbb{U} \} = \tilde{g}(-1). \] (4.7)

Indeed, for \( |z| \leq r < 1 \) we have
\[ \Re \frac{1 + Az}{1 + Bz} \geq \frac{1 - Ar}{1 - Br}, \]
and setting
\[ \chi(s, z) = \frac{1 + Az}{1 + Bz^2} \quad \text{and} \quad d\mu(s) = \frac{1 - \tau + \tau p}{\tau m} s^{\frac{1 - \tau + \tau p}{\tau m - 1}} \, ds \quad (0 \leq s \leq 1) \]
which is a positive measure on the closed interval \([0, 1]\) whenever \( \tau > 0 \), we get
\[ \tilde{g}(z) = \int_0^1 \chi(s, z) d\mu(s), \]
Let \( r \to 1^- \) in the above inequality we obtain the assertion (4.7) of our Theorem. The estimate in (4.4) is the best possible since the function \( \tilde{g} \) is the best dominant of (4.3).

Taking \( q = 2 \) and \( s = 1 \), for \( A_i = B_i = 1, \alpha_1 = 1, \alpha_2 = \beta_1 \) and \( A = 1 - \frac{2\alpha(p - j)!}{(1 - \tau + \tau p)p!} \) and \( B = -1 \) in Theorem 4.1 we get the following result:

**Corollary 4.1.** Let \( Qf(z) = (1 - \tau)f(z) + \tau zf'(z) \), where \( f \in \mathcal{A}_k(p) \). For \( \tau > 0 \)

\[
\text{Re} \left( \frac{Q^{(j)}f(z)(p - j)!}{z^{p-j}p!} \right) > \alpha, \ z \in U \quad (0 \leq \alpha < \frac{(1 - \tau + \tau p)p!}{(p - j)!}, \ 0 \leq j \leq p),
\]

implies that

\[
\text{Re} \left( \frac{f^{(j)}(z)}{z^{p-j}} \right) > \frac{\alpha}{1 - \tau + \tau p} + \left[ \frac{p!}{(p - j)!} \alpha + \frac{\alpha}{1 - \tau + \tau p} \frac{1}{2} \right] = \binom{1}{j} \left( \frac{1 - \tau + \tau p}{\tau m} + 1; \frac{1}{2} \right) - 1, \ z \in U.
\]

The above inequality is the best possible.

**Theorem 4.2.** For \( f \in \mathcal{A}_k(p) \) let the operator \( Q \) be given by (4.1), and let \( \tau > 0 \).

(i) If

\[
\text{Re} \left[ \frac{\theta^j_p(\alpha_1)f(z)}{z^{p-j}} \right] > \rho, \ z \in U \quad (\rho < \frac{p!}{(p - j)!}),
\]

then

\[
\text{Re} \left( \frac{Q^{(j)}f(z)}{z^{p-j}} \right) > \rho(1 - \tau + \tau p), \ |z| < R,
\]

where

\[
R = \left[ \sqrt{\frac{1 + \left( \frac{\tau m}{1 - \tau + \tau p} \right)^2}{1 - \tau + \tau p}} \right]^1 = \left( \frac{1}{m} \right).
\]

(ii) If

\[
\text{Re} \left[ \frac{\theta^j_p(\alpha_1)f(z)}{(-1)^jz^{p-j}} \right] < \rho, \ z \in U \quad (\rho > \frac{p!}{(p - j)!}),
\]

then

\[
\text{Re} \left( \frac{Q^{(j)}f(z)}{z^{p-j}} \right) < \rho(1 - \tau + \tau p), \ |z| < R.
\]

The bound \( R \) is the best possible.
Proof. (i) Defining the function $\Phi$ by

$$\frac{\theta_p^{\alpha_1}(f(z))^{(j)}}{z^{p-j}} = \rho + \left[ \frac{p!}{(p-j)!} - \rho \right] \Phi(z), \quad (4.9)$$

then $\Phi$ is an analytic function of the form (2.1) with positive real part in $U$. Differentiating (4.9) with respect to $z$ and using (4.5) we have

$$\frac{Q_j f(z)}{z^{p-j}} - \rho(1 - \tau + \tau p) = \left[ \frac{p!}{(p-j)!} - \rho \right] \left[ (1 - \tau + \tau p) \Phi(z) + \tau z \Phi'(z) \right]. \quad (4.10)$$

Now, by applying in (4.10) the following well-known estimate (MacGregor, 1963)

$$|z \Phi'(z)| \leq \frac{2mr^m}{1 - r^2}, \quad |z| = r < 1, \quad (4.11)$$

we have

$$\text{Re} \left[ \frac{Q_j f(z)}{z^{p-j}} - \rho(1 - \tau + \tau p) \right] \geq \text{Re} \Phi(z) \left[ \frac{p!}{(p-j)!} - \rho \right] \left[ (1 - \tau + \tau p) - \frac{2\tau mr^m}{1 - r^2} \right], \quad |z| = r < 1. \quad (4.12)$$

Now, it is easy to see that the right hand side of (4.12) is positive whenever $r < R$, where $R$ is given by (4.8). In order to show that the bound $R$ is the best possible, we consider the function $f \in A_k(p)$ defined by

$$\frac{\theta_p^{\alpha_1}(f(z))^{(j)}}{z^{p-j}} = \rho + \left[ \frac{p!}{(p-j)!} - \rho \right] \frac{1 + z^m}{1 - z^m}. \quad \text{(5.1)}$$

Then,

$$\frac{Q_j f(z)}{z^{p-j}} - \rho(1 - \tau + \tau p) = \frac{p!}{(p-j)! - \rho} \frac{1 + z^m}{1 - z^m} \left[ (1 - \tau + \tau p) (1 - z^m) + 2\tau m z^m \right] = 0,$$

for $z = R \exp^{\frac{i\pi}{m}}$, and the first part of the Theorem is proved. Similarly, we can prove part (ii) of the Theorem.

\(\square\)

5. An argument estimate

In this section we obtain an argument estimate involving the operator $\theta_p^{\alpha_1}(\alpha_1)$ and connected with the linear operator $Q$. 
Theorem 5.1. For \( f \in A_k(p) \), let the operator \( Q \) be defined by (4.1), and let \( 0 \leq \tau < \frac{1}{1 - p} \). If
\[
\left| \operatorname{arg} \frac{Q^j f(z)}{z^{p-j}} \right| < \frac{\pi \delta}{2}, \quad z \in \mathbb{U} \quad (\delta > 0, \ 0 \leq j \leq p),
\] (5.1)
then
\[
\left| \operatorname{arg} \left[ \frac{\theta^j_p(\alpha_1) f(z)}{z^{p-j}} \right] \right| < \frac{\pi \delta}{2}, \quad z \in \mathbb{U}.
\]

Proof. For \( f \in A_k(p) \), if we let
\[
q(z) := \left[ \frac{\theta^j_p(\alpha_1) f(z)}{z^{p-j}} \right] (p - j)! \frac{1}{p!},
\]
then \( q \) is of the form (2.1) and it is analytic in \( \mathbb{U} \). If there exists a point \( z_0 \in \mathbb{U} \) such that
\[
|\arg q(z_0)| < \frac{\pi \delta}{2}, \quad |z| < |z_0| \quad \text{and} \quad |\arg q(z_0)| = \frac{\pi \delta}{2} \quad (\delta > 0),
\]
then, according to Lemma 2.3 we have
\[
\frac{z_0 q'(z_0)}{q(z_0)} = ik \delta \quad \text{and} \quad q(z_0)^{1/\delta} = \pm ic \quad (c > 0).
\]
Also, from the equality (4.5) we get
\[
\frac{Q^j f(z_0)}{z_0^{p-j}} = \frac{p!}{(p - j)!} (1 - \tau + \tau p) q(z_0) \left[ 1 + \frac{\tau}{1 - \tau + \tau p} \frac{z_0 q'(z_0)}{q(z_0)} \right].
\]
If \( \arg q(z_0) = \frac{\pi \delta}{2} \), then
\[
\arg \frac{Q^j f(z_0)}{z_0^{p-j}} = \frac{\pi \delta}{2} + \arg \left( 1 + \frac{\tau}{1 - \tau + \tau p} ik \delta \right) = \frac{\pi \delta}{2} + \tan^{-1} \left( \frac{\tau}{1 - \tau + \tau p} \delta \right) \geq \frac{\pi \delta}{2},
\]
whenever \( k \geq \frac{1}{2} \left( c + \frac{1}{c} \right) \) and \( 0 \leq \tau < \frac{1}{1 - p} \), and this last inequality contradicts the assumption (5.1).

Similarly, if \( \arg q(z_0) = -\frac{\pi \delta}{2} \), then we obtain
\[
\arg \frac{Q^j f(z_0)}{z_0^{p-j}} \leq -\frac{\pi \delta}{2},
\]
which also contradicts the assumption (5.1).

Consequently, the function \( q \) need to satisfy the inequality \( |\arg q(z)| < \frac{\pi \delta}{2}, \quad z \in \mathbb{U} \), i.e. the conclusion of our theorem. \( \square \)
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