An Application for Certain Subclasses of $p$-Valent Meromorphic Functions Associated with the Generalized Hypergeometric Function

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Abstract

The main object of this paper is to give an application of a linear operator $H_{p,q,s}^{m,\mu}(\alpha)f(z)$ involving the generalized hypergeometric function. We define subclasses of the meromorphic function class $\Sigma_{p,m}$ by means of operator $H_{p,q,s}^{m,\mu}(\alpha)f(z)$.

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1. Introduction and definitions

Let $\Sigma_{p,m}$ denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \ldots\}),$$

(1.1)

which are analytic and $p$-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U\setminus\{0\}$. We also denote $\Sigma_{p,1-p} = \Sigma_p$.

A function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma S_p^*(\alpha)$ of meromorphically $p$-valent starlike functions of order $\alpha$ in $U$ if and only if

$$\Re \left( \frac{zf'(z)}{f(z)} \right) < -\alpha \quad (z \in U; 0 \leq \alpha < p).$$

(1.2)
Also a function $f \in \Sigma_{p,m}$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphically $p$-valent convex of order $\alpha$ in $U$ if and only if
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) < -\alpha \quad (z \in U; 0 \leq \alpha < p). \tag{1.3}
\]

It is easy to observe from (1.2) and (1.3) that
\[
f(z) \in \Sigma C_p(\alpha) \iff -\frac{zf'(z)}{p} \in \Sigma S_p^*(\alpha). \tag{1.4}
\]

For a function $f \in \Sigma_{p,m}$, we say that $f \in \Sigma K_p(\beta, \alpha)$ if there exists a function $g \in \Sigma S_p^*(\alpha)$ such that
\[
\Re \left( \frac{zf'(z)}{g(z)} \right) < -\beta \quad (z \in U; 0 \leq \alpha, \beta < p). \tag{1.5}
\]

Functions in the class $\Sigma K_p(\beta, \alpha)$ are called meromorphically $p$-valent close-to-convex functions of order $\beta$ and type $\alpha$. We also say that a function $f \in \Sigma_{p,m}$ is in the class $\Sigma K_p(\beta, \alpha)$ of meromorphically quasi-convex functions of order $\beta$ and type $\alpha$ if there exists a function $g \in \Sigma C_p(\alpha)$ such that
\[
\Re \left( \frac{(zf'(z))'}{g'(z)} \right) < -\beta \quad (z \in U; 0 \leq \alpha, \beta < p). \tag{1.6}
\]

It follows from (1.5) and (1.6) that
\[
f(z) \in \Sigma K_p^*(\beta, \alpha) \iff -\frac{zf'(z)}{p} \in \Sigma K_p(\beta, \alpha),
\]

where $\Sigma S_p^*(\alpha)$ and $\Sigma C_p(\alpha)$ are, respectively, the classes of meromorphically $p$-valent starlike functions of order $\alpha$ and meromorphically $p$-valent convex functions of order $\alpha$ ($0 \leq \alpha < p$)(see Aouf (Aouf, 2008) and Frasin (Frasin, 2012)).

For a function $f(z) \in \Sigma_{p,m}$, given by (1.1) and $g(z) \in \Sigma_{p,m}$ defined by
\[
g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k,
\]
we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by
\[
f(z) \ast g(z) = (f \ast g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k = (g \ast f)(z) \quad (p \in \mathbb{N}).
\]

For real or complex numbers $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \notin \mathbb{Z}^-_{0} = \{0, -1, -2, \ldots\}$; $j = 1, 2, \ldots, s$),
we consider the generalized hypergeometric function $\phi_{p,q}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, (Kiryakova, 2011, p.19))

$$\phi_{p,q}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k} \cdots (\alpha_q)_{k}}{(\beta_1)_{k} \cdots (\beta_s)_{k}} \frac{z^k}{k!}$$

where $\phi_{p,q}$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by

$$(\theta)_v = \frac{\Gamma(\theta + v)}{\Gamma(\theta)} = \begin{cases} 1 & (v = 0; \theta \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*), \\ \theta(\theta - 1) \cdots (\theta + v - 1) & (v \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases}$$

Corresponding to the function $\phi_{p}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ given by

$$\phi_{p}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-p} \phi_{p,q}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

we introduce a function $\phi_{p,q}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ defined by

$$\phi_{p}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * \phi_{p,q}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \frac{1}{z^{p}(1-z)^{\mu-p}}$$

$$(\mu > -p; z \in U^*).$$

We now define a linear operator $H_{p,q,s}^{\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) : \Sigma_{p,m} \rightarrow \Sigma_{p,m}$ by

$$H_{p,q,s}^{\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = \phi_{p,q}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z) \quad (1.7)$$

$$(\alpha_i, \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^*; i = 1, ..., q, j = 1, ..., s; \mu > -p, f \in \Sigma_{p,m}; z \in U^*).$$

For convenience, we write

$$H_{p,q,s}^{\mu}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) = H_{p,q,s}^{\mu}(\alpha_1)$$

and

$$H_{p,q,s}^{1-p,\mu}(\alpha_1) = H_{p,q,s}^{\mu}(\alpha_1) \quad (\mu > -p).$$

If $f(z)$ is given by (1.1), then from (1.7), we deduce that

$$H_{p,q,s}^{\mu}(\alpha_1) f(z) = z^{-p} + \sum_{k=m}^{\infty} \frac{(\mu + p)_{p+k}(\beta_1)_{p+k} \cdots (\beta_s)_{p+k}}{(\alpha_1)_{p+k} \cdots (\alpha_q)_{p+k}} a_k z^k \quad (\mu > -p; z \in U^*).$$

(1.8)

It is easily follows from (1.8) that

$$z \left( H_{p,q,s}^{\mu}(\alpha_1) f(z) \right)' = (\mu + p) H_{p,q,s}^{\mu+1}(\alpha_1) f(z) - (\mu + 2p) H_{p,q,s}^{\mu}(\alpha_1) f(z).$$

(1.9)
From the identity (1.9), we readily have
\[
z (H_{p,q,s}^m(\alpha_1)f(z))' = (\mu + p - 1)H_{p,q,s}^m(\alpha_1)f(z) - (\mu + 2p - 1)H_{p,q,s}^{m+1}(\alpha_1)f(z) \tag{1.10}
\]
and
\[
z (H_{p,q,s}^{m+1}(\alpha_1)f(z))' = (\mu + p + 1)H_{p,q,s}^{m+2}(\alpha_1)f(z) - (\mu + 2p + 1)H_{p,q,s}^{m+2}(\alpha_1)f(z). \tag{1.11}
\]
The linear operator $H_{p,q,s}^m(\alpha_1)$ was introduced by Patel and Palit (Patel & Palit, 2009).

We note that the linear operator $H_{p,q,s}^m(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator (Choi et al., 2002) for analytic functions and is essentially motivated by the operators defined and studied in (Cho & Noor, 2006) (see also, (Dziok & Srivastava, 1999), (Dziok & Srivastava, 2003), (Srivastava, 2007) and (Srivastava & Karlsson, 1985)).

Specializing the parameters $\mu, \alpha_i (i = 1, 2, ... q), \beta_j (j = 1, 2, ... s), q$ and $s$ we obtain the following:

(i) $H_{p,2,1}^{m,0}(p, p; p)f(z) = H_{p,2,1}^{m,1}(p, p; p)f(z) = f(z);
(ii) H_{p,2,1}^{m,1}(p, p; p)f(z) = \frac{2pf(z) + f'(z)}{p};
(iii) H_{p,2,1}^{m,2}(p, p; p)f(z) = \frac{2p+1f(z) + f'(z)}{p+1};
(iv) H_{p,1,1}^{m,1}(c + 1, 1; c)f(z) = J_{c,p}(f)(z) = \frac{c}{z+p} \int_0^z t^{c+p-1}f(t)dt \ (c > 0; z \in U^*), \text{ this integral operator is defined by}
\[
J_{c,p}(f)(z) = \frac{c}{z+p} \int_0^z t^{c+p-1}f(t)dt \ (c > 0; f \in \Sigma_{p,m}),
\]
(v) $H_{p,2,1}^{m,0}(p + 1, p; p)f(z) = \frac{p}{z^p} \int_0^z t^{2p-1}f(t)dt; \ (p \in \mathbb{N}; z \in U^*);
(vi) H_{p,2,1}^{1-p,n}(a, 1; a)f(z) = D^{p-n+1}f(z) (n > p), \text{ the operator } D^{n+p-1} \text{ studied by Ganigi and Uralegaddi (Ganigi & Uralegaddi, 1989), Yang (Yang, 1995), Aouf (Aouf, 1993), Aouf and Srivastava (Aouf & Srivastava, 1997) and Uralegaddi and Patil (Uralegaddi & Patil, 1989);}
(vii) $H_{p,2,1}^{m,1}(c, p + \mu; a)f(z) = L_p(a, c)f(z) \ (a, c \in \mathbb{R} \setminus \mathbb{Z}^*_0; \mu > -p)$ (see Liu (Liu, 2002));
(viii) $H_{1,2,1}^{m,1}(\mu + 1, n + 1; \mu)f(z) = I_{p, \mu}f(z) \ (\mu > 0; n > -1)$ (see Yuan et al. (Yuan et al., 2008)).

We also observe that, for $m = 0, p = 1$ replacing $\mu$ by $\mu - 1$, we have the operator $H_{p,q,s}^0(\alpha_1)f(z) = H_{p,q,s}^0(\alpha_1)f(z)$ defined by Cho and Kim (Cho & Kim, 2007).

The object of the present paper is to investigate some properties of meromorphic $p$-valent functions by the above operator $H_{p,q,s}^{m,0}(\alpha_1)f(z)$ given by (1.8).

**Definition 1.1.** Let $\mathcal{H}$ the set of complex valued functions $h(r, s, t) : \mathbb{C}^3 \to \mathbb{C}$ such that $h(r, s, t)$ is continuous in a domain $D \subset \mathbb{C}^3$;

$$(1, 1, 1) \in D \quad \text{and} \quad |h(1, 1, 1)| < 1;$$
Lemma 2.1. Let \( w(\mu, M, \delta, \lambda) \) (Miller & Mocanu, 1978).

2. The Main Result

and let \( h(r, s, t) \in \mathcal{H} \) and \( f \in \Sigma_{p,m} \) satisfies

\[
\begin{align*}
|h\left( e^{i\theta}, \frac{1}{\mu + p} + \frac{\mu + p - 1}{\mu + p} e^{i\theta} + \frac{1}{\mu + p} \delta, \frac{2}{\mu + p + 1} + \frac{\mu + p - 1}{\mu + p + 1} e^{i\theta} + \frac{1}{\mu + p + 1} \delta \right) &\geq 1 \\
\frac{1}{\mu + p + 1} \delta + \frac{(\mu + p - 1)\delta e^{i\theta} + [\delta + \beta - \delta^2]}{(\mu + p + 1) + (\mu + p - 1)(\mu + p + 1)\delta} &\in D
\end{align*}
\]

whenever

\[
\begin{align*}
\left( e^{i\theta}, \frac{1}{\mu + p} + \frac{\mu + p - 1}{\mu + p} e^{i\theta} + \frac{1}{\mu + p} \delta, \frac{2}{\mu + p + 1} + \frac{\mu + p - 1}{\mu + p + 1} e^{i\theta} + \frac{1}{\mu + p + 1} \delta \right) &\in D \\
\end{align*}
\]

with \( \Re(\beta \geq \delta(\delta - 1)) \) for real \( \theta, \delta \geq 1 \) and \( \lambda > 0 \).

2. The Main Result

In order to prove our main result, we recall the following lemma due to Miller and Mocanu (Miller & Mocanu, 1978).

Lemma 2.1. Let \( w(z) = a + w_n z^n + \ldots \) be analytic in \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \) with \( w(z) \neq a \) and \( n \geq 1 \). If \( z_0 = r_0 e^{i\theta} \) (\( 0 < r_0 < 1 \)) and \( |w(z_0)| = \max_{|z| \leq r_0} |w(z)| \). Then

\[ zw'(z_0) = \delta w(z_0) \tag{2.1} \]

and

\[ \Re\left( 1 + \frac{z_0 w''(z_0)}{w'(z_0)} \right) \geq \delta, \tag{2.2} \]

where \( \delta \) is a real number and

\[ \delta \geq n \frac{|w(z_0) - a|^2}{|w(z_0) - a|^2} \geq n \frac{|w(z_0)| - |a|}{|w(z_0) + |a|}. \]

Theorem 2.1. Let \( h(r, s, t) \in \mathcal{H} \) and let \( f \in \Sigma_{p,m} \) satisfies

\[
\begin{align*}
\begin{pmatrix}
H_{p,q,s}^{m,d}(\alpha_1)f(z) & H_{p,q,s}^{m,d+1}(\alpha_1)f(z) & H_{p,q,s}^{m,d+2}(\alpha_1)f(z) \\
H_{p,q,s}^{m,d-1}(\alpha_1)f(z) & H_{p,q,s}^{m,d}(\alpha_1)f(z) & H_{p,q,s}^{m,d+1}(\alpha_1)f(z)
\end{pmatrix} &\in D \subset \mathbb{C}^3
\end{align*}
\]

(2.3)
Using the identities (1.6) and (1.10), we have

\[ \left| h \left( \frac{H_{m,q,s}^m(\alpha_1)f(z)}{H_{m,q,s}^{m-1}(\alpha_1)f(z)}, \frac{H_{m,q,s}^{m+1}(\alpha_1)f(z)}{H_{m,q,s}^{m+1}(\alpha_1)f(z)} \right) \right| < 1 \quad (2.5) \]

for all \( z \in U \) and for some \( m \in \mathbb{N} \). Then we have

\[ \left| \frac{H_{m,q,s}^m(\alpha_1)f(z)}{H_{m,q,s}^{m-1}(\alpha_1)f(z)} \right| < 1 \quad (z \in U; \mu > -p, 0 \leqslant \alpha < p; p \in \mathbb{N}) \]

**Proof.** Let

\[ \frac{H_{m,q,s}^m(\alpha_1)f(z)}{H_{m,q,s}^{m-1}(\alpha_1)f(z)} = w(z). \quad (2.5) \]

Then it follows that \( w(z) \) is either analytic or meromorphic in \( U \), \( w(0) = 1 \) and \( w(z) \neq 1 \). Differentiating (2.5) logarithmically and multiply by \( z \), we obtain

\[ \frac{z(H_{m,q,s}^m(\alpha_1)f(z))'}{H_{m,q,s}^m(\alpha_1)f(z)} - \frac{z(H_{m,q,s}^{m-1}(\alpha_1)f(z))'}{H_{m,q,s}^{m-1}(\alpha_1)f(z)} = \frac{zw'(z)}{w(z)}. \]

Using the identities (1.6) and (1.10), we have

\[ \frac{H_{m,q,s}^{m+1}(\alpha_1)f(z)}{H_{m,q,s}^m(\alpha_1)f(z)} = \frac{1}{\mu + p} + \frac{\mu + p - 1}{\mu + p} w(z) + \frac{1}{\mu + p} zw'(z). \quad (2.6) \]

Differentiating (2.6) logarithmically and multiply by \( z \), we obtain

\[ \frac{z(H_{m,q,s}^{m+1}(\alpha_1)f(z))'}{H_{m,q,s}^{m+1}(\alpha_1)f(z)} - \frac{z(H_{m,q,s}^m(\alpha_1)f(z))'}{H_{m,q,s}^m(\alpha_1)f(z)} = \frac{1}{\mu + p} + \frac{\mu + p - 1}{\mu + p} w(z) + \frac{1}{\mu + p} zw'(z) \]

\[ = \frac{(\mu + p - 1)zw'(z) + \left[ \frac{zw'(z)}{w(z)} + \frac{zw''(z)}{w(z)} - \left( \frac{zw'(z)}{w(z)} \right)^2 \right]}{1 + (\mu + p - 1)w(z) + \frac{zw'(z)}{w(z)}}. \quad (2.7) \]
Further, an application of (2.2) in Lemma 2.1 given

Using the identities (1.9) and (1.11), we have

\[
\begin{align*}
\left(\mu + p + 1\right) \frac{H_{p,q,s}^{m+2}(\alpha_1) f(z)}{H_{p,q,s}^{m+1}(\alpha_1) f(z)} &= 1 + (\mu + p) \frac{H_{p,q,s}^{m+2}(\alpha_1) f(z)}{H_{p,q,s}^{m+1}(\alpha_1) f(z)} + \\
&= 1 + \left[ 1 + (\mu + p - 1)w(z) + \frac{z\theta'(z)}{w(z)} \right] + \\
&= 1 + (\mu + p - 1)z\theta'(z) + \left[ \frac{z\theta'(z)}{w(z)} + \frac{z^2 \theta''(z)}{w(z)} - \left(\frac{z\theta'(z)}{w(z)}\right)^2 \right] + \\
&= 1 + (\mu + p - 1)w(z) + \frac{z\theta'(z)}{w(z)}.
\end{align*}
\]

We claim that \(|w(z)| < 1\) for \(z \in U\). Otherwise there exists a point \(z_0 \in U\) such that \(\max_{|z| \leq r} |w(z)| = |w(z_0)| = 1\). Letting \(w(z_0) = e^{\mu\theta}\) and using Lemma 2.1 with \(a = 1\) and \(n = 1\), we have

\[
\begin{align*}
\frac{H_{p,q,s}^{m+1}(\alpha_1) f(z)}{H_{p,q,s}^{m}(\alpha_1) f(z)} &= e^{\mu\theta}, \\
\frac{H_{p,q,s}^{m+2}(\alpha_1) f(z)}{H_{p,q,s}^{m+1}(\alpha_1) f(z)} &= \frac{1}{\mu + p} + \frac{\mu + p - 1}{\mu + p} e^{\mu\theta} + \frac{1}{\mu + p} \delta, \\
\frac{2}{(\mu + p + 1)} + \frac{(\mu + p - 1)}{(\mu + p + 1)} e^{\mu\theta} + \frac{1}{(\mu + p + 1)} \delta \\
&+ \frac{(\mu + p - 1)\delta e^{\mu\theta} + [\delta + \beta - \delta^2]}{(\mu + p + 1) + (\mu + p + 1)(\mu + p - 1)e^{\mu\theta} + (\mu + p + 1)\delta},
\end{align*}
\]

where

\[
\beta = \frac{z^2 \theta''(z)}{w(z)} \quad \text{and} \quad \delta \geq 1.
\]

Further, an application of (2.2) in Lemma 2.1 given \(\Re(\beta \geq \delta)(\delta - 1)\). Since \(h(r, s, t) \in \mathcal{H}\), we have

\[
\begin{align*}
\left| h\left( \frac{H_{p,q,s}^{m}(\alpha_1) f(z)}{H_{p,q,s}^{m+1}(\alpha_1) f(z)} \frac{H_{p,q,s}^{m+1}(\alpha_1) f(z)}{H_{p,q,s}^{m+2}(\alpha_1) f(z)} \right) \right| \\
&= \left| h\left( e^{\mu\theta}, \frac{1}{\mu + p} + \frac{\mu + p - 1}{\mu + p} \right) + \frac{1}{\mu + p} \delta, \frac{2}{(\mu + p + 1)} + \frac{(\mu + p - 1)\delta e^{\mu\theta} + [\delta + \beta - \delta^2]}{(\mu + p + 1) + (\mu + p + 1)(\mu + p - 1)e^{\mu\theta} + (\mu + p + 1)\delta} \right) \right| \\
&\geq 1
\end{align*}
\]
which contradicts the condition (2.4) of Theorem 2.1. Therefore, we conclude that

\[
\left| \frac{H_{m,\mu}^{n}(\alpha_1)f(z)}{H_{m,\mu}^{n-1}(\alpha_1)f(z)} \right| < 1 \quad (z \in U).
\]

The proof is complete. \qed

Letting \( \mu = 1, q = 2, s = 1, \alpha_1 = p + 1, \alpha_2 = p \) and \( \beta_1 = p \) in Theorem 2.1, we have the following result.

**Corollary 2.1.** Let \( h(r,s,t) \in \mathcal{H} \) and let \( f(z) \in \Sigma_{p,m} \) satisfies

\[
\left( \frac{2pf(z) + zf'(z)}{pf(z)} \right) \frac{p \left[ (2p + 1)f(z) + zf'(z) \right]}{(p + 1) \left[ 2pf(z) + zf'(z) \right]},
\]

\[
\frac{(2p + 2)(2p + 1)f(z) + 4(p + 1)zf'(z) + 2f''(z)}{(2p + 2)(2p + 1)f(z) + zf'(z)} \in \mathbb{D} \subset \mathbb{C}^3
\]

and

\[
\left| h \left( \frac{2pf(z) + zf'(z)}{pf(z)} \right) \frac{p \left[ (2p + 1)f(z) + zf'(z) \right]}{(p + 1) \left[ 2pf(z) + zf'(z) \right]} \right| < 1
\]

for all \( z \in U \). Then we have

\[
\left| \frac{2pf(z) + zf'(z)}{pf(z)} \right| < 1 \quad (z \in U).
\]

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**References**


