COMPUTING WITH WORDS AND PERCEPTIONS (CWP) – A SHIFT IN DIRECTION IN COMPUTING AND DECISION ANALYSIS

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Abstract

In computing with words and perceptions, or CWP for short, the objects of computation are words, propositions and perceptions described in a natural language. In science, there is a deep-seated tradition of striving for progression from perceptions to measurements, and from the use of words to the use of numbers. Reflecting the bounded ability of sensory organs and, ultimately, the brain, to resolve detail, perceptions are intrinsically imprecise. Perceptions are f-granular in the sense that (a) the perceived values of attributes are fuzzy; and (b) the perceived values of attributes are granular, with a granule being a clump of values drawn together by indistinguishability, similarity, proximity or functionality.

F-granularity of perceptions is the reason why in the enormous literature on perceptions one cannot find a theory in which perceptions are objects of computation, as they are in CWP.

PNL (precisiated natural language) associates with a natural language, NL, a precisiation language, GCL (Generalized Constraint Language), which consists of generalized constraints and their combinations and qualifications.

The principal function of PNL is to serve as a system for computation and reasoning with perceptions. The need for redefinition arises because standard bivalent – classic-based definitions may lead to counterintuitive conclusions.

Computing with words and perceptions provides a basis for an important generalization of probability theory, namely, perception-based probability theory (PTp).

The importance of CWP derives from the fact that it opens the door to adding to any measurement-based theory.

Keywords: fuzzy, CWP (computing with words), PNL (precisiated natural language), probability theory.
COMPUTING WITH WORDS AND PERCEPTIONS (CWP)—A SHIFT IN DIRECTION IN COMPUTING AND DECISION ANALYSIS

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WHAT IS CWP?

THE BALLS-IN-BOX PROBLEM

Version 1. Measurement-based

- a box contains 20 black and white balls
- over 70% are black
- there are three times as many black balls as white balls

- what is the number of white balls?
- what is the probability that a ball drawn at random is white?
CONTINUED

Version 2. Perception-based

- a box contains about 20 black and white balls
- most are black
- there are several times as many black balls as white balls

- what is the number of white balls?
- what is the probability that a ball drawn at random is white?

CONTINUED

Version 3. Perception-based

- a box contains about 20 black balls of various sizes
- most are large
- there are several times as many large balls as small balls

- what is the number of small balls?
- what is the probability that a ball drawn at random is small?
**MEASUREMENT-BASED**
- A box contains 20 black and white balls
- Over seventy percent are black
- There are three times as many black balls as white balls
- What is the number of white balls?
- What is the probability that a ball picked at random is white?

**PERCEPTION-BASED** (version 1)
- A box contains about 20 black and white balls
- Most are black
- There are several times as many black balls as white balls
- What is the number of white balls?
- What is the probability that a ball drawn at random is white?

**COMPUTATION** (version 1)
- **Measurement-based**
  \[ X = \text{number of black balls} \]
  \[ Y = \text{number of white balls} \]
  \[ X \geq 0.7 \times 20 = 14 \]
  \[ X + Y = 20 \]
  \[ X = 3Y \]
  \[ X = 15 \quad ; \quad Y = 5 \]
  \[ P = \frac{5}{20} = .25 \]

- **Perception-based**
  \[ X = \text{number of black balls} \]
  \[ Y = \text{number of white balls} \]
  \[ X = \text{most } \times 20^* \]
  \[ X = \text{several } ^* Y \]
  \[ X + Y = 20^* \]
  \[ P = \frac{Y}{N} \]
BASIC PERCEPTIONS

attributes of physical objects
- distance
- time
- speed
- direction
- length
- width
- area
- volume
- weight
- height
- size
- temperature

sensations and emotions
- color
- smell
- pain
- hunger
- thirst
- cold
- joy
- anger
- fear

concepts
- count
- similarity
- cluster
- causality
- relevance
- risk
- truth
- likelihood
- possibility

DEEP STRUCTURE OF PERCEPTIONS

- perception of likelihood
- perception of truth (compatibility)
- perception of possibility (ease of attainment or realization)
- perception of similarity
- perception of count (absolute or relative)
- perception of causality

subjective probability = quantification of perception of likelihood
MEASUREMENT-BASED VS. PERCEPTION-BASED INFORMATION

**INFORMATION**

- **measurement-based**
  - numerical
    - it is 35°C
    - Eva is 28
    - probability is 0.8
    - 
    - 
    - 
  - perception-based
    - linguistic
      - it is very warm
      - Eva is young
      - probability is high
      - it is cloudy
      - traffic is heavy
      - it is hard to find parking near the campus

measurement-based information may be viewed as special case of perception-based information

MEASUREMENT-BASED VS. PERCEPTION-BASED CONCEPTS

<table>
<thead>
<tr>
<th>measurement-based</th>
<th>perception-based</th>
</tr>
</thead>
<tbody>
<tr>
<td>expected value</td>
<td>usual value</td>
</tr>
<tr>
<td>stationarity</td>
<td>regularity</td>
</tr>
<tr>
<td>continuous</td>
<td>smooth</td>
</tr>
</tbody>
</table>

Example of a regular process

\[ T = (t_0, t_1, t_2, \ldots) \]

\[ t_i = \text{travel time from home to office on day } i. \]
PERCEPTION OF MATHEMATICAL CONCEPTS: PERCEPTION OF FUNCTION

TEST PROBLEM

- A function, \( Y = f(X) \), is defined by its fuzzy graph expressed as

\[
\begin{align*}
  f_1 & : \text{if } X \text{ is small then } Y \text{ is small} \\
  & \quad \text{if } X \text{ is medium then } Y \text{ is large} \\
  & \quad \text{if } X \text{ is large then } Y \text{ is small}
\end{align*}
\]

(a) what is the value of \( Y \) if \( X \) is not large?

(b) what is the maximum value of \( Y \)
Computing with words and perceptions, or CWP for short, is a mode of computing in which the objects of computation are words, propositions and perceptions described in a natural language.
CONTINUED

- Perceptions play a key role in human cognition. Humans—but not machines—have a remarkable capability to perform a wide variety of physical and mental tasks without any measurements and any computations. Everyday examples of such tasks are driving a car in city traffic, playing tennis and summarizing a book.

COMPUTING WITH WORDS AND PERCEPTIONS (CWP)

Key points

- In computing with words and perceptions, the objects of computation are words, propositions, and perceptions described in a natural language

- A natural language is a system for describing perceptions

- In CWP, a perception is equated to its description in a natural language
CONTINUED

- in science, it is a deep-seated tradition to strive for the ultimate in rigor and precision

- words are less precise than numbers

- why and where, then, would words be used in preference to numbers?

CONTINUED

- when the available information is not precise enough to justify the use of numbers

- when precision carries a cost and there is a tolerance for imprecision which can be exploited to achieve tractability, robustness and reduced cost

- when the expressive power of words is greater than the expressive power of numbers
CONTINUED

• One of the major aims of CWP is to serve as a basis for equipping machines with a capability to operate on perception-based information. A key idea in CWP is that of dealing with perceptions through their descriptions in a natural language. In this way, computing and reasoning with perceptions is reduced to operating on propositions drawn from a natural language.

CONTINUED

• In CWP, what is employed for this purpose is PNL (Precisiated Natural Language.) In PNL, a proposition, \( p \), drawn from a natural language, NL, is represented as a generalized constraint, with the language of generalized constraints, GCL, serving as a precisiation language for computation and reasoning, PNL is equipped with two dictionaries and a modular multiagent deduction database. The rules of deduction are expressed in what is referred to as the Protoform Language (PFL).
KEY POINTS

- Decisions are based on information
- In most realistic settings, decision-relevant information is a mixture of measurements and perceptions
- Examples: buying a house; buying a stock
- Existing methods of decision analysis are measurement-based and do not provide effective tools for dealing with perception-based information
- A decision is strongly influenced by the perception of likelihoods of outcomes of a choice of action

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KEY POINTS

- In most realistic settings:
  a) The outcomes of a decision cannot be predicted with certainty
  b) Decision-relevant probability distributions are $f$-granular
  c) Decision-relevant events, functions and relations are $f$-granular
- Perception-based probability theory, $PT_p$, is basically a calculus of $f$-granular probability distributions, $f$-granular events, $f$-granular functions, $f$-granular relations and $f$-granular counts
**OBSERVATION**

- machines are driven by measurements
- humans are driven by perceptions
- to enable a machine to mimic the remarkable human capability to perform a wide variety of physical and mental tasks using perception-based information, it is necessary to have a means of converting measurements into perceptions

**BASIC PERCEPTIONS / F-GRANULARITY**

- temperature: warm + cold + very warm + much warmer + ...
- time: soon + about one hour + not much later + ...
- distance: near + far + much farther + ...
- speed: fast + slow + much faster + ...
- length: long + short + very long + ...

![Diagram of fuzzy set for size with labels: small, medium, large](image)
CONTINUED

- similarity: low + medium + high +...
- possibility: low + medium + high + almost impossible +...

- likelihood: likely + unlikely + very likely +...
- truth (compatibility): true + quite true + very untrue +...
- count: many + few + most + about 5 (5*) +...

subjective probability = perception of likelihood

CONTINUED

- function: if X is small then Y is large +...
  (X is small, Y is large)
- probability distribution: low \ small + low \ medium + high \ large +...
- Count \ attribute value distribution: 5* \ small + 8* \ large +...

PRINCIPAL RATIONALES FOR F-GRANULATION

- detail not known
- detail not needed
- detail not wanted
**Granular Computing**

**Generalized Valuation**

valuation = assignment of a value to a variable

- $X = 5$
- $0 \leq X \leq 5$
- $X$ is small
- $X$ is $R$
- $X$ is generalized
- Singular value
  - Measurement-based
- Granular values
  - Perception-based

**F-Generalization**

- $f$-generalization of a theory, $T$, involves an introduction into $T$ of the concept of a fuzzy set
- $f$-generalization of $PT$, $PT^*$, adds to $PT$ the capability to deal with fuzzy probabilities, fuzzy probability distributions, fuzzy events, fuzzy functions and fuzzy relations
F.G-GENERALIZATION

- f.g-generalization of $T$, $T^{++}$, involves an introduction into $T$ of the concept of a granulated fuzzy set
- f.g-generalization of $PT$, $PT^{++}$, adds to $PT^{++}$ the capability to deal with f-granular probabilities, f-granular probability distributions, f-granular events, f-granular functions and f-granular relations

**EXAMPLES OF F-GRANULATION (LINGUISTIC VARIABLES)**

- color: red, blue, green, yellow, ...
- age: young, middle-aged, old, very old
- size: small, big, very big, ...
- distance: near, far, very, not very far, ...

- humans have a remarkable capability to perform a wide variety of physical and mental tasks, e.g., driving a car in city traffic, without any measurements and any computations
- one of the principal aims of CTP is to develop a better understanding of how this capability can be added to machines
WHAT IS PRECISIATED NATURAL LANGUAGE (PNL)?

PRELIMINARIES

• A proposition, $p$, in a natural language, NL, is precisiable if it translatable into a precisiation language.

• In the case of PNL, the precisiation language is the Generalized Constraint Language, GCL.

• Precisiation of $p$, $p^*$, is an element of GCL (GC-form).
WHAT IS PNL?

- PNL is a sublanguage of precisiable propositions in NL which is equipped with two dictionaries: (1) NL to GCL; (2) GCL to PFL (Protoform Language); and (3) a modular multiagent database of rules of deduction (rules of generalized constrained propagation) expressed in PFL.

GENERALIZED CONSTRAINT

- standard constraint: \( X \in C \)
- generalized constraint: \( X \text{ isr } R \)

\[ X \text{ isr } R \]

- \( X = (X_1, \ldots, X_n) \)
- \( X \) may have a structure: \( X = \text{Location(Residence(Carol))} \)
- \( X \) may be a function of another variable: \( X = f(Y) \)
- \( X \) may be conditioned: \( (X/Y) \)

\( r := \{=, \leq, \geq, \neq, /, \text{blank}, \text{p}, \text{u}, rs, \text{fg}, ps, \ldots\} \)
GC-FORM (GENERALIZED CONSTRAINT FORM OF TYPE r)

\[ X \text{ isr} R \]

- \( r: e \)  equality constraint; \( X = R \) is abbreviation of \( X \text{is} R \)
- \( r: \leq \)  inequality constraint; \( X \leq R \)
- \( r: \subset \)  subsethood constraint; \( X \subset R \)
- \( r: \text{blank} \)  possibilistic constraint; \( X \text{is} R; \) \( R \) is the possibility distribution of \( X \)
- \( r: v \)  veristic constraint; \( X \text{isv} R; \) \( R \) is the verity distribution of \( X \)
- \( r: p \)  probabilistic constraint; \( X \text{isp} R; \) \( R \) is the probability distribution of \( X \)

CONTINUED

- \( r: rs \)  random set constraint; \( X \text{isrs} R; \) \( R \) is the set-valued probability distribution of \( X \)
- \( r: fg \)  fuzzy graph constraint; \( X \text{isfg} R; \) \( X \) is a function and \( R \) is its fuzzy graph
- \( r: u \)  usuality constraint; \( X \text{isu} R \) means usually \( (X \text{is} R) \)
- \( r: ps \)  Pawlak set constraint; \( X \text{isp} \{X, \overline{X}\} \text{ means that } X \text{is a set and } X \text{ and } \overline{X} \text{ are the lower and upper approximations to } X \)

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GENERALIZED CONSTRAINT LANGUAGE (GCL)

- GCL is generated by combination, qualification and propagation of generalized constraints
- In GCL, rules of deduction are the rules governing generalized constraint propagation
- Examples of elements of GCL
  - \((X \text{isp } R \text{ and } (X,Y) \text{is } S)\)
  - \((X \text{isr } R \text{ is unlikely}) \text{ and } (X \text{iss } S \text{ is likely})\)
  - If \(X\) is small then \(Y\) is large

- The language of fuzzy if-then rules is a sublanguage of PNL

THE BASIC IDEA

- Perception: \(p\) → \(NL(p)\) → \(GC(p)\)
- Precisiation of perception: \(NL(p)\) → \(GC(p)\)
- Abstraction: \(GC(p)\) → \(PF(p)\)

GCL (Generalized Constraint Language) is maximally expressive
WHAT IS A PROTOFORM?

- Informally, a protoform (abbreviation of “prototypical form”) is a symbolic expression which places in evidence the deep semantic structure of a proposition, question or command.

**Examples:**

\[ \begin{align*}
X & \text{ is } A \quad \text{(instantiation)} \quad \text{Speed is 150 km/h} \\
X & \text{ is a normally distributed random variable with mean } m \text{ and variance } \sigma^2 \\
A(B) & \text{ is } C \quad \text{(instantiation)} \quad \text{Age(Eva) is young} \quad (\text{Eva is young}) \\
\text{Prob} \ (X \text{ is } A) & \text{ is } B \quad \text{(instantiation)} \quad \text{Usually Robert returns from work at about 6 pm} \\
A(B, C) & \text{ is } D \quad \text{(instantiation)} \quad \text{Distance between Los Angeles and San Francisco is about 600 km.}
\end{align*} \]

CONTINUED

- A protoform of \( p \) defines its deep semantic structure

**Examples:**

- Allan is tall \( \rightarrow \) \( A(B) \) is \( R \)
- distance between New York and Boston is 200 miles \( \rightarrow \) \( A(B, C) \) is \( R \)
- Most Swedes are tall \( \rightarrow \) Count (A/B) is \( Q \)
- Usually Robert returns from work at about 6 pm \( \rightarrow \) \( \text{Prob} \ (X \text{ is } A) \) is \( B \)
**DICTIONARIES**

1:  
<table>
<thead>
<tr>
<th>proposition in NL</th>
<th>precisiation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( p^* ) (GC-form)</td>
</tr>
</tbody>
</table>

\[ \sum \text{Count (tall.Swedes/Swedes) is most} \]

2:  
<table>
<thead>
<tr>
<th>precisiation</th>
<th>protoform</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p^* ) (GC-form)</td>
<td>( PF(p^*) )</td>
</tr>
</tbody>
</table>

\[ \sum \text{Count (tall.Swedes/Swedes) is most} \]

Q A’s are B’s

**EXAMPLE OF TRANSLATION**

- \( P: \) usually Robert returns from work at about 6 pm
- \( P^*: \) Prob \((\text{Time(Return(Robert)) is 6 pm})\) is usually
- \( PF(p): \) \( \text{Prob \{X is A\} is B} \)
- \( X: \) Time(Return(Robert))
- \( A: \) 6 pm
- \( B: \) usually

\( p \in \text{NL} \)
\( p^* \in \text{GCL} \)
\( PF(p) \in \text{PFL} \)
TRANSLATION FROM NL TO PFL

Examples:

- Eva is young → A (B) is C
- Eva is much younger than Pat → (A (B), A (C)) is R

Usually Robert returns from work at about 6 pm

BASIC STRUCTURE OF PNL

- In PNL, deduction = generalized constraint propagation
- DDB: deduction database = collection of protoformal rules governing generalized constraint propagation
- WKDB: PNL-based
WORLD KNOWLEDGE

examples
- icy roads are slippery
- big cars are safer than small cars
- usually it is hard to find parking near the campus on weekdays between 9 and 5
- most Swedes are tall
- overeating causes obesity
- Ph.D. is the highest academic degree
- an academic degree is associated with a field of study
- Princeton employees are well paid

WORLD KNOWLEDGE

KEY POINTS

- world knowledge—and especially knowledge about the underlying probabilities—plays an essential role in disambiguation, planning, search and decision processes
- what is not recognized to the extent that it should, is that world knowledge is for the most part perception-based
WORLD KNOWLEDGE: EXAMPLES

specific:
- if Robert works in Berkeley then it is likely that Robert lives in or near Berkeley
- if Robert lives in Berkeley then it is likely that Robert works in or near Berkeley

generalized:
- if A/Person works in B/City then it is likely that A lives in or near B

precisiated:

\[ \text{Distance (Location (Residence (A/Person)), Location (Work (A/Person)) isu near} \]

protoform: \( F(A(B(C)), A(D(C))) \text{ isu R} \)

ORGANIZATION OF WORLD KNOWLEDGE

EPISTEMIC (KNOWLEDGE-DIRECTED) LEXICON (EL)
(ONTOLOGY-RELATED)

\[ R_{ij}, w_{ij}, K_i, K(j), w_{ij} \text{= granular strength of association between } i \text{ and } j \]

\[ \text{network of nodes and links} \]

\[ i \text{ (lexine): object, construct, concept \ (e.g., car, Ph.D. degree)} \]
\[ K(i): \text{world knowledge about } i \ \text{(mostly perception-based)} \]
\[ K(i) \text{ is organized into } n(i) \text{ relations } R_{ij} \ldots, R_{in} \]
\[ \text{entries in } R_{ij} \text{ are bimodal-distribution-valued attributes of } i \]
\[ \text{values of attributes are, in general, granular and context-dependent} \]
**EPISTEMIC LEXICON**

- $r_{ij}$: $i$ is an instance of $j$
- $r_{ij}$: $i$ is a subset of $j$
- $r_{ij}$: $i$ is a superset of $j$
- $r_{ij}$: $j$ is an attribute of $i$
- $r_{ij}$: $i$ causes $j$
- $r_{ij}$: $i$ and $j$ are related

**FORMAT OF RELATIONS**

**perception-based relation**

<table>
<thead>
<tr>
<th>lexine</th>
<th>$A_1$</th>
<th>...</th>
<th>$A_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td></td>
<td></td>
<td>$G_m$</td>
</tr>
</tbody>
</table>

- attributes
- granular values

**example**

<table>
<thead>
<tr>
<th>car</th>
<th>Make</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>ford</td>
<td></td>
<td>$G$</td>
</tr>
<tr>
<td>chevy</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$G$: 20% $\sqcup 15k^+ + 40% \sqcup [15k^+, 25k^+] + \cdots$

- granular count
PROTOFORM-BASED DEDUCTION

THE CONCEPT OF PROTOFORM AND RELATED CONCEPTS

Fuzzy Logic

protoform

Bivalent Logic

ontology

conceputal graph

skeleton

Montague grammar
THE CONCEPT OF A PROTOFORM

- Informally, a protoform—abbreviation of prototypical form—is an abstracted summary. More specifically, a protoform is a symbolic expression which defines the deep semantic structure of a construct such as a proposition, command, question, scenario, concept or a system of such constructs.

Example:

\[
\begin{align*}
\text{Eva is young} & \quad \rightarrow \quad A(B) \text{ is } C \\
\text{young} & \quad \rightarrow \quad C
\end{align*}
\]

TRANSFORMATION FROM NL TO PFL

Examples:

- Most Swedes are tall \( \rightarrow \) Count \((A/B)\) is \(Q\)
- Eva is much younger than Pat \( \rightarrow \) \((A(B), A(C))\) is \(R\)
- Usually Robert returns from work at about 6pm \( \rightarrow \) Prob \((A\text{ is }B)\) is \(C\)

\[\text{age} \quad \text{age} \quad \text{age} \quad \text{much} \quad \text{younger}\]

\[\text{usually} \quad \text{about 6 pm}\]
**EXAMPLE**

$p = \text{it is very unlikely that there will be a significant increase in the price of oil in the near future}$

$PF(p):$

- Prob($E$) is very unlikely → Prob($A$) is $B$
- $B$: Epoch ($E'$) is near future → Attr1 (C) is $D$
- $C$: significant increase in the price of oil → Attr2
- (Attr3(F))

**CONTINUED**

*semantic network representation of $E$*

$E^*$

- **modifier**
- **variation**
- **attribute**

- significant → increase → price → oil
- epoch → future
- mod → near

---

CONTINUED

Precision (f.b.-concept)

\[ E^*: \text{Epoch (Variation (Price (oil)) is significant increase) is near.future} \]

CONTINUED

Precision of very unlikely

\[ \mu_{\text{very unlikely}}(v) = (\mu_{\text{likely}}(1-v))^2 \]
THE CONCEPT OF i-PROTOFORM

- \( i \)-protoform: idealized protoform
- The key idea is to equate the grade of membership, \( \mu_A(u) \), of an object, \( u \), in a fuzzy set, \( A \), to the distance of \( u \) from an \( i \)-protoform.
- This idea is inspired by E. Rosch's work (ca 1972) on the theory of prototypes.

EXAMPLE: EXPECTED VALUE (f.f-concept)

- \( X \): real-valued random variable with probability density \( g \)
- Standard definition of expected value of \( X \):

\[
E(X) = \int_{u} u g(u) du
\]

\( E(X) = \text{average value of } X \)

- The label "expected value" is misleading.
- \( E(X) \) is a fuzzy set
- grade of membership of a particular function, \( E^*(X) \), in the fuzzy set of expected value of \( X \) is the distance of \( E^*(X) \) from best-fitting \( i\)-protoform
Scenario A:

Alan has severe back pain. He goes to see a doctor. The doctor tells him that there are two options: (1) do nothing; and (2) do surgery. In the case of surgery, there are two possibilities: (a) surgery is successful, in which case Alan will be pain free; and (b) surgery is not successful, in which case Alan will be paralyzed from the neck down.

Question: Should Alan elect surgery?
PF-EQUIVALENCE

Scenario B:
Alan needs to fly from San Francisco to St. Louis and has to get there as soon as possible. One option is fly to St. Louis via Chicago and the other through Denver. The flight via Denver is scheduled to arrive in St. Louis at time a. The flight via Chicago is scheduled to arrive in St. Louis at time b, with a < b. However, the connection time in Denver is short. If the flight is missed, then the time of arrival in St. Louis will be c, with c > b. Question: Which option is best?
PROTOFORMAL SEARCH RULES

example

query: What is the distance between the largest city in Spain and the largest city in Portugal?

protoform of query: ?Attr (Deso(A), Deso(B))

procedure

query: ?Name (A)|Desc (A)
query: Name (B)|Desc (B)
query: ?Attr (Name (A), Name (B))
PROTOFORM DEDUCTION RULE: GENERALIZED MODUS PONENS

classical
\[ A \rightarrow B \]
\[ B \]

fuzzy logic
\[ X \text{ is } A \]
\[ \text{If } X \text{ is } B \text{ then } Y \text{ is } C \]
\[ Y \text{ is } D \]

symbolic
\[ D = A \cdot (B \times C) \]
(fuzzy graph; Mamdani)

computational 1
\[ D = A \cdot (B \Rightarrow C) \]
(implication; conditional relation)

PROTOFORMAL RULES OF DEDUCTION

examples
\[ X \text{ is } A \]
\[ (X, Y) \text{ is } B \]
\[ Y \text{ is } A \cdot B \]
\[ \mu_{A,B}(v) = \max_i (\mu_A(u_i) \land \mu_B(u,v)) \]

\[ \mu_D(u) = \max_i (\mu_B(\int u \cdot g(u) g(u)du)) \]
subject to: \[ \int g(u) g(u) du = 1 \]
**PROTOFORM-BASED (PROTOFORMAL) DEDUCTION**

- Rules of deduction in the Deduction Database (DDB) are protoformal.
  - Examples: (a) Compositional rule of inference
    
    \[
    \begin{align*}
    X &\text{ is } A \\
    (X, Y) &\text{ is } B \\
    Y &\text{ is } A \lor B
    \end{align*}
    \]
    \[
    \mu_x(v) = \sup\{\mu_x(u) \land \mu_y(u, v)\}
    \]
  
    (b) Extension principle
    
    \[
    \begin{align*}
    X &\text{ is } A \\
    Y &= f(X) \\
    Y &= f(A)
    \end{align*}
    \]
    \[
    \mu_x(v) = \sup_x(\mu_x(u))
    \]

- Rules of deduction are basically rules governing generalized constraint propagation.
- The principal rule of deduction is the extension principle.

\[
\begin{align*}
X &\text{ is } A \\
Y &= f(X) \\
Y &= f(A)
\end{align*}
\]
\[
\mu_x(v) = \sup_x(\mu_x(u))
\]

Subject to: \( v = f(u) \)
GENERALIZATIONS OF THE EXTENSION PRINCIPLE

information = constraint on a variable

\[
\begin{align*}
\text{given information about } X & \quad f(X) \text{ is } A \\
\text{inferred information about } X & \quad g(X) \text{ is } B
\end{align*}
\]

\[\mu_a(v) = \sup_{u} \{ \mu_u(f(u)) \} \]
Subject to: \( v = g(u) \)

CONTINUED

\[
\begin{align*}
\frac{f(X_1, \ldots, X_n) \text{ is } A}{g(X_1, \ldots, X_n) \text{ is } B}
\end{align*}
\]

\[\mu_a(v) = \sup_{u} \{ \mu_u(f(u)) \} \]
Subject to: \( v = g(u) \)

\[
\begin{align*}
(X_1, \ldots, X_n) \text{ is } A \\
g_j(X_1, \ldots, X_n) \text{ is } Y_{j^*}, \quad j = 1, \ldots, n
\end{align*}
\]

\[\mu_a(v) = \sup_{u} \{ \mu_u(f(u)) \} \]
Subject to: \( v = g(u) \)

\[
\begin{align*}
(Y_1, \ldots, Y_n) \text{ is } B
\end{align*}
\]

\[j = 1, \ldots, n\]
COUNT-AND MEASURE-RELATED RULES

\[ Q \text{ A's are B's} \]
\[ \text{ant (Q) A's are not B's} \]

\[ Q \text{ A's are B's} \]
\[ Q^{1/2} \text{ A's are } \frac{2}{3} \text{B's} \]

most Swedes are tall
ave (height) Swedes is \( ?h \)

\[ Q \text{ A's are B's} \]
\[ \text{ave (B|A) is } \frac{?C}{h} \]

\[ \mu_{av}(v) = \sup_{a} \mu_{a} \left( \frac{1}{N} \sum_{i} \mu_{a}(a_{i}) \right) \]
\[ a = (a_{1}, ..., a_{N}) \]

\[ v = \frac{1}{N} \left( \sum_{i} a_{i} \right) \]

CONTINUED

not(QA's are B's) \leftrightarrow (not Q) A's are B's

\[ Q_{1} \text{ A's are B's} \]
\[ Q_{2} (A&B)'s are C's \]
\[ Q_{1} Q_{2} A's are (B&C)'s \]

\[ Q_{1} \text{ A's are B's} \]
\[ Q_{2} A's are C's \]
\[ (Q_{1} + Q_{2} - 1) A's are (B&C)'s \]
**PROBABILITY MODULE**

X: real-valued random variable

\( g: \) probability density function of \( X \)

\( A_1, \ldots, A_n, A: \) perception-based events in \( U \)

\( P_1, \ldots, P_n, P: \) perception-based probabilities in \( U \)

\[
\begin{align*}
\text{Prob} \{X \text{ is } A_1\} &= P_{j(1)} \\
&\quad \ldots \\
\text{Prob} \{X \text{ is } A_n\} &= P_{j(n)} \\
\text{Prob} \{X \text{ is } A\} &= P
\end{align*}
\]
CONTINUED

\[ \mu_p(v) = \sup_g \left( \mu_{P_1} \left( \int_0^1 g(u) \mu_{A_1}(u) du \right) \right) \land \cdots \land \mu_{P_m} \left( \int_0^1 g(u) \mu_{A_m}(u) du \right) \]

subject to:

\[ v = \int_0^1 g(u) \mu_A(u) du \]

PROBABILITY MODULE (CONTINUED)

\[ \begin{array}{c}
\text{X is} p \quad \text{Prob} (X \text{ is } A) \text{ is } P \\
\frac{X \text{ is } P}{Y = f(X)}\quad \frac{\text{Prob} (f(X) \text{ is } B) \text{ is } Q}{Y \text{ is } f(P)} \\
\text{X is} u \quad \text{X is } A \\
\frac{X \text{ is } u}{(X,Y) \text{ is } R} \quad \frac{Y = f(X)}{Y \text{ is } f(A)} \\
\frac{Y \text{ is } s}{Y \text{ is } s} \\
\end{array} \]
PROBABILISTIC CONSTRAINT PROPAGATION RULE
(a special version of the generalized extension principle)

\[ \int_U g(u) \mu_A(u) \, du \text{ is } R \]

\[ \int_U g(u) \mu_B(u) \, du \text{ is } ?S \]

\[ \mu_S(v) = \sup_B \{ \mu_B(\int_U g(u) \mu_A(u) \, du) \} \]

subject to

\[ v = \int_U g(u) \mu_B(u) \, du \]

\[ \int_U g(u) \, du = 1 \]

PROTOFORMAL DEDUCTION RULES

**Possibility Extension Principle**

\[ X \text{ is } (\Sigma_i \mu_i / u_i) \]

\[ Y = f(X) \]

\[ Y \text{ is } (\Sigma_i \mu_i / f(u_i)) \]

\[ \mu_i / u_i + \mu_j / u_i = (\mu_i \lor \mu_j) / u_i \]

**Probabilistic Extension Principle**

\[ X \text{ isp } (\Sigma_i p_i \setminus u_i) \]

\[ Y = f(X) \]

\[ Y \text{ isp } (\Sigma_i p_i \setminus f(u_i)) \]

\[ p_i \setminus u_i + p_j \setminus u_i = (p_i + p_j) \setminus u_i \]
COMPUTATION WITH PERCEPTIONS
PROTOFORMAL RULE OF DEDUCTION

\[ X \text{ is } A \]
\[ (X, Y) \text{ is } (\sum A_i \times B_i) \]
\[ Y \text{ is } (\sum m_i \wedge B_i) \]
\[ m_i = \sup (A_i \cap A_j) \]

PROTOFORMAL DEDUCTION RULE

\[ X \text{ is } pa (\sum P_i \wedge A_i) \]
\[ Y \text{ is } sfq (\sum B_i \times C_i) \]
\[ Y \text{ is } r ? D \]
**PROTOFORMAL DEDUCTION RULE**

\[
\begin{align*}
X \text{ ispa } (\Sigma_i P_i \mid A_i) \\
Y = f(X) & \quad \text{Y isr } ?B
\end{align*}
\]

\[
\begin{align*}
X \text{ ispb } (\Sigma_i P_i \mid \neg A_i) \\
Y = f(X) & \quad \text{Y isr } ?C
\end{align*}
\]

**PROTOFORMAL CONSTRAINT PROPAGATION**

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\text{GC}(p))</th>
<th>(\text{PF}(p))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dana is young</td>
<td>Age (Dana) is young</td>
<td>X is A</td>
</tr>
<tr>
<td>Tandy is a few years older than Dana</td>
<td>Age (Tandy) is (Age (Dana)) + few</td>
<td>Y is (X+B)</td>
</tr>
<tr>
<td>X is A</td>
<td>Age (Tandy) is (young + few)</td>
<td></td>
</tr>
<tr>
<td>Y is (X+B)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y is A+B</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\mu_{A+B}(v) = \sup_u (\mu_A(u) \land \mu_B(v-u))
\]
PNL AS A DEFINITION LANGUAGE

HIERARCHY OF DEFINITION LANGUAGES

PNL

F.G language

F language

B language

NL

fuzzy-logic-based

bivalent-logic-based

NL: natural language
B language: standard mathematical bivalent-logic-based language
F language: fuzzy logic language without granulation
F.G language: fuzzy logic language with granulation
PNL: Precisiated Natural Language

Note: the language of fuzzy if-then rules is a sublanguage of PNL

Note: a language in the hierarchy subsumes all lower languages
SIMPLIFIED HIERARCHY

PNL ← fuzzy-logic-based
B language ← bivalent-logic-based
NL

The expressive power of the B language – the standard bivalence-logic-based definition language – is insufficient

Insufficiency of the expressive power of the B language is rooted in the fundamental conflict between bivalence and reality

EVERYDAY CONCEPTS WHICH CANNOT BE DEFINED REALISTICALLY THROUGH THE USE OF B

- check-out time is 12:30 pm
- speed limit is 65 mph
- it is cloudy
- Eva has long hair
- economy is in recession
- I am risk averse
- ...

LAZ 6/27/2003
DEFINITION OF p: ABOUT 20-25 MINUTES

INSUFFICIENCY OF THE B LANGUAGE

Concepts which cannot be defined
- causality
- relevance
- intelligence

Concepts whose definitions are problematic
- stability
- optimality
- statistical independence
- stationarity
**DEFINITION OF OPTIMALITY**

**OPTIMIZATION = MAXIMIZATION?**

- **Yes**
  - $\text{gain} \rightarrow \text{max}$
  - Example: $a$ to $X$

- **Unsure**
  - $\text{gain} \rightarrow \text{max}$
  - Example: $a$ to $X$

- **No**
  - $\text{gain} \rightarrow \text{max}$
  - Example: $a$ to $b$

- **Hard to tell**
  - $\text{gain} \rightarrow \text{max}$
  - Example: $a$ to $b$ to $c$

- *definition of optimal $X$ requires use of PNL*

---

**MAXIMUM?**

- **a)** $\forall x \ (f(x) \leq f(a))$
- **b)** $\not\exists x \ (f(x) > f(a))$

---

**extension principle**

**Pareto maximum**

- **b)** $\exists x \ (f(x) \text{ dominates } f(a))$
**EXAMPLE**

- *I am driving to the airport. How long will it take me to get there?*
- *Hotel clerk’s perception-based answer: about 20-25 minutes*
- *“about 20-25 minutes” cannot be defined in the language of bivalent logic and probability theory*
- *To define “about 20-25 minutes” what is needed is PNL*
**EXAMPLE**

**PNL definition of “about 20 to 25 minutes”**

- Prob \{getting to the airport in less than about 25 min\} is unlikely
- Prob \{getting to the airport in about 20 to 25 min\} is likely
- Prob \{getting to the airport in more than 25 min\} is unlikely

---

**PNL-BASED DEFINITION OF STATISTICAL INDEPENDENCE**

![Diagram of PNL-based definition of statistical independence]

**Contingency table**

<table>
<thead>
<tr>
<th></th>
<th>L/S</th>
<th>L/M</th>
<th>L/L</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>M/S</td>
<td>M/M</td>
<td>M/L</td>
</tr>
<tr>
<td>2</td>
<td>S/S</td>
<td>S/M</td>
<td>S/L</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \Sigma (M/L) = \frac{\Sigma (M \times L)}{\Sigma (L)} \]

- degree of independence of Y from X = degree to which columns 1, 2, 3 are identical

---

LAZ 6/27/2003
LYAPOUNOV STABILITY IS COUNTERINTUITIVE

\[ x = f(x) \]
\[ x_0 = 0 : \text{equilibrium state} \]

- the system is stable no matter how large \( D \) is

PNL-BASED DEFINITION OF STABILITY

- a system is \( F \)-stable if it satisfies the fuzzy Lipshitz condition

\[ \| \Delta x \| \leq F \| \Delta x_i \| \]

- interpretation

\[ \| \Delta x_i \| \]

degree of stability = degree to which \( f \) is in \( \leq F \| \Delta x_i \| \)
What is not widely recognized is that some seemingly simple concepts, e.g., cluster and edge, are hard to define because they are higher-order concepts.

Informally, a concept is of order (level) \( k \) if its denotation is a set of order \( k \). A set whose elements are points is of order one. A set whose elements are sets of order one is of order two, etc.
There are four categories of second-order concepts: (1) b.b-concepts, i.e., bivalent (crisp) concepts whose instances are bivalent sets, e.g., convex set; (2) b.f-concepts, i.e., bivalent concepts whose instances are fuzzy sets, e.g., convex fuzzy sets; (3) f.b-concepts, i.e., fuzzy concepts whose instances are bivalent sets, e.g., small squares; and (4) f.f-concepts, i.e., fuzzy concepts whose instances are fuzzy sets. The concepts of cluster and edge are examples of f.f-concepts. That is why they are hard to define.
INTERPOLATION MODULE AND PROBABILITY MODULE

\[ Prob \{ X \text{ is } A_i \} = P_i, \quad i = 1, ..., n \]
\[ Prob \{ X \text{ is } A \} = Q \]

\[ \mu_u(v) = \sup_g \left( \mu_{p_i} \left( \int \mu_{A_i}(u)g(u)du \right) \right) \]

subject to

\[ U = \int \mu_A(u)g(u)du \]

PROBABILISTIC CONSTRAINT PROPAGATION RULE
(a special version of the generalized extension principle)

\[ \int g(u)\mu_A(u)du \quad is \ R \]

\[ \int g(u)\mu_B(u)du \quad is \ ?S \]

\[ \mu_S(v) = \sup_g \left( \mu_B \left( \int g(u)\mu_A(u)du \right) \right) \]

subject to

\[ v = \int g(u)\mu_B(u)du \]
\[ \int g(u)du = 1 \]
USUALITY SUBMODULE

CONJUNCTION

\[
\begin{align*}
X \text{ is } A \\
X \text{ is } B \\
\therefore X \text{ is } A \cap B
\end{align*}
\]

\[
\begin{align*}
X \text{ is } u \text{ A} \\
X \text{ is } u \text{ B} \\
\therefore X \text{ is } r \text{ A} \cap B
\end{align*}
\]

- determination of \( r \) involves interpolation of a bimodal distribution
**USUALITY — QUALIFIED RULES**

\[
\begin{align*}
X & \text{ isu } A \\
\frac{X \text{ isu } A}{X \text{ isu } \lnot A}
\end{align*}
\]

\[
\begin{align*}
X & \text{ isu } A \\
\frac{Y = f(X)}{Y \text{ isu } f(A)} \\
\mu_{f(A)}(v) &= \sup_{u \in f^{-1}(v)}(\mu_A(u))
\end{align*}
\]

**USUALITY — QUALIFIED RULES**

\[
\begin{align*}
X & \text{ isu } A \\
Y & \text{ isu } B \\
Z &= f(X,Y) \\
\frac{Z \text{ isu } f(A,B)}{Z \text{ isu } f(A,B)} \\
\mu_Z(w) &= \sup_{u,v \in f^{-1}(w)}(\mu_A(u) \land \mu_B(v))
\end{align*}
\]
EXTENSION PRINCIPLE MODULE

PRINCIPAL COMPUTATIONAL RULE IS
THE EXTENSION PRINCIPLE (EP)

point of departure: function evaluation

\[ X = a \\
\frac{Y = f(X)}{Y = f(a)} \]
VERSION EP(1,1) (COMPOSITIONAL RULE OF INFERENCE) (1965)

\[ X \text{ is } A \]
\[ (X, Y) \text{ is } R \]
\[ Y \text{ is } A \times R \]

\[ \mu_Y(v) = \sup_u (\mu_A(u) \wedge \mu_R(u, v)) \]

EXTENSION PRINCIPLE EP(2,0) (Mamdani)

\[ f^* = \sum_i A_i \times B \]
\[ X = a \]
\[ Y = \sum_i \mu_i A_i(a) \wedge B_i \]

(if \( X \) is \( A_i \), then \( Y \) is \( B_i \))
VERSION EP(2,1)

\[ f'(A) \rightarrow f'' (\text{granulated } f) \]

\[ X \text{ is } A \]
\[ (X, Y) \text{ is } R \]
\[ Y \text{ is } \sum_i m_i \wedge B_i \]

\[ R = \sum_i A_i \times B_i \]

\[ m_i = \sup_u \left( \mu_A(u) \wedge \mu_{A_i}(u) \right): \text{matching coefficient} \]

VERSION EP(1,1b) (DEMPSTER-SHAFER)

\[ X \text{ is } p_1 \mid u_1 + \ldots + p_u \mid u_n \]
\[ (X, Y) \text{ is } R \]
\[ Y \text{ is } p_1 \mid R(u_1) + \ldots + p_n \mid R(u_n) \]

Y is a fuzzy-set-valued random variable

\[ \mu_{R(u)}(v) = \mu_R(u_v, v) \]
VERSION GEP(0,0)

\[
\begin{align*}
  f(X) & \text{ is } A \\
  g(X) & \text{ is } g(f^{-1}(A))
\end{align*}
\]

\[
\mu_{g(f^{-1}(A))}(v) = \sup_u (\mu_A(f(u)))
\]

subject to

\[
  v = g(u)
\]

GENERALIZED EXTENSION PRINCIPLE

\[
\begin{align*}
  f(X) & \text{ is } A \\
  g(Y) & \text{ is } B \\
  Z & = h(X,Y)
\end{align*}
\]

\[
Z \text{ is } h(f^{-1}(A), g^{-1}(B))
\]

\[
\mu_Z(w) = \sup_{u,v} (\mu_A(f(u)) \wedge \mu_B(g(u)))
\]

subject to

\[
w = h(u,v)
\]
U-Qualified Extension Principle

If \( X \) is \( A_i \) then \( Y \) is u \( B_i \), \( i = 1, \ldots, n \)

\[ X \text{ is } A \]

\[ Y \text{ is } \sum_i m_i \wedge B_i \]

\[ m = \sup_u (\mu_A(u) \wedge \mu_A(u)) : \text{matching coefficient} \]
THE ROBERT EXAMPLE

- the Robert example relates to everyday commonsense reasoning— a kind of reasoning which is preponderantly perception-based

- the Robert example is intended to serve as a test of the deductive capability of a reasoning system to operate on perception-based information

THE ROBERT EXAMPLE

- the Robert example is a sequence of versions of increasing complexity in which what varies is the initial data-set (IDS)

  version 1

IDS: usually Robert returns from work at about 6 pm

questions:

  q₁ : what is the probability that Robert is home at t* (about 6 pm)?

  q₂ : what is the earliest time at which the probability that Robert is home is high?
CONTINUED

version 2:
IDS: usually Robert leaves office at about 5:30pm, and usually it takes about 30min to get home

$q_1, q_2$: same as in version 1

version 3: this version is similar to version 2 except that travel time depends on the time of departure from office.

$q_1, q_2$: same as version 1

THE ROBERT EXAMPLE (VERSION 3)

IDS: Robert leaves office between 5:15pm and 5:45pm. When the time of departure is about 5:20pm, the travel time is usually about 20min; when the time of departure is about 5:30pm, the travel time is usually about 30min; when the time of departure is about 5:40pm, the travel time is about 20min

- usually Robert leaves office at about 5:30pm
- What is the probability that Robert is home at about t pm?
THE ROBERT EXAMPLE

Version 4

• Usually Robert returns from work at about 6 pm
  Usually Ann returns from work about half-an-hour later
What is the probability that both Robert and Ann are home at about t pm?

THE ROBERT EXAMPLE

Version 1.

My perception is that Robert usually returns from work at about 6:00pm

$q_1$: What is the probability that Robert is home at about t pm?
$q_2$: What is the earliest time at which the probability that Robert is home is high?
PROTOFORMAL DEDUCTION

THE ROBERT EXAMPLE

IDS  p: usually Robert returns from work at about 6 pm.
TDS  q: what is the probability that Robert is home at about t pm?

1. precisation:
   \[ p \rightarrow \text{Prob (Time (Robert returns from work is about 6 pm) is usually } \]
   \[ q \rightarrow \text{Prob (Time (Robert is home) is about t pm)} \]
   \[ \text{?D} \]

2. calibration: \( \mu_{\text{usually}}, \mu_{t^*}, t^* = \text{about t} \)

3. abstraction:
   \[ p^* \rightarrow \text{Prob (X is A) is B} \]
   \[ q^* \rightarrow \text{Prob (Y is C) is ?D} \]

CONTINUED

4. search in Probability module for applicable rules (top-level agent)
   \[
   \begin{align*}
   \text{Prob (X is A) is B} \\
   \text{Prob (Y is C) is D}
   \end{align*}
   \]
   not found

found:
   \[
   \begin{align*}
   \text{Prob (X is A) is B} \\
   \text{Prob (X is C) is D}
   \end{align*}
   \]
   \[
   \begin{align*}
   \text{Prob (X is A) is B} \\
   \text{Prob (f(X) is C) is D}
   \end{align*}
   \]

5. back to IDS and TDS. Go to WKDB (top-level agent)
   - A/person is at home at time t if A returns before t
   - Robert is home at t^* = Robert returns from work before t

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THE ROBERT EXAMPLE

event equivalence

Robert is home at about t pm = Robert returns from work before about t pm

Before about t pm = ≤ about t pm

CONTINUED

6. back to Probability module

\[
\begin{align*}
\text{Prob} \{X \text{ is } A\} & = B \\
\text{Prob} \{X \text{ is } C\} & = D \\
\mu_x(v) & = \sup_y (\mu_x(\int u \mu_y(u)g(u)du)) \\
v & = \int u \mu_x(u)g(u)du
\end{align*}
\]

7. Instantiation:

\(D = \text{Prob} \{\text{Robert is home at about } 6:15\}\)
\(X = \text{Time (Robert returns from work)}\)
\(A = 6^*\)
\(B = \text{usually}\)
\(C = \leq 6:15^*\)
SUMMATION

KEY POINTS

- humans have a remarkable capability—a capability which machines do not have—to perform a wide variety of physical and mental tasks using only perceptions, with no measurements and no computations

- perceptions are intrinsically imprecise, reflecting the bounded ability of sensory organs, and ultimately the brain, to resolve detail and store information

CONTINUED

- imprecision of perceptions stands in the way of constructing a computational theory of perceptions within the conceptual structure of bivalent logic and bivalent-logic-based probability theory

- this is why existing scientific theories—based as they are on bivalent logic and bivalent-logic-based probability theory—provide no tools for dealing with perception-based information
CONTINUED

- in computing with words and perceptions (CWP), the objects of computation are propositions drawn from a natural language and, in particular, propositions which are descriptors of perceptions.

- computing with words and perceptions is a methodology which may be viewed as (a) a new direction for dealing with imprecision, uncertainty and partial truth; and (b) as a basis for the analysis and design of systems which are capable of operating on perception-based information.

STATISTICS

Count of papers containing the word “fuzzy” in the title, as cited in INSPEC and MATH.SCI.NET databases. (Data for 2002 are not complete)

Compiled by Camille Wanat, Head, Engineering Library, UC Berkeley, April 17, 2003

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