Some Concepts of Uniform Exponential Dichotomy for Skew-Evolution Semiflows in Banach Spaces

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Abstract

The exponential dichotomy is one of the most important asymptotic properties for the solutions of evolution equations, studied in the last years from various perspectives. In this paper we study some concepts of uniform exponential dichotomy for skew-evolution semiflows in Banach spaces. Several illustrative examples motivate the approach.

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1. Introduction

The property of exponential dichotomy is a mathematical domain with a substantial recent development as it plays an important role in describing several types of evolution equations. The literature dedicated to this asymptotic behavior begins with the results published in Perron (1930). The ideas were continued by in Massera & Schäffer (1966), with extensions in the infinite dimensional case accomplished in Daleckií & Krein (1974) and in Pazy (1983), respectively in Sacker & Sell (1994). Diverse and important concepts of dichotomy were introduced and studied, for example, in Appell et al. (1993), Babuţia & Megan (2015), Chow & Leiva (1995), Coppel (1978), Megan & Stoica (2010), Sasu & Sasu (2006) or Stoica & Borlea (2012).

The notion of skew-evolution semiflow that we study in this paper and which was introduced in Megan & Stoica (2008) generalizes the skew-product semiflows and the evolution operators. Several asymptotic properties for skew-evolution semiflows are defined and characterized see Viet Hai (2010), Viet Hai (2011), Stoica & Borlea (2014), Stoica & Megan (2010) or Yue et al. (2014).

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In this paper we intend to study some concepts of uniform exponential dichotomy for skew-evolution semiflows in Banach spaces. The definitions of various types of dichotomy are illustrated by examples. We also aim to give connections between them, emphasized by counterexamples.

2. Preliminaries

Let \((X, d)\) be a metric space, \(V\) a Banach space and \(\mathcal{B}(V)\) the space of all \(V\)-valued bounded operators defined on \(V\). Denote \(Y = X \times V\) and \(T = \{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\}\).

**Definition 2.1.** A mapping 
\[ \varphi : T \times X \to X \]

is called a skew-evolution semiflow on \(X\) if the following properties are satisfied:

1. \(\varphi(t, t, x) = x, \forall (t, x) \in \mathbb{R}_+ \times X\);
2. \(\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x), \forall (t, s, t_0) \in T, x \in X\).

**Definition 2.2.** A mapping \(\Phi : T \times X \to \mathcal{B}(V)\) is called evolution cocycle over an evolution semiflow \(\varphi\) if:

1. \(\Phi(t, t, x) = I, \forall t \geq 0, x \in X\) (I - identity operator).
2. \(\Phi(t, s, \Phi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x), \forall (t, s, t_0) \in T, \forall x \in X\).

Let \(\Phi\) be an evolution cocycle over an evolution semiflow \(\varphi\). The mapping \(C = (\varphi, \Phi)\), defined by:

\[ C : T \times Y \to Y, C(t, s, x, v) = (\varphi(t, s, x), \Phi(t, s, x)v) \]

is called skew-evolution semiflow on \(Y\).

**Example 2.1.** We will denote \(C = C(\mathbb{R}, \mathbb{R})\) the set of continuous functions \(x : \mathbb{R} \to \mathbb{R}\), endowed with uniform convergence topology on compact subsets of \(\mathbb{R}\). The set \(C\) is metrizable with the metric

\[ d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}, \text{ unde } d_n(x, y) = \sup_{t \in [-n, n]} |x(t) - y(t)|. \]

For every \(n \in \mathbb{N}^*\) we consider a decreasing function

\[ x_n : \mathbb{R}_+ \to \left(\frac{1}{2n + 1}, \frac{1}{2n}\right), \lim_{t \to \infty} x_n(t) = \frac{1}{2n + 1}. \]

We will denote

\[ x_n^s(t) = x_n(t + s), \forall t, s \geq 0. \]

Let be \(X\) the closure in \(C\) of the set \(\{ x_n^s, n \in \mathbb{N}^*, s \in \mathbb{R}_+ \}\). The application

\[ \varphi : T \times X \to X, \varphi(t, s, x) = x_{t-s}, \text{ unde } x_{t-s}(\tau) = x(t - s + \tau), \forall \tau \geq 0, \]

is an evolution cocycle over an evolution semiflow \(\varphi\).
is a evolution semiflow on $X$. Let consider the Banach space $V = \mathbb{R}^2$ with the norm $\|(v_1, v_2)\| = |v_1| + |v_2|$. Then, the application

$$\Phi : T \times X \to \mathcal{B}(V), \quad \Phi(t, s, x)v = \left(e^{\alpha_1 \int_s^t x(r-s)dr}v_1, e^{\alpha_2 \int_s^t x(r-s)dr}v_2\right),$$

where $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ is fixed, is a cocycle application of evolution over the semiflow $\varphi$, and $C = (\varphi, \Phi)$ is a evolution cocycle on $Y$.

Let us remind the definition of an evolution operator, followed by examples that punctuate the fact that it is generalized by an skew-evolution semiflows.

**Definition 2.3.** A mapping $E : T \to \mathcal{B}(V)$ is called **evolution operator** on $V$ if following properties hold:

(e1) $E(t, t) = I, \quad \forall t \in \mathbb{R}^+$;

(e2) $E(t, s)E(s, t_0) = E(t, t_0), \quad \forall (t, s), (s, t_0) \in T$.

**Example 2.2.** One can naturally associate to every evolution operator $E$ the mapping

$$\Phi_E : T \times X \to \mathcal{B}(V), \quad \Phi_E(t, s, x) = E(t, s),$$

which is an evolution cocycle on $V$ over every evolution semiflow $\varphi$. Therefore, the evolution operators are particular cases of evolution cocycles.

**Example 2.3.** Let $X = \mathbb{R}_+$. The mapping

$$\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+, \quad \varphi(t, s, x) = t - s + x$$

is an evolution semiflow on $\mathbb{R}_+$. For every evolution operator $E : T \to \mathcal{B}(V)$ we obtain that

$$\Phi_E : T \times \mathbb{R}_+ \to \mathcal{B}(V), \quad \Phi_E(t, s, x) = E(t - s + x, x)$$

is an evolution cocycle on $V$ over the evolution semiflow $\varphi$. It follows that an evolution operator on $V$ is generating a skew-evolution semiflow on $Y$.

### 3. Sequences of Invariant Projections for a Cocycle

**Definition 3.1.** A continuous map $P : X \to \mathcal{B}(V)$ which satisfies the following relation:

$$P(x)P(x) = P(x), \quad (\forall)x \in X$$

is called projection on $V$.

**Definition 3.2.** A projection $P$ on $V$ is called **invariant** for a skew-evolution semiflow $C = (\varphi, \Phi)$ if:

$$P(\varphi(t, s, x)) \Phi(t, s, x) = \Phi(t, s, x)P(x),$$

for all $(t, s) \in T$ and $x \in X$. 
Remark. If $P$ is a projection on $V$, than the map

$$Q : X \to \mathcal{B}(V), \quad Q(x) = I - P(x)$$

is also a projection on $V$, called complementary projection of $P$.

Remark. If the projection $P$ is invariant for $C$ then $Q$ is also invariant for $C$.

Definition 3.3. We will name $(C, P)$ a dichotomy pair where $C$ is a skew-evolution semiflow and $P$ is invariant or $C$.

4. Concepts of Uniform Exponential Dichotomy for Skew-Evolution Semiflows

Definition 4.1. Let $(C, P)$ be a dichotomy pair. We say that $(C, P)$ is uniformly strongly exponentially dichotomic (u.s.e.d) if there exist $N \geq 1$ and $\nu > 0$ such that:

(used1) $\|\Phi(t, s, x)P(x)\| \leq Ne^{-\nu(t-s)}$

(used2) $N\|\Phi(t, s, x)Q(x)\| \geq e^{\nu(t-s)}$

for all $(t, s) \in T$ and $x \in X$.

Definition 4.2. We say that $(C, P)$ is uniformly exponentially dichotomic (u.e.d) if there exist $N \geq 1$ and $\nu > 0$ such that:

(ued1) $\|\Phi(t, s, x)P(x)v\| \leq Ne^{-\nu(t-s)}\|P(x)v\|$

(ued2) $N\|\Phi(t, s, x)Q(x)v\| \geq e^{\nu(t-s)}\|Q(x)v\|$

for all $(t, x) \in T \times X$ and for all $v \in V$.

Definition 4.3. We say that $(C, P)$ is uniformly weakly exponentially dichotomic (u.w.e.d) if there exist $N \geq 1$ and $\nu > 0$ such that:

(uwed1) $\|\Phi(t, s, x)P(x)\| \leq Ne^{-\nu(t-s)}\|P(x)\|$

(uwed2) $N\|\Phi(t, s, x)Q(x)\| \geq e^{\nu(t-s)}\|Q(x)\|$

for all $(t, x) \in T_xX$ and for all $v \in V$.

Proposition 1. If $(C, P)$ is (s.u.e.d) then

$$\sup_{x \in X} \|P(x)\| < +\infty. \quad (4.1)$$

Proof. Consider in (used1) $t = s$. Then we have

$$\|\Phi(t, t, x)P(x)\| = \|P(x)\| = \|P(x)\| \leq N \quad (4.2)$$

for all $x \in X$. \qed
Proposition 2. If \((C, P)\) is (u.s.e.d) then \((C, P)\) is (u.w.e.d).

Proof. If \((C, P)\) is (u.s.e.d) then by (used1), for \(x \in X\), we have that \(\|P(x)\| \leq N\) and hence
\[
\|Q(x)\| = \|I - P(x)\| \leq 1 + \|P(x)\| \leq 2N.
\]
We have from (used1) and (used2) that:
\[
\|\Phi(t, s, x)\| \leq Ne^{-\nu(t-s)} \cdot 1 \leq Ne^{-\nu(t-s)}\|P(x)\| \leq 2N2e^{-\nu(t-s)}\|P(x)\|.
\]

\[2N^2\|\Phi(t, s, x)Q(x)\| \geq 2Ne^{\nu(t-s)} \geq e^{\nu(t-s)}\|Q(x)\|, \quad (4.5)
\]
hence \((C, P)\) is (u.w.e.d) \(\Box\)

Proposition 3. If \((C, P)\) is (u.e.d) then \((C, P)\) is also (u.w.e.d)

Proof. It follows immediately by taking the supremum over all \(v \in V\) with \(|v| = 1\). \(\Box\)

Definition 4.4. We say that \(C\) has a uniform exponential growth (u.e.g) if there exist \(M \geq 1\), \(\omega > 0\) such that
\[
\|\Phi(t, s, x)\| \leq Me^{\omega(t-s)},
\]
for all \((t, s) \in T\) and \(x \in X\).

Theorem 4.1. Assume that a dichotomy pair \((C, P)\) is (u.w.e.d) and \(C\) has a uniform exponential growth. Then:
\[
\sup_{x \in X} \|P(x)\| < +\infty.
\]

Proof. Let \(N, \nu\) given by the (u.w.e.d) property of \((C, P)\) and \(M, \omega\) given by the (u.e.g) of \(C\). Consider \(s \geq 0\) fixed, \(t \geq s\) and \(x \in X\).

\[
\left[\frac{1}{2N}e^{\nu(t-s)} - Ne^{-\nu(t-s)}\right]\|P(x)\| \leq \frac{1}{N}e^{\nu(t-s)}\|Q(x)\| - Ne^{-\nu(t-s)}\|P(x)\|
\leq \|\Phi(t, s, x)Q(x)\| - \|\Phi(t, s, x)P(x)\|
\leq \|\Phi(t, s, x)\| \leq Me^{\omega(t-s)}.
\]
Let \(t_0 > 0\) be such that
\[
\lambda_0 := \frac{1}{2N}e^{\nu t_0} - Ne^{-\nu t_0} > 0.
\]
From the above estimation is follows that for \(t = t_0 + s\),
\[
\|P(x)\| \leq \frac{Me^{\lambda_0}}{\lambda_0}, \quad (\forall)x \in X.
\]
\(\Box\)
from where the conclusion follows.

**Remark.** In the following section we will see that for a dichotoomic pair \((C, P)\):

1. \((\text{u.s.e.d})\) does not imply \((\text{u.e.d})\)
2. \((\text{u.e.d})\) does not imply \((\text{u.s.e.d})\)
3. \((\text{u.w.e.d})\) does not imply \((\text{u.e.d})\)
4. \((\text{u.w.e.d})\) does not imply \((\text{u.s.e.d})\)

5. **Examples and Counterexamples**

**Example 5.1.** Define, on \(\mathbb{R}^3\), the family of projections

\[
P(x)(v_1, v_2, v_3) = (v_1, 0, 0)
\]

and the evolution cocycle on \(\mathbb{R}^3\):

\[
\Phi(t, s, x)(v_1, v_2, v_3) = \begin{cases} (v_1, v_2, v_3), & t = s \\ (e^{s-t}v_1, e^{s-t}v_2, 0), & t > s, \end{cases}
\]

with the following norm:

\[
\|x\| = |x_1| + |x_2| + |x_3|, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.
\]

We have that for all \((t, s) \in T, x \in X\) and \(v \in \mathbb{R}^3\)

\[
\|\Phi(t, s, x)P(x)v\| = e^{s-t}v_1 = e^{s-t}\|P(x)v\|
\]

from where we get that

\[
\|\Phi(t, s, x)P(x)\| \leq e^{s-t}\|P(x)\|
\]

and

\[
\|\Phi(t, s, x)Q(x)v\| = \begin{cases} \|Q(x)v\|, & t = s \\ \|Q(0, e^{s-t}v_2, 0)\|, & t > s \end{cases} \leq e^{s-t}\|Q(x)v\|
\]

hence

\[
\|\Phi(t, s, x)Q(x)\| \leq \|Q(x)\|.
\]

Choose \((0, 1, 0) \in \mathbb{R}^3\). Then

\[
\|\Phi(t, s, x)Q(x)(0, 1, 0)\| = e^{s-t}\|Q(x)(0, 1, 0)\|
\]

from where we finally obtain that:

\[
\|\Phi(t, s, x)Q(x)\| = e^{s-t}\|Q(x)\|,
\]

hence \((C, P)\) is \((\text{u.w.e.d})\). Assume by a contradiction that \((C, P)\) is \((\text{u.e.d})\). Then there exists, \(N \geq 1, \nu > 0\) such that

\[
N\|\Phi(t, s, x)Q(x)(v_1, v_2, v_3)\| \geq e^{\nu(t-s)}\|Q(x)(v_1, v_2, v_3)\|. \quad (5.1)
\]
Put \( t > s \) and \( (v_1, v_2, v_3) = (0, 0, 1) \). Then \( \|Q(x)(v_1, v_2, v_3)\| = 1 \) and
\[
e^{v(t-s)} \leq \|\Phi(t, s, x)(v_1, v_2, v_3)\| = \|\Phi(t, s, x)(0, 0, 1)\| = 0,
\]
which is a contradiction.

**Example 5.2** (u.e.d does not imply u.s.e.d). On \( V = \mathbb{R}^2 \) and \((X, d) = (\mathbb{R}_+, d)\) endowed with the max - norm. Consider,
\[
P(x) : \mathbb{R}^2 \to \mathbb{R}^2, \quad P(x)(v_1, v_2) = (v_1 + xv_2, 0)
\]
it follows that
\[
\|P(x)\| = 1 + x, (\forall)x \geq 0
\]
Define the skew - evolutiv cocycle
\[
\Phi(t, s, x) = e^{x+t}P(x) + e^{x-s}Q(x).
\]
We have that
\[
\|\Phi(t, s, x)P(x)\| = e^{x-t}\|P(x)\| \text{ and }
\|\Phi(t, s, x)Q(x)\| \geq e^{x-s}\|Q(x)\|
\]
Hence \((C, P)\) is (u.e.d). It can not be (u.s.e.d) because of (5.2).

**Remark.** From the above example, by taking the sup norm in (5.3) over \( v = 1 \), we get that \((C, P)\) is also (u.w.e.d). Hence \((C, P)\) is (u.w.e.d) but not (u.s.e.d).

**Remark.** The connection between the three concepts studied in this paper is summarized in the below diagram
\[
(u.s.e.d) \Rightarrow (u.e.d) \Rightarrow (u.w.e.d) \Leftrightarrow (u.s.e.d)
\]
\[
(u.s.e.d) \Leftrightarrow (u.e.d) \Leftrightarrow (u.w.e.d) \Rightarrow (u.s.e.d).
\]

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References


