On Certain Properties for Hadamard Product of Uniformly Univalent Meromorphic Functions with Positive Coefficients

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Abstract

In this paper we study some results concerning the Hadamard product of certain classes related to uniformly starlike and convex univalent meromorphic functions with positive coefficients.

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1. Introduction

Throughout this paper, let the functions of the form

\[ \varphi(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n \quad (c_1 > 0; c_n \geq 0), \]  

and

\[ \psi(z) = d_1 z - \sum_{n=2}^{\infty} d_n z^n \quad (d_1 > 0; d_n \geq 0) \]  

which are analytic and univalent in the unit disc

\[ U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}; \]

also, let

\[ f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_0 > 0; a_n \geq 0), \]  

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Lemma 1.1. Let the function $f$ defined by (1.3). Then $f \in U\Sigma S^*_0(\alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} [n(1 + \beta) + (\alpha + \beta)]a_n \leq (1 - \alpha)a_0. \quad (1.11)$$
Lemma 1.2 (3). Let the function \( f \) defined by (1.3). Then \( f \in U\Sigma C_0 (\alpha, \beta) \) if and only if
\[
\sum_{n=1}^{\infty} n[n(1+\beta) + (\alpha + \beta)]a_n \leq (1 - \alpha)a_0.
\]

Definition 1.1. Let the function \( f \) defined by (1.3). Then \( f \in U\Sigma S_m (\alpha, \beta) \) if and only if
\[
\sum_{n=1}^{\infty} n^m[n(1+\beta) + (\alpha + \beta)]a_n \leq (1 - \alpha)a_0,
\]
where \( 0 \leq \beta < \infty \), \( 0 \leq \alpha < 1 \) and \( m \) any positive integer number.

We note that \( U\Sigma S_1 (\alpha, \beta) = U\Sigma C_0 (\alpha, \beta) \) and \( U\Sigma S_0 (\alpha, \beta) \) is equivalent to \( U\Sigma S_0^* (\alpha, \beta) \). Further, \( U\Sigma S_m (\alpha, \beta) \subset U\Sigma S_r (\alpha, \beta) \) if \( m > r \geq 0 \), the containment being proper. Whence, for any positive integer \( m \), we have the inclusion relation
\[
U\Sigma S_m (\alpha, \beta) \subset U\Sigma S_{m-1} (\alpha, \beta) \subset \ldots \subset U\Sigma S_2 (\alpha, \beta) \subset U\Sigma C_0 (\alpha, \beta) U\Sigma S_0^* (\alpha, \beta).
\]

Also, we note that for nonnegative real number \( m \) the class \( U\Sigma S_m (\alpha, \beta) \) is nonempty as the functions of the form
\[
f(z) = \frac{a_0}{z} + \sum_{n=1}^{\infty} \frac{(1 - \alpha)a_0}{n^m[n(1+\beta) + (\alpha + \beta)]} \lambda_n z^n,
\]
where \( a_0 > 0 \), and \( \sum_{n=1}^{\infty} \lambda_n \leq 1 \), satisfy the inequality (1.13). For the functions
\[
f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n (a_{n,j} \geq 0; j = 1, 2).
\]

We denote by \((f_1 * f_2)(z)\) the Hadamard product (or convolution) of functions \( f_1(z) \) and \( f_2(z) \), that is
\[
(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n.
\]

Similarly, we can define the Hadamard product of more than two functions. The quasi-Hadamard product of two or more functions \( \varphi(z) \) and \( \psi(z) \) given by (1.1) and (1.2), (see (Kumar, 1987)).
\[
(\varphi * \psi)(z) = c_1 d_1 z - \sum_{n=2}^{\infty} c_n d_n z^n
\]

In this paper, we can discuss certain results concerning the Hadamard product of functions in the classes \( U\Sigma S_0^* (\alpha, \beta) \), \( U\Sigma S_m (\alpha, \beta) \) and \( U\Sigma C_0 (\alpha, \beta) \).
2. Main results

**Theorem 2.1.** Let the functions $f_i(z)$ defined by (1.4) be in the class $UΣC_0 (α, β)$ for every $i = 1, 2, \ldots, m$, and suppose that the functions $g_j(z)$ defined by (1.6) be in the class $UΣS^*_0 (α, β)$ for every $j = 1, 2, \ldots, q$. Then the Hadamard product $(f_1 * f_2 \cdots * f_m * g_1 * g_2 \cdots g_q)(z)$ belongs to the class $UΣS_{2m+q-1} (α, β)$.

**Proof.** It is sufficient to show that

$$
\sum_{n=1}^{\infty} \left\{ n^{2m+q-1} \{ n(1 + β) + (α + β) \} \left[ \prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right] \right\} \leq (1 - α) \left[ \prod_{i=1}^{m} a_{0,i} \prod_{j=1}^{q} b_{0,j} \right].
$$

(2.1)

Since $f_i(z) \in UΣC_0 (α, β)$, we get

$$
\sum_{n=1}^{\infty} n \{ n(1 + β) + (α + β) \} a_{n,i} \leq (1 - α) a_{0,i} \quad (i = 1, 2, \ldots, m).
$$

(2.2)

Therefore,

$$
a_{n,i} \leq \frac{(1 - α)}{n \{ n(1 + β) + (α + β) \}} a_{0,i}
$$

(2.3)

which implies that

$$
a_{n,i} \leq n^{-2} a_{0,i} \quad (i = 1, 2, \ldots, m).
$$

(2.4)

Similarly, for $g_j(z) \in UΣS^*_0 (α, β)$, we obtain

$$
\sum_{n=1}^{\infty} \{ n(1 + β) + (α + β) \} b_{n,j} \leq (1 - α) b_{0,j},
$$

(2.5)

for $j = 1, 2, \ldots, q$. Hence we have

$$
b_{n,j} \leq n^{-1} b_{0,j} \quad (j = 1, 2, \ldots, q).
$$

(2.6)

Using (2.4) for $i = 1, 2, \ldots, m$, (2.6) for $j = 1, 2, \ldots, q - 1$, and (2.5) for $j = q$, we have

$$
\sum_{n=1}^{\infty} \left\{ n^{2m+q-1} \{ n(1 + β) + (α + β) \} \left[ \prod_{i=1}^{m} a_{n,i} \prod_{j=1}^{q} b_{n,j} \right] \right\}
\leq \sum_{n=1}^{\infty} \left\{ n^{2m+q-1} \{ n(1 + β) + (α + β) \} \left[ n^{-2m} n^{-(q-1)} \prod_{i=1}^{m} a_{0,i} \prod_{j=1}^{q-1} b_{0,j} \right] b_{n,q} \right\}
= \left[ \prod_{i=1}^{m} a_{0,i} \prod_{j=1}^{q-1} b_{0,j} \right] \sum_{n=1}^{\infty} \left\{ n \{ n(1 + β) + (α + β) \} b_{n,q} \right\} \leq (1 - α) \left[ \prod_{i=1}^{m} a_{0,i} \prod_{j=1}^{q} b_{0,j} \right].
$$

Hence $(f_1 * f_2 \cdots * f_m * g_1 * g_2 \cdots g_q)(z) \in UΣS_{2m+q-1} (α, β)$. The proof of Theorem 1 is completed. }
Theorem 2.2. Let the functions $f_i(z)$ defined by (1.4) be in the class $USC_0(\alpha, \beta)$ for every $i = 1, 2, ..., m$, then the Hadamard product $(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $USS_{2m-1}(\alpha, \beta)$.

Proof. It is sufficient to show that

$$
\sum_{n=1}^{\infty} \left\{ n^{2m-1} \{ \sum_{i=1}^{m} a_{n,i} \} \right\} \leq (1 - \alpha) \left\{ \prod_{i=1}^{m} a_{0,i} \right\}.
$$

(2.7)

Since $f_i(z) \in USC_0(\alpha, \beta)$, the inequalities (2.1) and (2.2) hold for every $i = 1, 2, ..., m$.

Using (2.2) for $i = 1, 2, ..., m - 1$, and (2.1) for $i = 1, 2, ..., m$, we have

$$
\sum_{n=1}^{\infty} \left\{ n^{2m-1} \{ \sum_{i=1}^{m} a_{n,i} \} \right\} \leq \sum_{n=1}^{\infty} \left\{ n^{2m-1} \{ \sum_{i=1}^{m} a_{n,i} \} \right\}
$$

$$
= \left\{ \prod_{i=1}^{m} a_{0,i} \right\} \sum_{n=1}^{\infty} \left\{ n^{2m-1} \{ \sum_{i=1}^{m} a_{n,i} \} \right\} \leq (1 - \alpha) \left\{ \prod_{i=1}^{m} a_{0,i} \right\}.
$$

Hence $(f_1 * f_2 * ... * f_m)(z) \in USS_{2m-1}(\alpha, \beta)$. The proof of Theorem 2 is completed. \(\square\)

Theorem 2.3. Let the functions $f_i(z)$ defined by (1.4) be in the class $USS_0^*(\alpha, \beta)$ for every $i = 1, 2, ..., m$, then the Hadamard product $(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $USS_{m-1}(\alpha, \beta)$.

Proof. Since $f_i(z) \in USS_0^*(\alpha, \beta)$, we have

$$
\sum_{n=1}^{\infty} \left\{ n^{2m-1} \{ \sum_{i=1}^{m} a_{n,i} \} \right\} \leq (1 - \alpha) a_{0,i},
$$

(2.8)

for every $i = 1, 2, ..., m$. Therefore, we obtain $a_{n,i} \leq \frac{(1 - \alpha)}{m(1 + \beta + (\alpha + \beta))} a_{0,i}$ which implies that

$$
a_{n,i} \leq n^{-1} a_{0,i} \quad (i = 1, 2, ..., m).
$$

(2.9)

Using (2.9) for $i = 1, 2, ..., m - 1$, and (2.8) for $i = 1, 2, ..., m$, we have

$$
\sum_{n=1}^{\infty} \left\{ n^{2m-1} \{ \sum_{i=1}^{m} a_{n,i} \} \right\} \leq \sum_{n=1}^{\infty} \left\{ n^{2m-1} \{ \sum_{i=1}^{m} a_{n,i} \} \right\}
$$

$$
= \left\{ \prod_{i=1}^{m} a_{0,i} \right\} \sum_{n=1}^{\infty} \left\{ n^{2m-1} \{ \sum_{i=1}^{m} a_{n,i} \} \right\} \leq (1 - \alpha) \left\{ \prod_{i=1}^{m} a_{0,i} \right\}.
$$

Hence $(f_1 * f_2 * ... * f_m)(z) \in USS_{m-1}(\alpha, \beta)$, which completes the proof of Theorem 3. \(\square\)

Remark. Taking $\beta = 0$ in our main results, we obtain the results obtained by Mogra (Mogra, 1991).
References


