Abstract
In this paper, we have obtained sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ belonging to a new subclass of generalized Sakaguchi type functions introduced by (Frasin, 2010).

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1. Introduction
Let $A$ be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \quad (z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}) \tag{1.1}$$

and $S$ be the subclass of $A$ consisting of univalent functions. For two functions $f, g \in A$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\Delta$ and write $f \prec g$, or $f(z) \prec g(z)$; ($z \in \Delta$) if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \Delta$), such that $f(z) = g(w(z))$, ($z \in \Delta$). In particular, if the function $g$ is univalent in $\Delta$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Denote by $S^*_s$ the subclass of $S$ consisting of functions given by (1.1) satisfying $Re\left[\frac{zf'(z)}{f(z)-1}\right] > 0$ for $z \in \Delta$. These functions introduced by (Sakaguchi, 1959) are called functions starlike with respect to symmetric points.

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Recently (Frasin, 2010) introduced and studied a generalized Sakaguchi type class $S(\alpha, s, t)$ if it satisfies

$$\text{Re} \left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} > \alpha \tag{1.2}$$

for some $0 \leq \alpha < 1$, $s, t \in C$ with $s \neq t$ and for all $z \in \Delta$. Also denote by $T(\alpha, s, t)$ the subclass of $A$ consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, s, t)$. The class $S(\alpha, 1, t)$ was introduced and studied by Owa et al. (Owa et al., 2005, 2007). If $t = -1$, the class $S(\alpha, 1, -1) \equiv S(\alpha)$ (Sakaguchi, 1959) is called Sakaguchi function of order $\alpha$ (see (Cho et al., 1993; Owa et al., 2005)), where as $s(0) = S^*(\alpha)$ (Sakaguchi, 1959).

Note that $S(\alpha, 1, 0) \equiv S^*(\alpha)$ and $T(\alpha, 1, 0) \equiv C(\alpha)$ which are, respectively, the familiar classes of starlike functions of order $\alpha \ (0 \leq \alpha < 1)$ and convex functions of order $\alpha \ (0 \leq \alpha < 1)$. Mathur & Mathur (Trilok Mathur & Ruchi Mathur, 2012) investigated the classes $S^*_\phi(\phi, s, t)$ and $T(\phi, s, t)$ defined as follows.

**Definition 1.1.** Let $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$ be univalent starlike function with respect to 1 which maps the unit disk $\Delta$ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$. The function $f \in A$ is in the class $S^*_\phi(\phi, s, t)$ if

$$\left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} < \phi(z), \ s \neq t.$$

**Remark.** $T(\phi, s, t)$ denotes the subclass of $A$ consisting functions $f(z)$ such that $zf'(z) \in S^*_\phi(\phi, s, t)$. Observe that $S^*_\phi(\phi, 1, 0) \equiv S^*_\phi(\phi)$ and $T(\phi, 1, 0) \equiv C(\phi)$, which are the classes introduced and studied by Ma and Minda (Ma & Minda, 1994). Also note that $S^*_\phi(\phi, 1, -1) \equiv S^*_\phi(\phi)$, Shanmugam et al. (Shanmugham et al., 2006).

The $q^\text{th}$ Hankel determinant for $q \geq 1$ and $n \geq 0$ is stated by Noonan and Thomas (Noonan & Thomas, 1976) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q} \\ a_{n+1} & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & \cdots & a_{n+2q-2} & \end{vmatrix}$$

This determinant has also been considered by several authors. For example, Noor (Noor, 1983) determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f$ given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles (Hayami & Owa, 2010; Janteng et al., 2008; Kharudin et al., 2011; Noor, 1983; Selvaraj & Vasanthi, 2010) for different classes of functions.

Easily, one can observe that the Fekete-Szego functional is $H_2(1)$. Fekete-Szego then further generalised the estimate $|a_3 - \mu a_2^2|$ where $\mu$ is real and $f \in S$. For our discussion in this paper, we consider the Hankel determinant in the case $q = 2$ and $n = 2$,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$
2. Preliminary Results

Let $P$ denote the class of functions

$$p(z) = 1 + c_1z + c_2z^2 + \cdots$$

which are regular in $\Delta$ and satisfy $\text{Re } p(z) > 0, z \in \Delta$. Throughout this paper we assume that $p(z)$ is given by (2.1) and $f(z)$ is given by (1.1). To prove the main results we shall require the following lemmas.

**Lemma 2.1.** (Duren, 1983) Let $p \in P$, then $|c_k| \leq 2, k = 1, 2, \ldots$ and the inequality is sharp.

**Lemma 2.2.** (Libera & Zlotkiewicz, 1982, 1983) Let $p \in P$, then

$$2c_2 = c_1^2 + 4c_1(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2y(1 - |x|^2)(4 - c_1^2)$$

for some $x, y$ such that $|x| \leq 1$ and $|y| \leq 1$.

3. Main Results

**Theorem 3.1.** If $f \in S_\phi^*(\phi, s, t)$, then

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{(3 - s^2 - st - t^2)^2}, \text{ provided } s + t \neq 2.$$ (3.1)

The result obtained is sharp.

**Proof.** Let $f \in S_\phi^*(\phi, s, t)$. Then there exists a Schwarz function $w(z) \in A$ such that

$$\begin{cases} (s - t)zf'(z) & = \phi(w(z)), \quad (z \in \Delta, s \neq t) \\ f(sz) - f(tz) & \end{cases}$$ (3.2)

If $P_1(z)$ is analytic and has positive real part in $\Delta$ and $P_1(0) = 1$, then

$$P_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots$$ (3.3)

From (3.3) we obtain

$$w(z) = \frac{c_1}{2}z + \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots$$ (3.4)

Let

$$p(z) = \frac{(s - t)zf'(z)}{f(sz) - f(tz)} = 1 + b_1z + b_2z^2 + \cdots; \quad (z \in \Delta)$$ (3.5)
which gives

\[ b_1 = (2 - s - t)a_2, \]
\[ b_2 = (3 - s^2 - st - t^2)a_3 + (s + t)(s + t - 2)a_2^2, \]
\[ b_3 = (4 - s^3 - st - s t^2 - t^3)a_4 + 2(s^2 + st + t^2)(s + t - 1)a_2a_3 \]
\[ + 2a_2^2(s + t)^2 - 3a_2a_3(s + t). \]

Since \( \phi(z) \) is univalent and \( P < \phi \), therefore using (3.4) we obtain

\[ P(z) = \phi(w(z)) = 1 + \frac{B_1c_1}{2} z + \left\{ \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}c_1^2B_2 \right\} z^2 \]
\[ + \frac{B_1}{2} \left\{ c_3 + c_1 \left( \frac{c_1^2}{2} - c_2 \right) - c_1c_2 \right\} \]
\[ + \frac{B_1c_1}{2} \left( \frac{c_2 - c_1^2}{2} + \frac{B_3c_1^3}{8} \right) z^3 + \cdots \quad (3.6) \]

Now from (3.5) and (3.6) we have

\[ \frac{(s - t)z'f(z)}{f(sz)-f(tz)} = 1 + \frac{B_1c_1}{2} z + \left\{ \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}c_1^2B_2 \right\} z^2 + \cdots \quad (3.7) \]

On equating the coefficient of \( z, z^2 \) and \( z^3 \) in (3.7) we obtain

\[ a_2 = \frac{B_1c_1}{2(2 - s - t)}. \]
\[ a_3 = \frac{1}{(3 - s^2 - st - t^2)} \left\{ \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}c_1^2B_2 + \frac{(s + t)B_1c_1^2}{4(2 - s - t)} \right\}. \]
\[ a_4 = \frac{1}{(4 - s^3 - st - s t^2 - t^3)} \left[ B_1c_3 + c_1^3 \right. \]
\[ \left. + B_1 \left( \frac{B_1}{4} - \frac{B_2}{4} + \frac{B_3}{8} - \frac{B_1^2(s + t)^2}{4(2 - s - t)^3} \right. \right] \]
\[ + \left. \frac{(s + t + 2(s^2 + st + t^2))}{16(2 - s - t)(3 - s^2 - st - t^2)} \right] \]
\[ + c_1c_2 \left[ - \frac{3B_1}{2} + \frac{B_2}{2} \right] \]
\[ + B_1^2 \frac{3(s + t) - 2(s^2 + st + t^2)(s + t - 1)}{8(2 - s - t)(3 - s^2 - st - t^2)} \]
Thus we have,

\[
|a_2 a_4 - a_3^2| = \frac{B_1 c_1}{2P_1} \left\{ B_1 c_3 + c_1^3 \left[ \frac{2(B_1 - B_2) + B_3}{8} - \frac{B_1^3(s + t)^2}{4(2 - s - t)^3} + \frac{3(s + t) - 2(s^2 + st + t^2)(s + t - 1)}{16(2 - s - t)(3 - s^2 - st - t^2)} \left\{ B_1 B_2 - B_1^2 + \frac{B_1^3}{(2 - s - t)} \right\} \right\} \\
+ c_1 c_2 \left[ -\frac{3B_1}{2} + B_2 + \frac{B_1^2}{8(2 - s - t)(3 - s^2 - st - t^2)} \right] \\
+ \frac{P_3}{c_1^4 f_1(s, t) + c_1^2 g_1(s, t) + \frac{c_1^2 B_1^2}{16} + \frac{c_1^2 B_1^3}{4}}.
\]

Suppose now that \(c_1 = c\). Since \(|c| = |c_1| \leq 2\), using the Lemma 2.1, we may assume without restriction \(c \in [0, 2]\). Substituting for \(c_2\) and \(c_3\), from Lemma 2.2 and applying the triangle inequality with \(\rho = |x|\), we obtain

\[
|a_2 a_4 - a_3^2| \leq c^4 \left[ \frac{f(s, t)}{P_1} + \frac{B_1 g(s, t)}{4P_1} - \frac{f_1(s, t)}{P_3} - \frac{g_1(s, t)}{2P_3} + \frac{B_1}{16P_3} \right] \\
+ c^2(4 - c^2)\rho \left[ \frac{B_1}{4P_1} + \frac{B_1^2}{4P_1} - \frac{B_1^2\rho}{8P_1} - \frac{g_1(s, t)}{2P_3} + \frac{B_1}{8P_3} \right] \\
+ \frac{c^2 B_1^2}{16P_3} + \frac{B_1^2 c(4 - c^2)(1 - \rho^2)}{4P_1} + \frac{B_1^2 \rho^2(4 - c^2)^2}{16P_3} \\
= F(\rho)
\]

(3.8)

where,

\[
P_1 = (2 - s - t)(4 - s^3 - st^2 - s^2 t - t^3),
\]

\[
P_2 = 8(2 - s - t)(3 - s^2 - st - t^2),
\]

\[
P_3 = (3 - s^2 - st - t^2)^2,
\]

\[
f_1(s, t) = \frac{B_1^2}{16} + \frac{(s + t)^2 B_1^4}{16(2 - s - t)^2} - \frac{B_1 B_2}{8} + \frac{B_1^2 B_2(s + t)}{8(2 - s - t)} + \frac{(s + t)B_1^3}{8(2 - s - t)},
\]

\[
g_1(s, t) = \frac{(s + t)B_1^2}{4(2 - s - t)^2} + \frac{B_1 B_2}{4} - \frac{B_1^2}{4},
\]

\[
f(s, t) = \frac{B_1^2}{4} - \frac{B_1 B_2}{8} + \frac{B_1 B_3}{8} - \frac{B_1^2(s + t)^2}{8(2 - s - t)^3} + \frac{[(B_1 B_2 - B_1^2)(2 - s - t)] + B_1^2(3(s + t) - 2(s^2 + st + t^2)(s + t - 1))}{32(2 - s - t)^2(3 - s^2 - st - t^2)},
\]

\[
g(s, t) = -B_1 + \frac{B_2}{2} - \frac{B_1^2(3(s + t) - 2(s^2 + st + t^2)(s + t - 1))}{2}.
\]
with $\rho = |x| \leq 1$. Furthermore

$$F'(\rho) = c^2(4 - c^2) \left[ \frac{B_1}{4P_1} + \frac{B_1^2}{4P_1} - \frac{B_1^2\rho}{8P_1} - \frac{B_1}{2P_3} - \frac{B_1}{8P_3} \right]$$

$$+ \frac{B_1^2c(4 - c^2)(4 - c)}{8P_1} + \frac{B_1^2(4 - c^2)^2c}{8P_3}.$$ 

For a $c \in [0, 2]$, $F(\rho) \leq F(1)$, that is

$$|a_2a_4 - a_3^2| \leq c^4 \left[ \frac{f(s, t)}{P_1} + \frac{B_1g(s, t)}{4P_1} - \frac{f_1(s, t)}{P_3} - \frac{g_1(s, t)}{2P_3} - \frac{B_1}{16P_3} \right]$$

$$+ c^2(4 - c^2) \left[ \frac{B_1}{4P_1} + \frac{B_1^2}{4P_1} - \frac{B_1^2c(4 - c^2)(4 - c)}{8P_1} - \frac{B_1(4 - c^2)^2c}{16P_3} \right]$$

$$= G(c).$$

By elementary calculus we have $G''(c) \leq 0$ for $0 \leq c \leq 2$ and $G(c)$ has real critical point at $c = 0$. Thus the upper bound of $F(\rho)$ corresponds to $\rho = 1$ and $c = 0$. Thus the maximum of $G(c)$ occurs at $c = 0$. Hence,

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{(3 - s^2 - st - t^2)^2}.$$

If $p(z) \in P$ with $c_1 = c = 0$, $c_2 = 2$, $c_3 = 1$, then we obtain

$p(z) = (1 - z) + \frac{z}{(1 - z)'} = 1 + 2z^2 + z^3 + \cdots \in P$. The result is sharp for the functions defined by

$$\left\{ \begin{array}{l} (s - t)z f'(z) \\ f(sz) - f(tz) \end{array} \right\} = \phi(z), \ s \neq t$$

and

$$\left\{ \begin{array}{l} (s - t)z f'(z) \\ f(sz) - f(tz) \end{array} \right\} = \phi(z^2), \ s \neq t.$$

Remark. If $f \in S_3^2(\phi, 1, -1)$, then

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{2}.$$

Since $f(z) \in T(\phi, s, t)$ if and only if $z f'(z) \in S_3^2(\phi, s, t)$, proceeding on similar lines as in Theorem 3.1 we obtain the upper bound for the functional $|a_2a_4 - a_3^2|$ belonging to the class $T(\phi, s, t)$ which is stated below without proof.

**Theorem 3.2.** If $f \in T(\phi, s, t)$, then

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{9(3 - s^2 - st - t^2)^2}, \text{ provided } s + t \neq 2.$$

(3.9)

The result obtained is sharp.
Remark. If $f \in T(\phi, 1, -1)$, then
\[ |a_2a_4 - a_3^2| \leq \frac{B_2^2}{36}. \]

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References


